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A NOVEL-TYPE ANALYTICAL SOLUTION OF THE PROBLEM OF SIMULTANEOUS CONVECTION AND MULTI COMPONENT DIFFUSION PROCESS THROUGH POROUS MEDIA

Ágnes Bálint[1,2*], István Nikolényi[3], Csaba Mészáros[3]

[1]*Institute of Environmental Engineering and Natural Sciences,*
[2]*Hydro-Bio-Mechanical Systems ResearchCentre, Óbuda University, Budapest, Hungary, e-mail:* balint.agnes@uni-obuda.hu
[3]*Department of Mathematics and Basic Sciences, MATE, Gödöllő, Hungary, e-mail:* Meszaros.Csaba@uni-mate.hu, Nikolenyi.Istvan.Robert@uni-mate.hu

Abstract: *A novel-type analytical solution is proposed for simultaneous convection-diffusion processes taking place through porous media. It is shown, that the Riccati-type ordinary differential equation, frequently applied at modelling of such processes, may be interpreted in a more general manner, as it has been done. For simultaneous convection and diffusion of multi-component, chemically interacting materials, a completely new modelling type approach is proposed and described in the present work. Then, the integration factor function method is applied, and completely novel-type formulae are derived for solving of the adequate types of the Riccati's ordinary differential equation. Finally, the new analytical results are compared to the earlier analytical solutions and their relevance for accurate future modelling of such types of drying processes is also discussed concisely.*

Keywords *Fractional-order derivatives, Percolative-fractal processes, Riccati-type ordinary differentialequations, Simultaneous convection and diffusion*

INTRODUCTION

It is known, that in the contemporary theories of coupled transport phenomena through porous media methods of both non-equilibrium thermodynamics [1] and many ones from the confident, well-established results of the contemporary theories of critical phenomena relevant for percolative systems [2],[3],[4] should simultaneously be taken into account. Although many of these methods have also been applied in the contemporary modelling and simulation of drying processes, there are further possibilities for their common new applications, to which our work is devoted, too.

The mathematical modelling of convection-diffusion processes is of importance from the points of view of both fundamental research and solving of engineering problems, whose complexity is frequently manifested by the nonlinear character of the ordinary differential equations (ODEs) and partial differential equations (PDEs) to be solved during the modelling procedure. The basic PDE for the convection-diffusion problems is [5]:

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$$\frac{\partial c}{\partial t} - \nabla\left(D\left(c\right)\nabla c\right) + \frac{dK}{dc}\cdot\frac{\partial c}{\partial z} = 0, \tag{1}$$

where c(r,t) denotes the concentration distribution function to be determined, D(c,T) is the (usually:) thermodynamically state-dependent diffusion coefficient, K(c) is the concentration-dependent hydraulic conductivity coefficient and z-axis corresponds to the direction of the gravitational acceleration. The general solution of the basic problem for a large class of convection - type transport processes through porous media can be suitably explained in the following finite sum form [6],[7]:

$$\begin{gathered} c\left(\zeta\right) = a_0 + \sum_{i=1}^{q}\left(a_i\cdot\omega^i + b_i\cdot\omega^{-i}\right), \\ a_0, a_i, b_i = \text{const.}, \left(1\le i\le q\right), q\in\mathbb{N} \end{gathered} \tag{2}$$

and it is explained by use of D'Alembert-type independent variables ζ instead of simple spatial coordinates. (This variable is defined as $\zeta := x - v\cdot t$, with a constant value "v" of the convection flow velocity). The physical meaning of the formula (2) is, that it gives analytical solution of the basic convection-diffusion equation (1) in the form of multiple travelling waves. It is a very important for the solution formula (2), that some very different types of effects (e.g. dispersion, dissipation and nonlinearities of various types of origin), which all may change forms of the multiple wave forms being discussed have to be balanced out. Furthermore, the functions of the separate modes entering the general solution (2), must obey the following Riccati-type ODE:

$$\frac{d\omega}{d\zeta} = \omega^2 + k, \tag{3}$$

with "k" as a parameter depending on the "actual experimental conditions" [6]. The solutions of this basic convection-diffusion problem may be even of solitonic type. For our present study, it is of crucial importance, that we are always faced with applications of the Riccati's ODE. In order to realize our modelling work, we specify the eq. (3) as follows:

$$\begin{gathered} \omega' = \omega^2 + k \to \omega' = \omega^2 + \sum_m a_m\left(p\right)\zeta^m, \\ \left(\omega' \equiv \frac{d\omega}{d\zeta}\right) \end{gathered} \tag{4}$$

i.e. the constant coefficient "k" originally used in [6] has been replaced here by a series $\sum_m a_m\left(p\right)\cdot\zeta^m$, where the series expansion coefficients are assumed to be of stochastic character, due to the genuine percolative-fractal character of the dried bulk porous material to be observed at mesoscopic level.

MATERIALS AND METHODS

Following the general modelling concept described previously, in the present section we will justify concisely the most essential mathematical features of the modelling methods to be applied, following mainly the technique discussed in detail in our own earlier study [7]. Accordingly, as an essential new feature of technique initiated by us (which is nevertheless of very simplifying character), we would like to point out here, that firstly, the following reduced form of (4) will be applied:

$$\frac{d\omega}{d\zeta} - \omega^2 - a_n(p)\cdot\zeta^n = 0, \tag{5}$$

i.e. only the "dominant convective term" has been selected in the ODE (4), characterized by the expression $a_n(p)\cdot\zeta^n$. Then, after linearizing this last ODE (5), we got a second-order ODE expressed by

$$\frac{d^2\omega}{d\zeta^2} - a_n(p)\cdot\zeta^n\cdot\omega(\zeta) = 0. \tag{6}$$

The solution of this types of ODEs can be expressed in a closed form by use of Bessel-functions of the first-, and second kind, respectively, as (for the sake of generality, the dependent variable change notation $\omega(\zeta)\mapsto y(\zeta)$ has also been applied here):

$$\begin{aligned} y(\zeta) &= C_1\sqrt{\zeta}\cdot J_{\frac{1}{n+2}}\left(\frac{2\sqrt{-a_n(p)}\cdot\zeta^{\frac{n}{2}+1}}{n+2}\right)+ \\ &+C_2\sqrt{\zeta}\cdot N_{\frac{1}{n+2}}\left(\frac{2\sqrt{-a_n(p)}\zeta^{\frac{n}{2}+1}}{n+2}\right), \\ &C_1, C_2 = \text{const.}, \end{aligned} \tag{7}$$

i.e. the final result is explained by Bessel-functions, which are of different kind, but of the same order (1/(n+2)). The graphical representation of this solution function is given (in relative units) on the next figure.

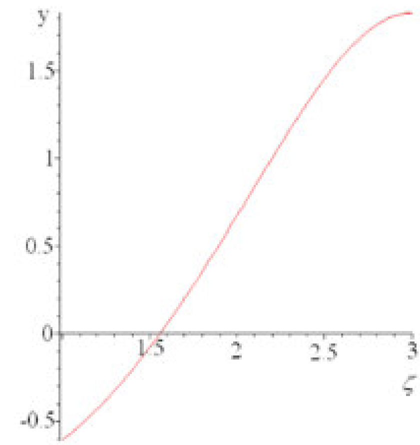

Figure (1) Graphical presentation of solution of the ODE (5) after linearization

Then, by taking into account the genuine very dissipative character of different types of convection-diffusion processes through porous media ranging from sub-, to superdiffusion type [8-9], a further, physically justified modification of the above proposed general solution must also be mentioned here. Namely, at a complete treatment of dispersion phenomena, the basic solution of the simultaneous convection-diffusion-type transfer processes through porous media in form of travelling waves, supplementary time-dependent factor functions imposed on each mode from (2), and ensuring decaying character of the amplitudes of such waves must also be taken into account [10].

Although the solution (7) represents a refined variant of the direct "tanh-function type solution" proposed initially by Fan, the method (explained essentially by the ODEs (4) and (5)) leading to it, will be improved in a slightly different way.

Namely, despite of the fact, that we have obtained solution of the problem we are investigating in a closed, analytical form, it is necessary to extend the calculation method applied, in order to open possibility for including further terms (in the sense of the eq. (4)) beyond the "dominant term" into the general modelling procedure we are developing, i.e. by

$$\frac{d^2\omega}{d\zeta^2} - a_n(p)\cdot\zeta^n\cdot\omega(\zeta) = a_r(p)\cdot\zeta^r. \tag{8}$$

Therefore, having solved the "basic homogeneous problem" (6), we may directly apply the well-known basic general formula (based on the classical Lagrange's method of variation of constants) for solution of the relevant inhomogeneous second-order ODE (6), and by this way (using the MAPLE computer algebra system [11]) we have also derived the following general solution formula:

$$
\begin{aligned}
\omega(\zeta) = {} & C_1\sqrt{\zeta}\cdot J_{\frac{1}{n+2}}(\xi) + C_2\sqrt{\zeta}\cdot N_{\frac{1}{n+2}}(\xi) + \\
& + \zeta^{\frac{1}{2}}\cdot J_{\frac{1}{n+2}}(\xi)\times\int \frac{(-1)\cdot a_r(p)\cdot\zeta^r\cdot N_{\frac{1}{n+2}}(\xi)d\zeta}{J_{\frac{1}{n+2}}(\xi)\left[\zeta^{\frac{1}{2}}N_{\frac{1}{n+2}}(\xi)\right]' - N_{\frac{1}{n+2}}(\xi)\left[\zeta^{\frac{1}{2}}J_{\frac{1}{n+2}}(\xi)\right]'} \\
& + \zeta^{\frac{1}{2}}\cdot N_{\frac{1}{n+2}}(\xi)\times\int \frac{a_r(p)\cdot\zeta^r\cdot J_{\frac{1}{n+2}}(\xi)d\zeta}{J_{\frac{1}{n+2}}(\xi)\left[\zeta^{\frac{1}{2}}N_{\frac{1}{2}}(\xi)\right]' - N_{\frac{1}{n+2}}(\xi)\left[\zeta^{\frac{1}{2}}J(\xi)\right]'},
\end{aligned}
\tag{9}
$$

where again, the independent variable defined by $\xi := \dfrac{2\sqrt{-a_n(p)}}{(n+2)\cdot\zeta^{-\left(\frac{n}{2}+1\right)}}$ has been introduced and applied.

Then, the whole procedure applied here may be continued (and therefore: algorithmized in a suitable manner) by using gradually further terms from (4), having « less-dominant » character compared to the previously included ones.

One concrete explicit solution form corresponding to the subdiffusion limit situation ([1],[2]) is the following one. Accordingly, by direct use of the MAPLE computer algebra system, we arrive at (the exact meaning all of the coefficients in it, is also specified in the basic reference works [1],[2] we have mentioned):

$$
\begin{aligned}
& \frac{A^2(3-q)(q-2)}{B\cdot\left(A^2\right)^{1/(q-2)}}\cdot c_{\mathrm{SubDiff}}(x,t) = \\
& = -3\cdot t^{-1/q}\cdot F\left(\left[\frac{1}{2},-\frac{1}{q-2}\right],\left[\frac{3}{2}\right],\frac{t^{-2/q}\cdot x^2}{A^2}\right)A^2q^2 + \\
& +2\cdot t^{-3/2}x^2q\cdot F\left(\left[\frac{q-3}{q-2},\frac{3}{2}\right],\left[\frac{5}{2}\right],\frac{t^{-2/q}\cdot x^2}{A^2}\right) + \\
& +6v\cdot t^{\frac{q-3}{q}}xq\cdot F\left(\left[\frac{3-q}{2},\frac{q-3}{q-2}\right],\left[\frac{5-q}{2}\right],\frac{t^{-2/q}\cdot x^2}{A^2}\right) \\
& +15\cdot t^{-1/q}\cdot F\left(\left[\frac{1}{2},-\frac{1}{q-2}\right],\left[\frac{3}{2}\right],\frac{t^{-2/q}\cdot x^2}{A^2}\right)A^2q + \\
& -6\cdot t^{-3/q}x^2\cdot F\left(\left[\frac{q-3}{q-2},\frac{3}{2}\right],\left[\frac{5}{2}\right],\frac{t^{-2/q}\cdot x^2}{A^2}\right) + \\
& -18\cdot t^{-1/q}\cdot F\left(\left[\frac{1}{2},-\frac{1}{q-2}\right],\left[\frac{3}{2}\right],\frac{t^{-2/q}\cdot x^2}{A^2}\right).
\end{aligned}
$$

RESULTS AND DISCUSSION

The spreading out of the basic solution curve (i. e. dispersion) in the form of travelling waves can be assigned to the convection-diffusion equation, which is in the present study applied by stochastic change of the dominant convective term figuring in it. This procedure may also be directly generalized to the case of simultaneous convection and diffusion of several components, and the well-known Lagrangian representation of continuum mechanics has proven to be suitable from this point of view. Having discussed the essential basic general features of the simultaneous convection-diffusion processes taking place through porous media, we came to position for applying of the Riccati's ordinary differential equation in a novel manner, which method is suitable for modelling of this basic problem of drying processes under very different types of outer conditions. This modelling result, as well as another recent one [12], emanating from an attempt of introducing of the most essential thermodynamic features of the anomalous diffusion-type transfer processes (which represent a macroscopic manifestation of the genuine percolative-fractal character of the porous materials to be dried at mesoscopic level) into the recent drying theoretical formalisms.

CONCLUSIONS

In the present study a novel-type general method has been described for modelling of simultaneous convection-diffusion processes taking place in bulk porous media. The relevance of the Riccati-type ordinary differential equation is accepted as a crucial mathematical background necessary for modelling of such types of transport processes. According to this novel-type algorithm for accurate modelling of the simultaneous convection-diffusion processes through bulk porous media, the possible problems appearing because of the inclusion of more than the first dominant convective term into the linearized Riccati's ordinary differential equation is also eliminated.

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