E.Imre, Q.P.Trang: Some notes on the dry density for sands

E. Imre, Q. P. Trang*,
Óbuda University,Kandó Kálmán Faculty of Electrical Engineering Budapest, Hungary & Geotechnical Department, BME, Budapest, Hungary
*Bachy Soletanche, 126 Nguyen Thi Minh Khai St., Dist. 3, Ho Chi Minh City, Vietnam

Abstract: A transfer function generation method is applied in the simplest form to find a relationship between the grading curve and the dry density. The method is based on the grading entropy concept and uses laboratory emax test data measured on some "optimal" grading curves. The goodness of results is tested using some additional laboratory emax test data of some "non-optimal" soils mixtures. As a by-product of the research, some comments are made on the "densest packing problem".

Keywords: grading curve, grading entropy, void ratio, density, sand, transfer function

Introduction

The aim of the ongoing research is to elaborate a grading curve – dry density relationship. Being the ratio of the smallest and largest dry density constant for sands (Kabai, 1972), the research is dealing with the largest dry density only.
It is a general view that the structure of granular materials and multi-component mixtures is complex and possibly intractable with a direct geometrical model (see e.g. Dexter, Tanner (1971)). Some recent approaches use semi-analytical models (Yu, Standish (1988)) based on the analogy with solutions.
In this paper a different approach, the transfer function generation method (Imre et al, 2008) is applied for the smallest dry density. The method is based on the grading entropy concept and the non-normalised entropy diagram (Lőrincz et al, 2005, Fig 1) which is briefly described in Appendix 1. Some laboratory e_{max} test data of "optimal" soil mixtures are used for the generation of an approximate dry density transfer function. First result are presented and, some comments are made concerning the densest packing problem, too.

Methods

For the elaboration of a dry density transfer function, laboratory e_{max} test data were made on some optimal soil mixtures, composed of maximum 5 fractions (i.e. N=1, 2, 3 and 5, Fig 2, A-1).
For the testing of the validity of the transfer function, some additional e_{max} tests of non-optimal mixtures were made in this research (Fig A-2) and were reused from earlier studies (Kabai, 1972, App 3). The earlier non-optimal grading curve series, denoted by A_0, B_0, B_1, B_2, B_3, defined in terms of U, d_{max}, d_{10} are shown in Table 2, Figures 3 and A-3.
In the e_{max} test, the sand is poured through a funnel into a cylinder (10 cm high and 10 cm diameter Fig 4). During this process the bottom of the funnel is positioned just above the soil surface, so that the opportunity for grains to rearrange and pack is minimised.

Table 1 The grading curve series of Kabai (see also in Figure A- 3)

series	B_0	B_1	B_2	B_3	C_0
d_{max} [mm]	0.145-20.0	0.29-4.74	0.58-1.54	1.24-9.42	4.74
D_{10} [mm]	0.145	0.29	0.58	1.24	

U [-]	1-30	1-5	1-5	1-3	1-8

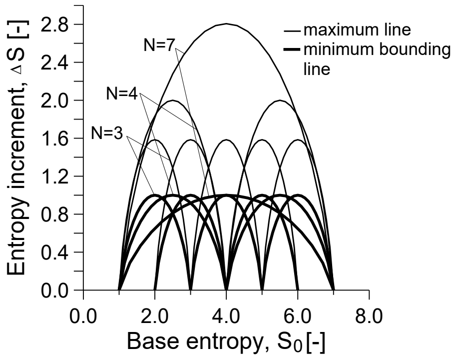


Figure 1. Unified non-normalized entropy diagram.

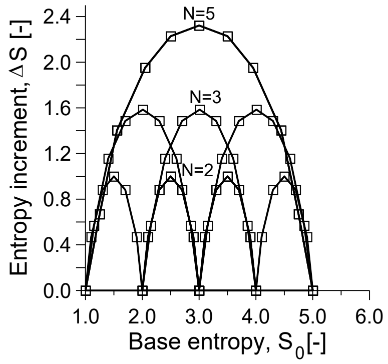


Figure 2. The optimal soils tested in the non-normalized diagram

Results

The results of the laboratory e_{max} tests are shown in Tables A-1 to A-5 (in App 2) in terms of the minimum dry solid volume ratio [s_{min}], with the corresponding entropy parameter values given. The results can be expressed in terms of the void ratio in the loosest state e_{max}=(1- s_{min})/ s_{min} or the smallest dry density $\rho_{dmin}= s_{min} \cdot \rho_S$ or the specific volume v_{max} =1/s_{min}.

Densest packing problem

For the continuous, three-fraction soils (N=3), the iso-density lines (i.e. s_{min} level lines) were constructed in the classical triangle diagram (Fig 5(a)). Noting that, x_1, x_2, x_3 denotes the proportion of fine fraction C, middle fraction D and coarse fraction E; the following regular shape was observed at the upper part of the triangle diagram. The densest state down to the horizontal line (x_2=x_2*) was found always at about x_2=x_2* and x_1=1/3 (1- x_2*). The densest packing occurred at x_2*=0, x_3=1/3, x_1=2/3 (i.e. gap-graded grading curve). In general, the s_{min} values of the optimal grading curve series for any tested N>1 values showed a regular pattern. The maximum s_{min} value was found at about A=2/3 (Fig 6).

The image of the level lines

For the three-fraction soils (N=3), the simplex can be represented in the two dimensional space and the image entropy map is a “two sheeted cover” (Figs A-1(c), 5(b)). This means that on the entropy diagram, each point will belong to two s_{min} level lines from the simplex or that each point on the entropy diagram has 2 discrete inverse points on the simplex. The s_{min} value is not the same on these. For example, a diagram point may belong to both the s_{min}=0.55 and s_{min}=0.57 level lines. The slopes of the two sheets at each point are different (generally one positive and one negative) and the negative slope is more uniform (Fig 5(b)).

The level surfaces can not be visualized in the entropy diagram for larger dimensions. Whereas for N=3, the inverse of the entropy diagram point was generally 2 points in the simplex (“0 dimensional circle”), for N, the inverse image appeared as “N-3 dimensional circle” on the simplex which means infinite many points. For N=4, the s_{min} values were determined in some points of an inverse image “1-circle” shown in Figure A-2(c). The measured values were not constant, as it can be seen in Figure 7.

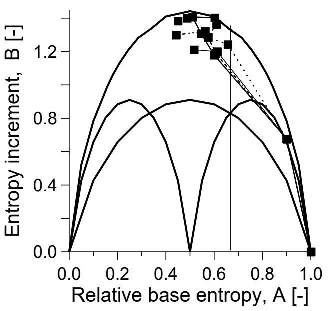


Figure 3. The non-optimal soils of Kabai in the normalized diagram

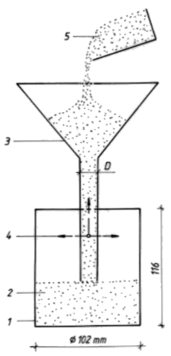
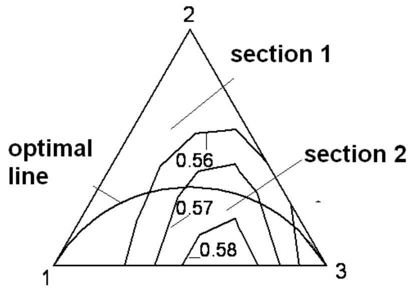


Figure 4. The definition of the e_{max} test

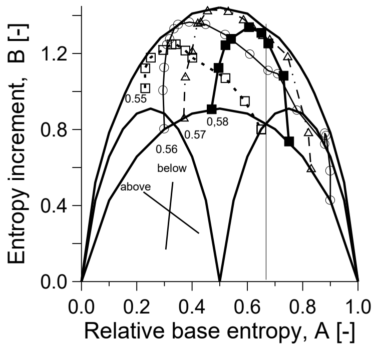
Dry density transfer function

Some s_{min} data measured in some optimal grading curves for N=2, 3 and 5 were used for the transfer function definition. Iso-density lines (i.e. s_{min} level lines) were constructed by graphical interpolation in the non-normalized entropy diagram, as about parallel straight lines with negative slope (Fig 8). According to the results, the fractions with increasing diameter had greater density and, the density increased with the entropy increment in about equal rate in the tested part of the non-normalized entropy diagram.

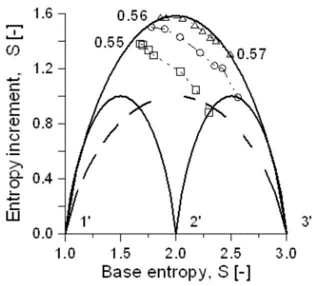
To test the validity of the transfer function, iso-density lines (i.e. s_{min} level lines) were constructed using the non-optimal grading curve data of Kabai (1972). The similar pattern of the s_{min} level lines indicated that the transfer function is acceptable for some class of the non-optimal grading curves in this preliminary form (Fig 9).



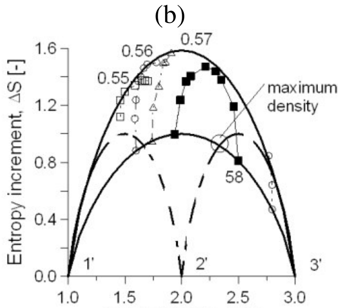
(a)



(b)



(c).



(d)

Figure 9. Example for N=5. The dry density transfer function defined by the s_{min} iso-lines. (a) Simplex representation. (b) Entropy diagram representation.

(c) The sheet mapped from above the optimal line (d) The sheet mapped from below the optimal line.

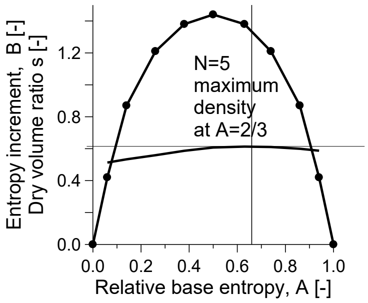


Figure 6. The $A - s_{min}$ relation showing the tested grading curves in the normalized diagram for N=5

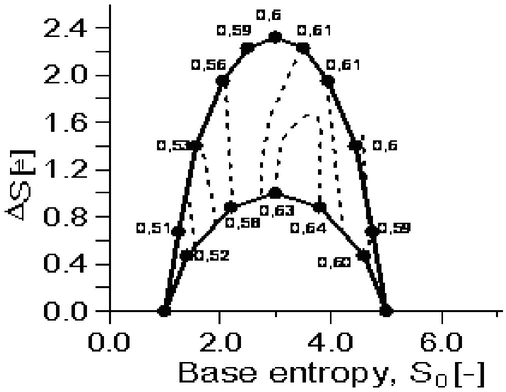


Figure 7. The iso-density (s_{min}) level lines, maximum continuous and gap-graded mixtures N=5.

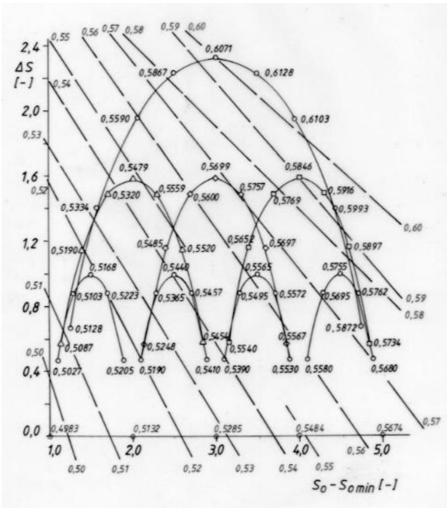


Figure 8. The dry density transfer function defined by the iso-density (s_{min}) level lines, continuous mixtures.

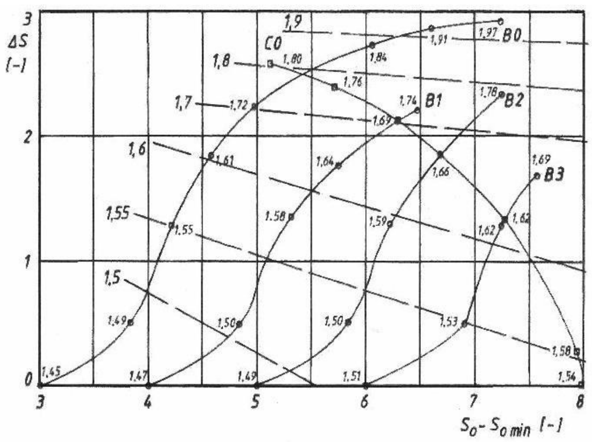


Figure 9. The iso-density ($\sqcup_{dmin}$) level lines.

Discussion, conclusion

Dry density transfer function

A preliminary transfer function was determined between the grading curve and the dry density of sands using the method described in the companion paper (Imre et al, 2008). The iso-density (s_{min}) lines of the transfer function were constructed by interpolation based on some s_{min} data of optimal grading curves with fraction number of N=2, 3 and 5 in the non-normalized entropy diagram. The s_{min} level lines were about parallel straight lines with negative slope (Fig 6).

The slope of the iso-lines can be estimated as the slope of the tangent of the maximum ΔS line at A=2/3 (i.e. using the fact that the measured s_{min} is maximal within an optimal grading curve series at A=2/3, see Fig 6). The so-computed tangent is equal to -1 for N=2.

It is assumed for the transfer function generation method, that the scalar property is the same at the nearly coinciding parts of the image of the optimal grading curves in the non-normalized entropy diagram. According to the results, this assumption was basically met.

For N=3, the entropy map is a two sheeted cover (Fig 5(b)) which can be used to illustrate the main features of the transfer function approach. The image of the true s_{min} level lines (constructed in the triangle diagram, using optimal grading curve data) mapped from the upper part of the triangle diagram (i.e. from above "line *a*", Fig A-1(c)) are the part of the iso-density lines of the transfer function. However, the s_{min} level lines mapped from the lower part of the triangle diagram (i.e. from below "line *a*", Fig A-1(c)) are missing from the transfer function. As a result, for example, the "densest point information" is missing from the transfer function in its present form.

These features can be explained by the facts that the transfer function is constructed on the basis of optimal grading curve data reflecting a kind of mean behavior (i.e. using the data related to line "a", Fig A-1(c)) and, the interpolation in between these data seems to be made in "downwards direction" in terms of *N*. The effect of these approximations can be compensated in further research if the transfer function is extended in a systematic way onto the inverse image of the non-normalized entropy diagram points.

To test the validity of the transfer function, the iso-density (s_{min}) lines were constructed in a similar way in the non-normalized entropy diagram from non-optimal grading curve data (Fig 9). The similar pattern of the s_{min} level lines indicated that the preliminary transfer function presented here was possibly acceptable for some non-optimal grading curves, too.

Densest packing problem

It was originally assumed that the solution of the densest packing problem in terms of entropy coordinates is either the maximum entropy point or the maximum entropy increment point with *A*= 2/3. These points – situated on the maximum *B* line - differ slightly.

The results of this study do not fully support these assumptions. For *N*=3, maximum density point was found for a gap-graded mixture with about 1/3 finest- 2/3 coarsest ratio (Fig 5(a)). The corresponding entropy diagram point is situated on the minimum *B* line at about *A*=2/3. The s_{min} values of the optimal grading curves were represented in the function of *A* for every *N* value separately. The global maximum of the *A* – *s* representation was found at about *A*=2/3 for the tested *N* values (e.g. Fig 6 for *N*=5). The corresponding point in the entropy diagram is situated on the maximum *B* line at about *A*=2/3.

References

Dexter, A.R; Tanner, D.W; (1971). Nature. 230:177.

Imre, E; Lőrincz, J; Rózsa, P (2008). Characterization of some sand mixtures. Proc. of the 12th Int. Conference of International Association for Computer Methods and Advances in Geomechanics (IACMAG) 1-6 October, 2008 Goa, India (accepted).

Lőrincz, J (1986). Grading entropy of soils Doctoral Thesis, Technical Sciences, TU of Budapest.

Lőrincz, J; Imre, E; Gálos, M; Trang, Q.P; Telekes, G; Rajkai, K; Fityus, I. (2005). Grading entropy variation due to soil crushing. Int. Journ. of Geomechanics. Vol 5. Number 4. p. 311-320.

Kabai, I. (1972). Relationship between the grading curve and the compactibility (A szemcseeloszlás és tömöríthetőség összefüggése). University doctoral thesis BME, Budapest, Hungary (in Hungarian).

Yu, A.B; Standish, N. (1988). An Analytical-Parametric Theory of the Random Packing Particles. Powder Technology, 55:171-186.

Table A-1. Fraction (eigen-) entropies

Fraction number i	1	...	23	24	25
d [mm] limits	2^{-22}-2^{-21}		1-2	2-4	4-8
S_{0i} [-]	0		22	23	24

Appendix 1. Grading entropy concept, transfer function generation method

Grading entropy coordinates

Two statistical cell systems are used in the definition of the grading entropy (Lőrincz et al. 20051). As it can be seen in Table 1, the fractions are defined by successive multiplication by a factor of 2, starting from an arbitrary diameter d_0. The following equation is met for the relative frequencies of the fractions denoted by x_i (i $=1..N$):

$$\sum_{i=1}^{N} x_i = 1, \qquad x_i \geq 0 \tag{1}$$

where N is the number of the fractions between the finest and coarsest ones. There is a natural one-to-one relationship between the space of the possible grading curves with N fractions and an N-1 dimensional closed simplex (denoted by □) since the relative frequencies of the fractions can be identified with the coordinates of the simplex points. (The N-1 dimensional simplex is the N-1 dimensional analogy of the triangle which is a 2 dimensional simplex.) The fractions are "embedded" into a smaller cell system with uniform width of d_0.

It is assumed that the distribution is uniform within a fraction. The grading entropy S is the statistical entropy of the grading curve in terms of the elementary statistical cell system. It can be split into two parts $S= S_o+\Delta S$. The S_o part is the base entropy given by the following function:

$$S_o = \sum_{k=i_1}^{i_N} x_k S_{ok} \tag{2}$$

using S_{ok} for the eigen-entropy of the k-th fraction (Table a-1) which is the grading entropy S assuming N=1. The entropy increment ΔS is given by the following function:

$$\Delta S = -\frac{1}{\ln 2} \sum_{i=1}^{N} x_i \ln x_i \quad , x_i > 0 \tag{3}$$

The normalised coordinates, the relative base entropy A and the normalised entropy increment B are:

$$A = \frac{\sum_{i=1}^{N} x_i (S_o - S_{o\,\min})}{N-1} = \frac{\sum_{i=1}^{N} x_i (i-1)}{N-1} \tag{4}$$

$$B = \frac{\Delta S}{\ln N} \tag{5}$$

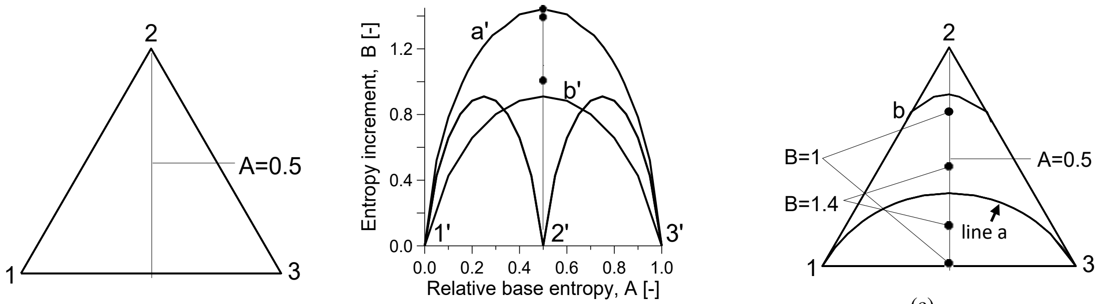
Grading entropy map

There is a natural one-to-one relationship between the grading curves with N fractions and the points of an N-1 dimensional simplex (i.e. the N-1 dimensional analogy of the triangle which is a 2 dimensional simplex). The relative frequencies of the fractions can be identified with the barycentre coordinates of the simplex points.

The entropy map is defined between the N -1 dimensional simplex and the two dimensional entropy space (□□[A,B] or □□□[S_0,□S]). Normalised grading entropy diagrams are shown in Figures A-1(b) and A-2(b) for a 2 dimensional simplex (Figure A-1(a)) and a 3 dimensional simplex (Figure A-2(a)), respectively. The critical values of the entropy map are the points of the maximum B line, whilst the regular values are the inner diagram points.

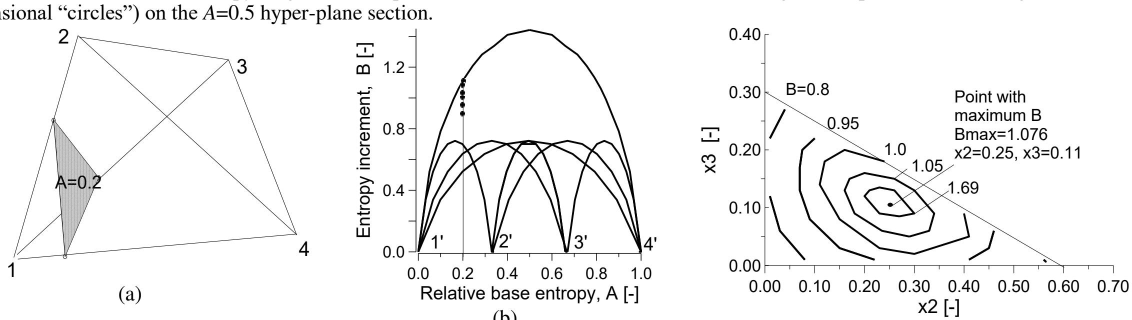
The inverse image of a regular value of [A,B] is similar to (some parts) of an N-3 dimensional circle being centred to the so called "optimal" point on the A=const hyper-plane section of the simplex (e.g. Figs A-1c and A-2c). The A=const hyper-planes are parallel to each-others and on a specifed section of the simplex the integral of every grading curve is the same. The optimal point is the inverse image of [A,B_{max}], the corresponding grading curve has fractal distribution, being a kind of mean grading curve concerning the inverse image of a regular value. The optimal points constitute a line along these parallel hyper-plane sections series of the simplex, depending on the value of parameter A ("line a" in Fig A-1c) for any value of N.

The transfer function generation method is based on the strict classification of the grading curves given by the entropy map (i.e. those grading curves are treated together that maps into the same entropy diagram point), the continuous extension property of the non-normalized entropy map and, the geometry of the non-normalized entropy diagram.



(a) (b) (c)

Figure A-1. N=3, the normalized entropy map of the 2 dimensional simplex. (a.) The triangle diagram and the A=0.5 hyper-plane section. (b) The normalized entropy diagram with a point series with A=0.5. (c) . The inverse image of the points shown in Fig 1b (0 dimensional "circles") on the A=0.5 hyper-plane section.



(a) (b) (c)

Figure A-2. N=4, the normalized entropy map of the 3 dimensional simplex. (a.) The simplex and the A=0.2 hyper-plane section. (b) The normalized entropy diagram with a point series with A=0.2. (c) . The inverse image of the foregoing points (shown in Fig 2b) (1 dimensional "circles") on the A=0.2 hyper-plane section of the simplex.

The transfer function is generated in two steps. First a preliminary transfer function F_1: $[S_0,S]R$ is constructed in the non-normalised grading entropy diagram by interpolation using some optimal grading curve data being related to the maximum S lines. Then the resulting preliminary transfer function is extended onto the inverse image of the non-normalised diagram points F_2: R. The simplex s expressed as the direct sum of the entropy diagram points $[S_0,S]$and the inverse image of the entropy diagram points $f^{1}([S_0,S]$. In the simplest case the constant function is used for the inverse image of the entropy diagram points

The following approximations and assumptions are involved in the first step. (i) Those grading curves are treated together during the interpolation that maps into the same non-normalised entropy diagram point. (ii) The interpolation is based on optimal grading curves which represent some kind of mean behaviour for a specified A and N. (iii) It is assumed that the behaviour is the same at the nearly coinciding parts of the optimal grading curves. As a result, it is be assumed that for an entropy diagram point falling in between the image of optimal soils with fractions $N-k$ and $N-k$ -1, the soil behaves as a soil with fraction number $N-k$. (iv) The point of an N-fraction soil may fall in between the optimal 1 to N-fraction soils and may never fall in between the optimal N- to N+1-fraction soils in the non-normalized diagram. It follows that the interpolation is always made in "downwards direction" in terms of N.

Appendix 2 Data of the soils tested here

Table A- 1 Loosest state for the fractions tested here

Fraction	A	B	C	D	E
D [mm]	0.071-0.125	0.125-0.25	0.25-0.5	0.5-1.0	1.0-2.0
s_{min} [-]	0.4983	0.5132	0.5285	0.5484	0.5674

Table A- 2 Loosest state for continuous 2-mixtures

Sample		212	213	214	215	221	222	223	224	225
A [-]	0.10	0.30	0.50	0.70	0.90	0.10	0.30	0.50	0.70	0.90
B [-]	0.68	1.27	1.44	1.27	0.68	0.68	1.27	1.44	1.27	0.68
s_{min} [-]	0.50	0.51	0.52	0.52	0.52	0.52	0.54	0.54	0.55	0.54

Table A- 2 cont.

Sample	231	232	233	234	235	241	242	243	244	245
A [-]	0.10	0.30	0.50	0.70	0.90	0.10	0.30	0.50	0.70	0.90
B [-]	0.68	1.27	1.44	1.27	0.68	0.68	1.27	1.44	1.27	0.68
s_{min} [-]	0.54	0.55	0.56	0.56	0.55	0.56	0.57	0.58	0.58	0.57

Table A- 3 Loosest state for optimal 3-mixtures

Sample	311	312	313	314	315	316	317
A [-]	0.08	0.20	0.35	0.51	0.65	0.80	0.93
B [-]	0.52	1.05	1.35	1.44	1.35	1.05	0.52
s_{min} [-]	0.51	0.52	0.53	0.55	0.56	0.55	0.55

Table A- 3 cont.

Sample	321	322	323	324	325	326	327
A [-]	0.08	0.20	0.35	0.51	0.65	0.80	0.93
B [-]	0.52	1.05	1.35	1.44	1.35	1.05	0.52
s_{min} [-]	0.52	0.55	0.56	0.57	0.58	0.57	0.56

Table A- 4 cont.

Sample	331	332	333	334	335	336	337
A [-]	0.08	0.20	0.35	0.51	0.65	0.80	0.93
B [-]	0.52	1.05	1.35	1.44	1.35	1.05	0.52
s_{min} [-]	0.55	0.57	0.58	0.58	0.59	0.59	0.57

Table A- 4 Loosest state for optimal 5-mixtures

Sample	41	42	43	44	45	46	47	48	49
A [-]	0.06	0.14	0.26	0.38	0.50	0.63	0.74	0.86	0.94
B [-]	0.42	0.87	1.21	1.38	1.44	1.38	1.21	0.87	0.42
s_{min} [-]	0.51	0.53	0.56	0.59	0.61	0.61	0.61	0.60	0.59

Table A- 5 Loosest state for gap-graded 3-mixtures

Sample	511	512	513	514	515	521	522	523	524	525
A [-]	0.1	0.3	0.5	0.7	0.9	0.1	0.3	0.5	0.7	0.9
B [-]	0.47	0.88	1	0.88	0.47	0.47	0.88	1	0.88	0.47
s_{min} [-]	0.51	0.54	0.56	0.57	0.55	0.53	0.56	0.58	0.59	0.56

Table A- 5 cont.

Sample	531	532	533	534	535
A [-]	0.1	0.3	0.5	0.7	0.9
B [-]	0.47	0.88	1	0.88	0.47
s_{min} [-]	0.55	0.58	0.60	0.60	0.58

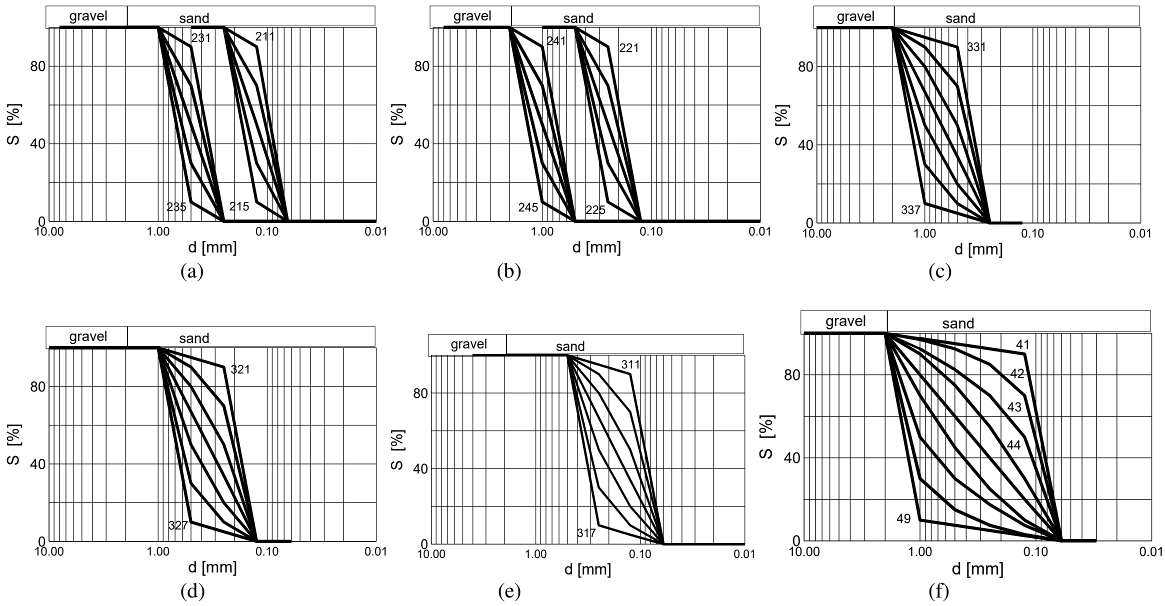


Figure A-1 a to f: The optimal soil tested here

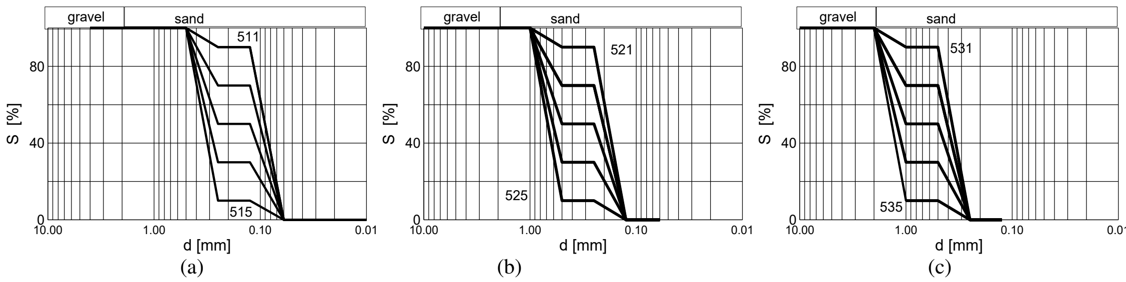


Figure A-2 a to c: The gap-graded soil tested here

Appendix 3 Data of the soils tested by Kabai

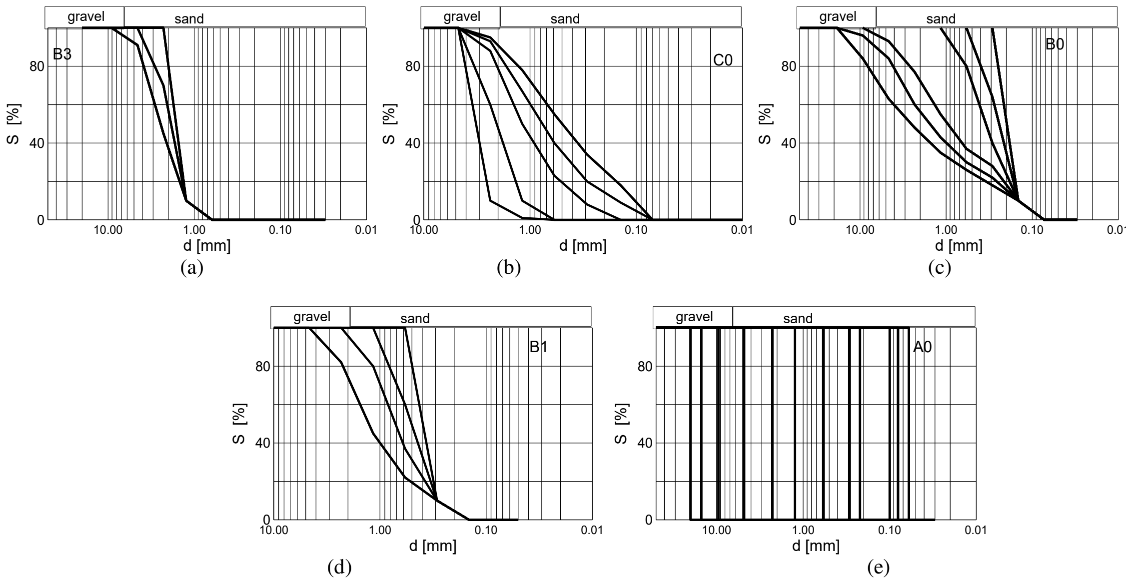


Figure A-3 a to e: The soil mixture series tested by Kabai