Conversion of Laplace equation from the spherical coordinate description to Cartesian coordinate system

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***Abstract*—**

In this article, we show how the solution of the Laplace's equation in spherical coordinate and its coefficients can be rewritten in Cartesian coordinates, and we show the necessary normalization of these coefficients to ensure the identity of these solutions with the pure Cartesian solution.

The solution of the Laplace equation in spherical coordinates is obtained by separating the longitude, colatitude and radius variables. An elementary solution – a basis function - is the product of the negative integer exponent of the radius coordinate, the associated Legendre function of the colatitude coordinate, and the harmonic function of the longitude coordinate.

Using symbolic computations in MATLAB©, we generated these functions. The associated Legendre functions were created with the Rodrigues formula. Within the symbolic module we were able to simplify the functions with trigonometric transformations and convert the spherical coordinates into Cartesian coordinates.

As a result, the obtained formulas that could be directly compared to the pure Cartesian solution based on the 3D Taylor series, and this enabled us to calculate the normalization coefficients. The coefficients we obtained were the Gauss normalization coefficients, instead of Schmidt normalization coefficients, regularly used in geoid calculations.

Keywords — Laplace-equation, Cartesian coordinates, Legendre equation, Rodrigues formula, Matlab© symbolic tool

I. INTRODUCTION

The Laplace equation is a partial differential equation that is used to describe many phenomena in physics.

Laplace considered Newton's relation to gravity, the power law of the form $\sim 1/r^2$, which states that the gravitational acceleration produced by a mass placed at the origin of a coordinate system at a given point in space can be described as a vector whose direction from the given point is toward the origin of the coordinate system, and whose magnitude is proportional to the reciprocal of the square of the distance of the point from the origin.

Laplace, in considering this vector, found that the vector at any point in space can be produced as a vector formed from the partial derivatives of a scalar function with respect to the Cartesian coordinates. He also made another observation: the sum of the second derivatives of this scalar function (with respect to Cartesian coordinates) is zero at any points where there are no masses. The expression of this last relation is the Laplace-Poisson differential equation.

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0$$

The Laplace equation helps to describe many phenomena in physics. The gravity example mentioned above is valid not only for a single point of mass, but the basis for describing the gravitational field of bodies with complex density distributions, such as the Earth or the planets of the Solar System. Laplace's equation can also be used to describe the Earth's magnetic field, since a magnetic scalar potential can be defined, the gradient of which gives rise to the magnetic induction field. In classical physics, the Laplace equation is used to describe static electric fields [1].

II. LAPLACE'S EQUATION IN SPHERICAL COORDINATES

In the description of Laplace's equation in spherical coordinates, the coordinates are the radius (r - the distance from the origin), the polar angle (ϑ - colatitude) and the longitude (λ - azimuth) angle. The equation takes the following form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)+\frac{1}{r^2 sin\vartheta}\left(sin\vartheta\frac{\partial U}{\partial \vartheta}\right)+\frac{1}{r^2 sin^2\vartheta}\left(\frac{\partial^2 U}{\partial \lambda^2}\right)=0$$

This is solved by the method of separation of variables, (Fourier method) i.e., a solution is sought which is the product of a r-dependent function R(r), a ϑ-dependent function $P(\vartheta)$ and a λ-dependent function $\Lambda(\lambda)$. Substituting back and performing the necessary transformations, we obtain the following conditions for each function:

The form of the function $R(r)$ must be either r^2 or r^{-n-1}, where n is an integer.

The function $\Lambda(\lambda)$ must be either $\cos(m\cdot\lambda)$ or $\sin(m\cdot\lambda)$, and since the function must be periodic in 2π, m is also an integer. From these two conditions it follows that the function $P(\vartheta)$ must be given by the following, so called Legendre's differential equation:

$$\frac{1}{sin\vartheta}\frac{\partial}{\partial\vartheta}\left(sin\vartheta\frac{\partial P(\vartheta)}{\partial\vartheta}\right)+\left[n\cdot(n+1)-\frac{m^2}{sin^2\vartheta}\right]P=0$$

Its solution involves two integers. This occurred to Schrödinger, who was able to relate it to the concept of electron shells, where n is the principal quantum number and m is the azimuthal quantum number. In this way, the solutions of the Laplace equation with spherical coordinates are also solutions of the time-independent Schrödinger equation [2].

The solution of the Legendre differential equation is called a Legendre polynomial for given n and $m = 0$, and an associated Legendre function, for $m \neq 0$, which can be obtained by the Rodriguez formula:

$$P_{n,m} = \frac{(1-cos^2\vartheta)^{m/2}}{2^n \cdot n!} \cdot \frac{\partial^{n+m}(cos^2\vartheta - 1)^n}{\partial(cos\vartheta)^{n+m}}$$

The consequence of the derivation in the formula shows that non-zero elements are obtained only for $m \leq n$ [3].

III. Schmidt normalization and quasi-normalization

Schmidt proved in an article [4] about geomagnetism, that the Legendre polynomials besides being orthogonal should also be normalized. The Schmidt quasi-normalisation factors can be calculated as follows:

if m ≠ 0 if m = 0

$$k_n^m = \sqrt{2\frac{(n-m)!}{(n+m)!}} \qquad k_n^0 = \sqrt{\frac{(n-m)!}{(n+m)!}}$$

The Schmidt quasi-normalization factors are not completely normalized harmonics. This condition is satisfied if we multiply the normalizing factor by the square of the Legendre function and then take its integral over the range [-1 1] to obtain 1. This can be obtained in spherical coordinates as follows.

$$k_n^m = 4\pi / \int_0^{2\pi} \int_0^{\pi} P_n^{m\,2} \sin\vartheta \, d\vartheta d\lambda$$

That means, the completed Schmidt normalization is:

if m≠0 if m=0

$$k_n^m = \sqrt{(2n+1) * 2\frac{(n-m)!}{(n+m)!}} \quad k_n^0 = \sqrt{(2n+1) * \frac{(n-m)!}{(n+m)!}}$$

IV. Laplace's equation in Cartesian coordinates

To get the potential of a mass distributed around the origin of the coordinate system, a 3D Taylor series approximation can be applied. The potential is the infinite sum of n^{th} order multipole sources multiplied by the appropriate basis functions, which are the n^{th} partial derivatives of the $1/r$ function. If only one unit mass point is considered, whose coordinates are x,y and z, the n^{th} order member of the series is the following:

$$U^{(n)} = (-1)^n \sum_{i+j+k=n} x^i y^j z^k \frac{1}{n!} \cdot \frac{\partial^n}{\partial X^i \partial Y^j \partial Z^k}\left(\frac{1}{\sqrt{X^2+Y^2+Z^2}}\right)$$

Where X, Y, Z are the locations where we want to find the potential of the mass [5].

V. Methods

To be able to compare the solution of Laplace equation in spherical and Cartesian coordinates, we transform the spherical coordinates to Cartesian coordinates. By using the appropriate trigonometric functions and ordering the equations, both the coefficients and the basis functions can be transformed. The transformation between the Cartesian and spherical coordinates is described by the following equations:

$$x = r * \sin(\vartheta) * \cos(\lambda)$$

$$y = r * \sin(\vartheta) * \sin(\lambda)$$

$$z = r * \cos(\vartheta)$$

in addition:

$$r = (x^2 + y^2 + z^2)^{0.5}$$

A. Theoretical solution

The solution of Laplace's equation with spherical coordinates is described by the following equation:

$$U(r,\vartheta,\lambda) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^{-(n+1)}[C_{n,m}\cos(m\lambda) +]S_{n,m}\sin[[(m\lambda)]\, P_{n,m}(\vartheta)$$

The coefficients $C_{n,m}$ and $S_{n,m}$ contain the coordinates of the source of the potential. The coefficients can be calculated by the following method:

$$C_{n,m} = r^{(n)} * \cos(m\lambda') * P_{n,m}(\vartheta')$$

$$S_{n,m} = r^{(n)} * \sin(m\lambda') * P_{n,m}(\vartheta')$$

where r, ϑ' and λ' are the spherical coordinates of the unit mass. The following operations are recommended to convert the solutions of coefficients and basis functions to the Cartesian coordinate system:

- In the case where m is greater than 1, it is important to convert the trigonometric functions of multiple angles into single angles.
- Solve the following equation $1\text{-}\cos^2(\vartheta'))^{(k/2)} = \sin^k(\vartheta')$, and then convert every $r * \sin(\vartheta') * \cos(\lambda')$ and $r * \cos(\vartheta') * \cos(\lambda')$ expressions into x and y coordinates.
- After calculating all possible x and y coordinates, the $\sin^k(\vartheta')$ should be converted into $1\text{-}\cos^2(\vartheta'))^{(k/2)}$. Then it is possible to convert every $r * \cos(\vartheta')$ into z coordinate.
- Finally, the remaining r's in the equation must be converted to $(x^2 + y^2 + z^2)^{0.5}$

B. Solutions using Matlab©

The calculations were done within Matlab© R2022a software. Within the symbolic module, the expressions mentioned in the previous subsection can be obtained.

Within the symbolic module we used the following commands:

- syms: create symbolic variables
- symplifyFraction: Simplify symbolic rational expressions. The symplifyFraction command is more useful than the simplify command in most cases. The simplify command created unfavorable trigonometric transformations, like multiple angles.
- subs: Symbolic substitution. We use this command to perform the converting operations.
- expand: Expand expressions and simplify inputs of functions by using identities [6].

First, we had to express the Rodrigues formula:

$$f = (costheta^2 - 1)^n;$$
$$f2 = diff(f, costheta, n+m)$$
$$f2 = subs(f2, costheta, cos(\vartheta));$$
$$P(n,m) = (-1)^m * \left(\frac{(1 \cos^2(\vartheta))^{\frac{m}{2}}}{2^n * factorial(n)} \right) * f2;$$

As mentioned before, besides the basis functions, the $C_{n,m}$ and $S_{n,m}$ coefficients also must be obtained. To convert to cartesian coordinates, according to the previous chapters, we performed the following operations:

$$C_{n,m} = subs\left(C_{n,m}, (\sin^2(\vartheta))^{\frac{m}{2}}, (\sin(\vartheta))^m\right);$$
$$C_{n,m} = \left(subs\left(C_{n,m}, \left[(r * \sin(\vartheta) * \cos(\lambda))^{\frac{m}{2}}, (r * \sin(\vartheta) * \sin(\lambda))^{\frac{m}{2}}\right], [x^m, y^m]\right)\right)$$
$$C_{n,m} = subs\left(C_{n,m}, \sin^{2*k}(\vartheta), (1 - \cos^{2*k}(\vartheta))\right);$$
$$C_{n,m} = \left(subs\left(C_{n,m}, \left[(r * \cos(\vartheta))^{\frac{m}{2}}\right], [z^m]\right)\right)$$
$$C_{n,m} = subs\left(C_{n,m}, r, (x^2 + y^2 + z^2)^{0.5}\right)$$

VI. Results

Table 1. shows some of the $C_{n,m}$ and $S_{n,m}$ coefficients expressed with spherical and Cartesian coordinates.

Coeffs	*Spherical coordinates*	*Cartesian coordinates*
$C_{1,0}$	$r * \cos(\vartheta)$	z
$C_{1,1}$	$-r * \cos(\lambda) * \sin(\vartheta)$	-x
$S_{1,1}$	$-r * \sin(\lambda) * \sin(\vartheta)$	-y
$C_{2,0}$	$\frac{3 * r^2 * \cos^2(\vartheta)}{2} - \frac{r^2}{2}$	$-\frac{x^2}{2} - \frac{y^2}{2} + z^2$
$C_{2,1}$	$-3 * r^2 * \cos(\lambda) * \cos(\vartheta) * \sin(\vartheta)$	$-3 * x * z$
$S_{2,1}$	$-3 * r^2 * \cos(\vartheta) * \sin(\lambda) * \sin(\vartheta)$	$-3 * y * z$
$C_{2,2}$	$3 * r^2 * \cos^2(\lambda) * \sin^2(\vartheta) - 3 * r^2 * \sin^2(\lambda) * \sin^2(\vartheta)$	$3 * (x^2 - y^2)$
$S_{2,2}$	$6 * r^2 * \cos(\lambda) * \sin(\lambda) * \sin^2(\vartheta)$	$6 * x * y$

Table 1. $C_{n,m}$ and $S_{n,m}$ coefficients expressed in spherical and Cartesian coordinates

Substituting back the coefficient to the potential formula we obtained the following equation:

$$U(R, \theta, \Lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} R^{-(n+1)} [C_{n,m} \cos(m\Lambda)]\, P_{n,m}(\Lambda)] + R^{-(n+1)} [S_{n,m} \sin(m\Lambda)]\, P_{n,m}(\Lambda)]$$

The coefficient contains the coordinates of the mass point, i.e. the source, and R, θ, Λ are the locations where we want to get the potential of the mass.

Now we were able to transform any expression written in spherical coordinates into Cartesian coordinates.

For example, let's compare the second order term of the potential of a unit mass located in (x, y, z) point. From one side this can be expressed using the coefficient and basis functions according to Table 1, as this solution is based on the spherical coordinates formulation.

From other side this second order term can be expressed using the Taylor series approximation:

$$U^2 = \sum_{i+j+k=2} x^i y^j z^k \cdot \frac{1}{2} \cdot \frac{\partial^2}{\partial X^i \partial Y^j \partial Z^k} \left(\frac{1}{R}\right) =$$
$$= \frac{\begin{pmatrix} 2 * X^2 * x^2 - X^2 * y^2 - X^2 * z^2 + 6 * X * Y * x * y \\ + 6 * X * Z * x * z - Y^2 * x^2 + 2 * Y^2 * y^2 - Y^2 * z^2 \\ + 6 * Y * Z * y * z - Z^2 * x^2 - Z^2 * y^2 + 2 * Z^2 * z^2 \end{pmatrix}}{2 * (X^2 + Y^2 + Z^2)^{\frac{5}{2}}}$$

However, these two solutions should be identical, and so the solutions of the spherical coordinate solution should be normalized.

To ensure the identity of the two approaches, the spherical coordinate solutions should have been multiplied with the subsequent coefficients:

$$\frac{{k_2^0}^2 * (x^2 + y^2 - 2 * z^2) * (X^2 + Y^2 - 2 * Z^2)}{4}$$
$$+ 9 * {k_2^2}^2 * (X^2 - Y^2) * (x^2 - y^2) + 36 * X * Y * k2^2 * x * y$$
$$+ 9 * X * Z * {k_2^1}^2 * x * z + 9 * Y * Z * k1^2 * y * z$$

where k_0, k_1 and k_2 are the normalizing factors, in this case:

$$k_2^0 = 1, \qquad k_2^1 = \sqrt{\frac{1}{3}}, \qquad k_2^2 = \sqrt{\frac{1}{12}}$$

These normalizing factors proved to be the Gaussian normalizing factors.

The advantage of the solution starting with the spherical coordinate description is that it requires less terms to be summed, 2n+1 elements for a given order, while, if n>2 the Cartesian coordinate solution requires more elements $(0.5n^2 + 1.5n + 1)$ to be summed. Table 2. shows some examples, how many terms has to be summed for the n^{th} order solution.

n	**number of added elements using the Taylor series based Cartesian coordinates**	**number of added elements using the spherial coordinate system**
1	3	3
2	6	5
3	10	7
4	15	9
5	21	11
6	28	13
n	$0{,}5n^2 + 1{,}5n + 1$	$2(n+1)$

Table 2. Number of the independent terms for a given order using the Taylor series based Cartesian coordinates and using the spherical coordinate system

As the chosen example for $n=2$ shows, the spherical method consists of $2n+1 = 5$ elements, while the Cartesian method consists of $0{,}5n^2 + 1{,}5n + 1 = 6$ independent elements.

We could also check the correctness of the quasi-normalization factor in Matlab© symbolic tools. The quasi-normalization solution was the same as the Taylors series based Cartesian solution when the equations were properly solved.

VII. Conclusions

We have shown in the Results (Table 2), that we could successfully express the coefficients using Cartesian coordinates. The expressions written in Cartesian coordinates are simpler and are easier to handle.

We got the result, that instead of the Schmidt normalization – widely used in gravity problems – the Gauss normalization should be used to ensure the identity of the two approaches. To understand the consequences, further investigation is needed.

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