Solving Olympiad Problems Via Lagrange Multipliers

Borbély József
Óbudai Egyetem, Székesfehérvár, Hungary
borbely.jozsef@uni-obuda.hu

***Abstract*— There are a lot of hard problems concerning inequalities in mathematical olympiads which have an elegant elementary solution. In this paper we will do the contrary of the traditional approach: we will try to give a method which solves a large number of these problems using the same basic idea (but the solution will not always be less complicated).**

I. Introduction

In this paper we will prove four olympiad level inequalities using Lagrange multipliers. These solutions were not published anywhere else, and use the same basic idea.

II. About Lagrange Multipliers

We will need the following theorem of Weierstrass (see [2]):

Theorem (Weierstrass): *Let n be a positive integer, and let D be a closed, bounded subset of the real n-dimensional space. If the function f: D→R is continuous , then it has a minimum and a maximum.*

A closed set is a set which contains all its limit points, which means that every convergent seqence of points in a closed set converges to a point of this given set.
Let us consider the following problem: let us take a real valued function g which is defined on a subset S of the real n-dimensional space. Furthermore let us assume that the functions g_1, g_2, ..., g_m all vanish on the set S (these are functions with n real variables as well). Using the conditions concerning the g_i's we want to determine the extreme values of the function g.
Here the so called method of Lagrange multipliers comes in handy: if these functions are differentiable by all of the n variables, then we define the modified function

$$G=g+\sum_{i=1}^{m} \lambda_i \cdot g_i.$$

Here the λ_i's are called the Lagrange multipliers which will be calculated later.
But why even this function? Because this function G is identical to g on the set S, and in this function we encoded our conditions in some way.
We will determine the partial derivatives of G, which must equal zero at any extremal points. By solving this sytem of equalities we will obtain a set of potential extremal points.

This is the basic idea of Lagrange multiplier method, which can be combined with Weierstrass theorem, if S is a closed and bounded set.

In the following section we demonstrate this method by proving olympiad level inequalities.

III. Olympiad Level Inequalities

We begin with an inequality which was set in the Hungarian journal KÖMAL in 2007 (problem A.433., see [5]):

Inequality 1: *If the sum of the squares of the real numbers*

a, b, c equals 1 then

$a+b+c \le 2abc + \sqrt{2}$.

Proof:

According to the condition we have

$a^2+b^2+c^2=1$. The real triples of numbers (a,b,c) satisfying

this condition define a closed, bounded set, so by Weierstrass theorem the continouos function

f(a,b,c)=a+b+c-2abc

must have a maximal and a minimal value on this set. Let us introduce the function

F(a,b,c)= a+b+c-2abc+λ·$(a^2+b^2+c^2-1)$, where λ is the Lagrange multiplier.

The partial derivatives of F(a,b,c) are the following functions:

F_a' = 1-2bc+2λa

F_b' = 1-2ac+2λb

F_c' = 1-2ab+2λc

Thus one has to solve the system of equations

0 = 1-2bc+2λa

0 = 1-2ac+2λb

0 = 1-2ab+2λc

in order to determine the potential extreme points.

By substracting the second equation from the first one we get

0=$(a-b)\cdot(2c+2\lambda)$. Similarly we get

0=$(b-c)\cdot(2a+2\lambda)$ and

0=$(c-a)\cdot(2b+2\lambda)$.

If there are two different ones among the numbers a,b,c, then at least two of the differences $(a-b)$, $(b-c)$, $(c-a)$ is nonzero. We can assume that a-b≠0 and b-c≠0. This implies that 2c+2λ=2a+2λ =0 which means that a=c= -λ. By substitution we get the system of equations

$0 = 1\text{-}2bc+2\lambda a=1+2\lambda b\text{-}2\lambda^2$,

$2\lambda^2+b^2=1$.

So $2\lambda b = 2\lambda^2 - 1$ and

$4\lambda^2=4\lambda^2\cdot(2\lambda^2+b^2)=8\lambda^4+4\lambda^2 b^2=8\lambda^4+(2\lambda^2-1)^2$, thus we get

$8\lambda^4+(2\lambda^2-1)^2\text{-}4\lambda^2 = 0$, which is equivalently

$12\lambda^4\text{-}8\lambda^2+1=0$. We get the solutions

$\lambda=\sqrt{\frac{1}{2}},\ -\sqrt{\frac{1}{2}},\ \sqrt{\frac{1}{6}},\ -\sqrt{\frac{1}{6}}$.

So if there are two different ones among the numbers a, b, c then the potential extreme points will be

$(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, 0)$, $(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0)$, $(\sqrt{\frac{1}{6}}, \sqrt{\frac{1}{6}}, \sqrt{\frac{4}{6}})$, $(-\sqrt{\frac{1}{6}}, -\sqrt{\frac{1}{6}}, \sqrt{\frac{4}{6}})$, $(-\sqrt{\frac{1}{6}}, -\sqrt{\frac{1}{6}}, -\sqrt{\frac{4}{6}})$, and the points we get by permutation of the coordinates of these vectors. If the coordinates are the same, then the possible extreme points are

$(\sqrt{3}, \sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3}, -\sqrt{3})$.

By substitution we can check that the maximal value of the expression f(a,b,c)=a+b+c-2abc with respect to the given condition equals $\sqrt{2}$.

The following inequality is a shortlisted problem for the International Mathematical Olympiad from the year 2010. This problem was posed at the Surányi János Emlékverseny in 2011 (see [3]):

Inequality 2: *If a,b,c,d are real numbers satisfying the relations a+b+c+d=6 and $a^2+b^2+c^2+d^2=12$ then*

$36 \leq 4(a^3+b^3+c^3+d^3)\text{-}(a^4+b^4+c^4+d^4) \leq 48.$

Proof:

The vectors (a, b, c, d) staisfying the two given conditions clearly determine a closed bounded set, thus by the theorem of Weierstrass the expression

$4(a^3+b^3+c^3+d^3)\text{-}(a^4+b^4+c^4+d^4)$

has a minimal and a maximal value with respect to the given conditions.

Let us define F(a,b,c,d) in the following way:

$F(a,b,c,d)= 4(a^3+b^3+c^3+d^3)\text{-}(a^4+b^4+c^4+d^4)+$

$+\lambda_1(a+b+c+d-6)+\lambda_2(a^2+b^2+c^2+d^2-12)$,

where λ_1 and λ_2 are the Lagrange multipliers.

The partial derivatives of F(a,b,c,d) are the following functions:

$F_a' = -4a^3+12a^2+2\lambda_2 a+\lambda_1$

$F_b' = -4b^3+12b^2+2\lambda_2 b+\lambda_1$

$F_c' = -4c^3+12c^2+2\lambda_2 c+\lambda_1$

$F_d' = -4d^3+12d^2+2\lambda_2 d+\lambda_1$

In order to determine the potential extremal points we have to solve the sytem of equalities

$F_a'=F_b'=F_c'=F_d'=0$.

First we will prove that for these solutions at most two of the numbers a, b, c, d can be different. We prove this statement indirectly. Let us assume to the contrary that there are three different ones among the numbers a, b, c, d satsisfying the system of equalities. Without loss of generalization we can assume that a, b, c are different numbers.

$0=F_a'\text{-}F_b'=2(b-a)(2a^2+2ab+2b^2-6a-6b-\lambda_2)$.

Due to the assumption a≠b we have

$2a^2+2ab+2b^2-6a-6b-\lambda_2 = 0$.

Similarly we get

$2b^2+2bc+2c^2-6b-6c-\lambda_2 = 0$.

By taking the difference of these equations we get

$0=(2a^2+2ab+2b^2-6a-6b-\lambda_2)\text{-}$

$\text{-}(2b^2+2bc+2c^2-6b-6c-\lambda_2)=$

$=2(a-c)(a+b+c-3)$.

Due to the assumption a≠c we have

a+b+c=3, which implies d=3.

In this case $a^2+b^2+c^2=3$.

It is easy to see that for real numbers a, b, c we have

$(\boldsymbol{a}-\boldsymbol{b})^2 + (\boldsymbol{b}-\boldsymbol{c})^2 + (\boldsymbol{c}-\boldsymbol{a})^2 \geq 0$, and we have equality if

and only if a=b=c. This inequality is equivalent to the following one:

$(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})^2 \leq 3(\mathbf{a^2}+\mathbf{b^2}+\mathbf{c^2})$.

Each side of this inequality equals 9, which means the only solution for real numbers in the case d=3 is a=b=c=1. But this contradicts the assumption that there are three different ones among the numbers a, b, c, d.

Thus we know that among the numbers a, b, c, d there are at most two different ones. If a=b=c=d then all of them must equal 3/2, but in this case the sum of their squares is different from 12.

Thus there are exactly two different ones among the numbers a,b,c,d. If three of them are identical and the fourth one is different, then we get the point (0, 2, 2, 2), (3, 1, 1, 1), and the points we get by permutation of the coordinates of these vectors.

If the two groups consist of two-two identical numbers then the system of equation does not have a solution in real numbers.

By substitution we can check that the minimal value is 36, the maximal value is 48.

Inequality 3 was a problem at the International Mathematical Olympiad in 1984:

Inequality 3: *If the sum of the nonnegative numbers x, y, z equals 1 then*

$0 \leq xy+yz+zx-2xyz \leq \frac{7}{27}$.

Proof:

For some real numbers a, b, c we have

x=a^2, y=b^2, z=c^2 and a^2+b^2+c^2=1.

Clearly the vectors (a, b, c) in question determine a closed, bounded set, so by Weierstrass theorem the function

f(a,b,c)=a^2b^2+b^2c^2+c^2a^2-2$a^2b^2c^2$ has a minimal and a maximal value with respect to the condition a^2+b^2+c^2=1.

Let us introduce the function

F(a,b,c)=a^2b^2+b^2c^2+c^2a^2-2$a^2b^2c^2$+λ·$(a^2+b^2+c^2-1)$, where λ is the Lagrange multiplier.

The partial derivatives of F(a,b,c) are the following functions:

F_a' = 2a·$(b^2+c^2-2b^2c^2+\lambda)$

F_b' = 2b·$(a^2+c^2-2a^2c^2+\lambda)$

F_c' = 2c·$(b^2+a^2-2b^2a^2+\lambda)$

Thus one has to solve the system of equations

0 = 2a·$(b^2+c^2-2b^2c^2+\lambda)$

0 = 2b·$(a^2+c^2-2a^2c^2+\lambda)$

0 = 2c·$(b^2+a^2-2b^2a^2+\lambda)$

If abc≠0 then

-λ=$b^2+c^2-2b^2c^2$ = $a^2+c^2-2a^2c^2$, so

b^2-a^2+2$c^2(a^2-b^2)$= $(a^2-b^2)(2c^2-1)$ =0.

In the case abc≠0 similarly we get the equations

$(b^2-c^2)(2a^2-1)$ =0 and

$(c^2-a^2)(2b^2-1)$ =0.

If at least two of the numbers $(2a^2-1)$, $(2b^2-1)$, $(2c^2-1)$ equal zero then one of the numbers a, b, c must equal zero which contradicts our assumption abc≠0. Thus at least two of the numbers (a^2-b^2), (b^2-c^2), (c^2-a^2) must equal zero, which means that a^2=b^2=c^2.

Thus the three dimensional points having the coordinates $\sqrt{3}$ or (- $\sqrt{3}$) are potential extreme points. By substitution we can check that the desired inequalities hold.

In the case abc=0 one of the numbers a, b, c must equal 0. Let us assume that c=0. In this case a^2+b^2=1 and so

xy+yz+zx-2xyz=a^2b^2=$a^2(1-a^2)$. Of course a^2b^2 is

nonnegative and $a^2(1-a^2)$ cannot be greater than $\frac{1}{4}$ which is less than $\frac{7}{27}$.

Inequality 4 is an own problem of the author of this paper which was set at the Nemzetközi Magyar Matematikaverseny in 2010 (class 11). Here we give a solution with Lagrange multipliers (which is more complicated than the original elementary solution):

Inequality 4: Let n be a positive integer number, and let c be a positive constant. If the nonnegative numbers x_1, x_2, ..., x_n satisfy the condition

$\sum_{i=1}^{n}({x_i}^2+ix_i)$*=c, then the inequalities*

$\sqrt{c+\frac{n^2}{4}} - \frac{n}{2} \leq \sum_{i=1}^{n} x_i \leq \sqrt{n} \cdot \sqrt{c+\sum_{j=1}^{n}\frac{j^2}{4}} - \sum_{j=1}^{n}\frac{j}{2}$.

hold.

Proof:

For some real numbers y_1, y_2,..., y_n we have ${y_i}^2$=x_i for every 1≤i≤n. Thus we have to give the minimal and maximal value of $\sum_{i=1}^{n} {y_i}^2$ with respect to the condition

c=$\sum_{i=1}^{n}({y_i}^4 + i{y_i}^2)$.

The existence of the extreme values is guaranteed by the theorem of Weierstrass.

Using the method of Lagrange multipliers we define the function

F$(y_1, y_2, \ldots, y_n)$=$\sum_{i=1}^{n} {y_i}^2$ + λ·$(\sum_{i=1}^{n}({y_i}^4 + i{y_i}^2) - c)$,

where λ is the Lagrange multiplier.

Taking the derivative of F by the i-th variable we get

$4\lambda {y_i}^3 + (2\lambda i + 2)y_i = 2y_i(2\lambda {y_i}^2 + \lambda i + 1)$.

In order to determine the potential extremal points we have to solve the system of equations

0=$2y_i(2\lambda {y_i}^2 + \lambda i + 1)$, where i=1, 2, ..., n.

For every possible index i we have either y_i=0 or

$2\lambda {y_i}^2 + \lambda i + 1$=0.

Thus if for some index i the number y_i is different from zero then we have

$2\lambda {y_i}^2 + \lambda i + 1$=0, which is equivalent to the equation

-1=λ$(2{y_i}^2 + i)$.

Here λ must be different from zero, thus

${y_i}^2 + \frac{i}{2} = \frac{-1}{2\lambda}$, and ${y_i}^4 + i{y_i}^2 + \frac{i^2}{4} = \frac{1}{4\lambda^2}$.

Let 1 ≤ i_1 < i_2 < ... < i_t ≤ n be the indices of the numbers in the set {y_1, y_2,..., y_n} which are different from zero.

Using this notation we have

$\sum_{j=1}^{t}\left({y_{i_j}}^2 + \frac{i_j}{2}\right) = \frac{-t}{2\lambda}$,

which is equivalent to the equation

$\sum_{j=1}^{t} {y_{i_j}}^2 = \frac{-t}{2\lambda}$ - $\sum_{j=1}^{t}\frac{i_j}{2}$.

Similarly we get

c = $\sum_{i=1}^{n}({y_i}^4 + i{y_i}^2) = \sum_{j=1}^{t}\left({y_{i_j}}^4 + i_j {y_{i_j}}^2\right)$ =

= t·$\frac{1}{4\lambda^2} - \sum_{j=1}^{t}\frac{{i_j}^2}{4}$.

Thus c + $\sum_{j=1}^{t}\frac{{i_j}^2}{4}$ = t·$\frac{1}{4\lambda^2}$, and

$tc + t\sum_{j=1}^{t}\frac{{i_j}^2}{4} = \frac{t^2}{4\lambda^2}$.

The square root of $\frac{t^2}{4\lambda^2}$ is $\frac{-t}{2\lambda}$, because λ must be positive if the numbers y_1, y_2,..., y_n are real (namely in the equation ${y_i}^2 + \frac{i}{2} = \frac{-1}{2\lambda}$ the left hand side is positive).

Thus we have

$\sqrt{t}\cdot\sqrt{c + \sum_{j=1}^{t}\frac{{i_j}^2}{4}}$ - $\sum_{j=1}^{t}\frac{i_j}{2} = \sum_{j=1}^{t} {y_{i_j}}^2 = \sum_{i=1}^{n} {y_i}^2$.

We have to determine the minimal and maximal value of this expression.

We will prove that for every positive integer k the inequality

$\sqrt{t}\cdot\sqrt{c + \sum_{j=1}^{t}\frac{{i_j}^2}{4}}$ - $\sum_{j=1}^{t}\frac{i_j}{2}$ <

< $\sqrt{t+1}\cdot\sqrt{c + \frac{k^2}{4} + \sum_{j=1}^{t}\frac{{i_j}^2}{4}}$ - $\sum_{j=1}^{t}\frac{i_j}{2}$ - $\frac{k}{2}$ holds.

This inequality is equivalent to the following one:

$\sqrt{tc + t\sum_{j=1}^{t}\frac{{i_j}^2}{4}}$ - $\sum_{j=1}^{t}\frac{i_j}{2}$ <

<$\sqrt{tc + \frac{tk^2}{4} + t\sum_{j=1}^{t}\frac{{i_j}^2}{4} + c + \frac{k^2}{4} + \sum_{j=1}^{t}\frac{{i_j}^2}{4}}$ - $\sum_{j=1}^{t}\frac{i_j}{2}$ - $\frac{k}{2}$,

which is equivalently

$\sqrt{tc + t\sum_{j=1}^{t}\frac{{i_j}^2}{4}}$ +$\frac{k}{2}$ <

<$\sqrt{tc + \frac{tk^2}{4} + t\sum_{j=1}^{t}\frac{{i_j}^2}{4} + c + \frac{k^2}{4} + \sum_{j=1}^{t}\frac{{i_j}^2}{4}}$.

By taking the squares on both sides we have to prove that

$tc + t\sum_{j=1}^{t}\frac{{i_j}^2}{4} + \frac{k^2}{4} + k\sqrt{tc + t\sum_{j=1}^{t}\frac{{i_j}^2}{4}}$ <

< $tc + \frac{tk^2}{4} + t\sum_{j=1}^{t}\frac{{i_j}^2}{4} + c + \frac{k^2}{4} + \sum_{j=1}^{t}\frac{{i_j}^2}{4}$, which inequality is equivalent to the following one:

0 < $c + \sum_{j=1}^{t}\frac{{i_j}^2}{4}$ - $k\sqrt{t}\sqrt{c + \sum_{j=1}^{t}\frac{{i_j}^2}{4}} + \frac{tk^2}{4} + c$

On the right hand side we have

$\left(c + \sum_{j=1}^{t}\frac{{i_j}^2}{4} - \frac{k\sqrt{t}}{2}\right)^2$ +c, which is clearly positive, because c is a positive number. Thus we proved that

$$\sqrt{t}\cdot\sqrt{c+\sum_{j=1}^{t}\frac{{i_j}^2}{4}}-\sum_{j=1}^{t}\frac{i_j}{2}<$$
$$<\sqrt{t+1}\cdot\sqrt{c+\frac{k^2}{4}+\sum_{j=1}^{t}\frac{{i_j}^2}{4}}-\sum_{j=1}^{t}\frac{i_j}{2}-\frac{k}{2}.$$

This (using induction) implies that $\sum_{i=1}^{n} {y_i}^2$ is maximal if all the numbers y_1, y_2,..., y_n are nonzero. Solving the equalities we get that the maximal value is

$$\sqrt{n}\cdot\sqrt{c+\sum_{j=1}^{n}\frac{j^2}{4}}-\sum_{j=1}^{n}\frac{j}{2}.$$

Similarly, the minimal value will be attained if exactly one of the numbers y_1, y_2,..., y_n is nonzero. It is easy to check that in this case $y_n \neq 0$, y_1=y_2=... =y_{n-1}=0, and the minimal possible value is

$$\sqrt{\mathbf{c}+\frac{n^2}{4}}-\frac{n}{2}.$$

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