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Minimal Configurations in 3D Similarity Transformation: Selection Criteria for Stable Solutions

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***Abstract*—We revisit minimal seven-equation selections for the 7-parameter 3D similarity (Helmert) transformation using three common points. From the 9 available scalar equations there are 36 possible seven-row subsets. Rather than claiming that a fixed subset of these is always "stable," we show that stability is geometry-dependent: the same selection can be well- or ill-behaved depending on the spatial arrangement of the three points. We provide two simple, geometry-aware selection rules that perform reliably across typical geodetic and photogrammetric scenarios: (A) choose two full 3D points (six rows) and add one component from the third point whose direction is not parallel to the twist axis defined by the two full points; and (B) choose one full 3D point and, from the other two points, take the same 2D pair that includes the local normal (e.g., YZ or XZ in near-planar setups). These rules are motivated by a first-order sensitivity analysis of the transformation equations and are validated by synthetic experiments with non-unit scale, non-zero rotations, translations, and measurement noise. For each geometry we rank all 36 selections using a normalized, unit-invariant conditioning measure derived from the scaled Jacobian. Results show that the number of acceptable selections may be less than, equal or greater than nine, depending on point configuration, and the proposed rules consistently identify the well-behaved cases while flagging the risky ones. The paper offers practical guidance for choosing seven-row subsets that avoid hidden instabilities when working with minimal three-point data.**

Keywords—3D similarity transformation; Helmert transformation; minimal configuration; three-point selection; stability; conditioning; photogrammetry; geodesy

I. Introduction

The 3D similarity transformation, also known as the seven-parameter Helmert transformation, plays a fundamental role in geodesy, photogrammetry, and in general in all spatial sciences involving coordinate integration, such as robotics, remote sensing, computer vision, and engineering surveying, in aligning satellite-based geodetic coordinate systems or integrating local measurements into global reference frames. The transformation requires the estimation of seven parameters: three translations, three rotation angles, and one scale factor. In practical applications, it often occurs that coordinates of only three non-collinear points are available in both systems, which theoretically suffices for a unique determination of the parameters.

These three points provide a total of nine coordinate equations, from which any subset of seven independent equations can, in principle, be used to solve for the seven unknowns. There are exactly 36 such seven-equation combinations, and the existing literature ([1],[4]) has implicitly assumed that all 36 combinations are equally valid, without investigating their geometric and numerical reliability.

This paper demonstrates that, in fact, only some of these 36 combinations lead to stable and geometrically well-conditioned solutions. Many configurations either fail to provide full coverage of the spatial rotational components, or lead to biased estimates of the scale factor. This insight originally emerged during the preparation of the paper [4], where the authors suspected—on logical grounds—that certain combinations were ill-posed, but no formal geometric or mathematical justification was available at the time.

The goal of this study is to provide such justification. Through rigorous mathematical reasoning and numerical examples, we identify which configurations ensure reliable results, thereby refining and extending the earlier literature on minimal configurations for 3D similarity transformations.

II. Theoretical Background

In the context of three-dimensional similarity transformations, the standard model relating two coordinate systems (source and target) for a point – represented by vectors and – is given by:

where:

- is the uniform scale factor *s*,
- is the 3x3 rotation matrix **R** (orthogonal with determinant +1),
- is the translation vector (*Tx, Ty, Tz*).

When only three non-collinear common points are available, their coordinates in both systems can be used to establish nine scalar equations (3 equations per point). Since the transformation has seven unknown parameters, any selection of seven out of these nine equations yields a formally solvable system. There are such combinations.

However, the choice of which seven equations are used can significantly affect the geometric validity and numerical stability of the resulting solution. If certain spatial directions (e.g., components along one of the coordinate axes) are underrepresented or missing altogether, the rotation matrix may not be properly determined, and the scale factor may be distorted.

In earlier works, these 36 combinations were generally treated as equivalent from a computational standpoint [2]. Nevertheless, without ensuring proper spatial coverage, some combinations may become ill-conditioned or even singular in the estimation process.

The following sections address the geometric conditions required for stable estimation and categorize the 36 combinations accordingly.

III. Geometric Conditions for Stability

For a minimal three-point setup to produce a numerically stable and geometrically valid seven-parameter Helmert solution, algebraic independence of seven rows is not sufficient. The chosen seven coordinate equations must also make the transformation parameters observable in first order: they must constrain translations on all three axes and rotations about three independent directions.

Minimal decomposition and geometric intuition

It is convenient to think of the parameters in two intuitive groups:

In-plane (2D-like) effects: two translations and one rotation that primarily act within a data plane, plus the scale.

Out-of-plane (depth) effects: the remaining translation and the two rotations that primarily act orthogonal to that plane.

In practice, a “data plane” is not a fixed XY/XZ/YZ plane of the global frame. It is induced by the geometry of the selected equations. Most importantly, when two full 3D points are used, they determine a twist axis (the line through the two points in the target frame). Any residual ambiguity after using those two points is a rotation around this axis; a seventh equation must be chosen so that it is sensitive to this twist (i.e., its measured direction is not parallel to the axis). If it is parallel, the first-order sensitivity to twist vanishes and the configuration becomes fragile.

Practical stability rules (for three points, 36 selections)

We use two simple, geometry-aware rules that work robustly across typical layouts:

Rule A — Two full points + one component (twist-aware).

Pick two full 3D points (6 rows) and add one coordinate from the third point whose direction is not parallel to the twist axis defined by the two full points. Intuition: the extra component must “see” the remaining twist DOF.

Rule B — One full point + matched 2D pairs that include the normal.

Pick one full 3D point (3 rows). From the other two points take the same 2D pair (two rows each) that includes the local normal of the data plane (e.g., YZ or XZ for near-XY geometries). This reliably fixes two rows of the rotation; the third follows from orthonormality.

Plane-based restatement.

Instead of naming a global plane (XY/XZ/YZ), first infer the effective plane from the geometry (or from two full points). Then ensure that at least one measured component per “non-full” point lies orthogonal to that plane. Merely picking “XY” because it is convenient in the global axes can misclassify cases—what matters is the geometry-induced plane and axis.

Examples (illustrative, geometry-dependent)

Near-XY layouts (points almost planar with a small Z relief) typically favor Z as the out-of-plane direction:

Rule A (twist-aware):

A(x,y,z), B(x,y,z), C(z)

A(x,y,z), C(x,y,z), B(z)

B(x,y,z), C(x,y,z), A(z)

Rule B (matched pairs incl. normal):

A(x,y,z), B(y,z), C(y,z)

B(x,y,z), A(y,z), C(y,z)

C(x,y,z), A(y,z), B(y,z)

If the effective plane is closer to XZ (resp. YZ), exchange Z for Y (resp. X) in the examples above. These form the canonical families that most geometries elevate to the top when all 36 selections are evaluated.

Configurations that omit the normal direction (e.g., only x,y rows from two points and an x from the third in a near-XY geometry), or that add a seventh row parallel to the twist axis, tend to be ill-posed in first order and numerically fragile in practice.

Implications for the 36 selections

The two rules above provide a clear checklist to screen the 36 possible seven-row selections. For many near-planar geometries this screening reduces the practically reliable choices to the canonical families illustrated in section III.C (often nine, depending on which global axis aligns with the geometry-induced normal). For more three-dimensional point arrangements, additional selections may also behave well, but those identified by Rules A–B remain consistently robust. In the next section we enumerate all 36 selections for the test geometries, mark the stable ones according to these rules, and quantify their numerical behavior using a unit-invariant conditioning measure introduced later.

IV. Evaluation Protocol and Analysis of the 36 Selections

This section defines the unit-invariant metric we use to judge numerical stability, explains how the 36 seven-row selections are generated and solved, and reports how the geometry-aware rules of Section III manifest in practice on two synthetic three-point cases with non-unit scale, non-zero rotations and translations, and additive noise.

Normalized conditioning of the scaled Jacobian

Let $r \in \mathbb{R}^7$ collect the seven residual equations selected from the nine scalar coordinate equations:

$$r_{i,c} = \mathrm{P}_{i,c} - (s\,\mathrm{R}\,\mathrm{p}_i + t)_c \qquad c \in \{X, Y, Z\},\ i \in \{A, B, C\} \tag{1}$$

Linearizing around the solution $(s, \mathrm{R}, \mathrm{t})$ gives $r \approx \mathrm{J}\,\eth\theta$, where $\eth\theta \in \mathbb{R}^7$ are the parameter increments (scale, three rotation parameters in radians, and three translations) and $\mathrm{J} \in \mathbb{R}^{7\times 7}$ is the Jacobian assembled from the seven chosen rows.

To make conditioning independent of units, we scale the translation parameters by a characteristic length

$$L_0 = \frac{1}{3}\left(\|p_A - p_B\| + \|p_A - p_C\| + \|p_B - p_C\|\right) \tag{2}$$

and keep scale and rotations dimensionless (scale) / radians (rotations). With

$$\mathrm{S} = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ L_0,\ L_0,\ L_0) \tag{3}$$

the normalized condition number is

$$K = \frac{\sigma_{max} JS}{\sigma_{min} JS} \quad (4)$$

Small K indicates a well-conditioned selection (robust to noise and modeling error); large K signals fragility. In noisy tests we also report the data misfit via RMS residuals.

Practical thresholds used in this paper.

- Stable: $K \leq 10$
- Usable with caution: $10 < K \leq 30$
- Unstable: $K > 30$

These bands proved informative across our experiments and can be tightened/relaxed per application.

Enumerating and solving the 36 seven-row selections

From the nine scalar equations $\{A.X, \ldots, C.Z\}$ we enumerate all $\binom{9}{7} = 36$ subsets. For each subset:

1. Initialization. We compute an initial similarity estimate with a closed-form method (e.g., Umeyama) on all nine rows to obtain (s_0, R_0, t_0).
2. Nonlinear fit on the selected seven rows. We solve $\min_{s,R,t} \| r \|_2^2$ with rotation parametrized by a unit quaternion (or any minimal 3-parameter update); constraints $\det R = +1$, $R^T R = I$ are enforced through the parametrization.
3. Diagnostics. We record K from the scaled Jacobian JS at the solution, the orthonormality error $\| R^T R - I \|_F$, $\det R$, and two RMS values: – RMS7: over the seven rows used in the fit; – RMS9: over all nine rows (consistency check). Selections are then ranked by K, and labeled according to the thresholds above.

Test geometries and ground truth

We consider two representative three-point layouts; in both cases the target coordinates are synthesized by a non-trivial similarity transform and optional Gaussian noise.

The transformed coordinates are computed using the exact Helmert formula:

$$X_i = s \cdot R \cdot x_i + t \quad (5)$$

The rotation matrix R is constructed from the given Euler angles.

- Case A — Near-planar geometry (highly discriminative).

$p_A = (0,\ 0,\ 0), p_B = (100,\ 0,\ 10^{-4}), p_C = (0,\ 100,\ 10^{-4})$, scale $s = 1.0025$, $(\omega, \phi, \kappa) = (+1.5^o,\ -2.3^o,\ +3.7^o)$, $t = (300,\ 700,\ 1200)^T$.

We set $P_i = s\, R(\omega, \phi, \kappa)\, p_i + t + \varepsilon_i$, with $\varepsilon_i \sim \mathcal{N}(0, \sigma^2 I)$, $\sigma = 5$ mm unless noted. This layout is almost XY-planar, so "out-of-plane" information aligns with Z.

- Case B — General 3D geometry (less discriminative, more tolerant).

$p_A = (2,\ 10,\ 15), p_B = (120,\ 20,\ 5), p_C = (35,\ 130,\ 60)$, $s = 1.0008$, $(\omega, \phi, \kappa) = (-0.8^o,\ +1.9^o,\ -2.6^o)$, $t = (500.5,\ 999.3,\ 1502.1)^T$, $\sigma = 5$–10 mm.

Here no global axis coincides with the "effective normal"; thus more selections behave acceptably.

The root-mean-square (RMS) error is computed as:

$$\mathrm{RMS9} = \sqrt{\frac{1}{9} \sum_{i=1}^{3} (\Delta X_i^2 + \Delta Y_i^2 + \Delta Z_i^2)} \quad (6)$$

where ΔX_i, ΔY_i , ΔZ_i d are the differences between the observed and the transformed coordinates of point i. The denominator 9 reflects that RMS9 is computed per coordinate component (three points × three components).

RMS7 is computed only for the selected 7 coordinate components:

$$\mathrm{RMS7} = \sqrt{\frac{1}{7} \sum_{(i,c) \in S} (\Delta c_i)^2} \quad (7)$$

where S is the set of the seven chosen coordinate components ($|S| = 7$, $c \in \{X, Y, Z\}$).

How the 36 selections separate in practice

Across both cases, the rankings consistently reflect the geometry-aware rules of Section III:

- Rule A (two full points + one twist-aware component). Selections of the form $\{A(x,y,z), B(x,y,z), C(c)\}$ with c not parallel to the twist axis of AB (and cyclic permutations) systematically fall in the stable band for Case A, and remain among the best in Case B.
- Rule B (one full point + matched 2D pairs including the normal). Selections like $\{A(x,y,z), B(y,z), C(y,z)\}$ (or xz/xz , depending on the effective normal) also rate stable in Case A and near-stable in Case B.
- Omitting the normal or adding a row parallel to the twist axis (e.g., $\{A(x,y), B(x,y), C(x)\}$ in a near-XY geometry, or "Rule A" with the third component aligned with the twist axis) drives $\sigma_{\min}(JS)$ down and pushes K into the unstable band, often with RMS9 noticeably larger than RMS7 (indicating overfitting of the restricted subset).

In the near-planar Case A this leads to a clear separation: the canonical families predicted by Section III dominate the top of the list, while many of the remaining combinations cluster at large K . In the more volumetric Case B, additional selections may enter the "usable" range, but Rule-A and Rule-B families remain the most reliable.

Table organization and labeling

For each case we provide two tables:

- Table I. The top-ranked selections with brief descriptors (row set), $\det R$, orthonormality error (onErr), RMS7/RMS9, and K for Case A.
- Table II. The full list of all 36 selections, sorted by K for Case B.

Takeaways for three-point practice

1. Geometry matters. Stability is not a property of a fixed nine selections, but of how the chosen rows interact with the twist axis and the effective normal of the data.
2. Rules A–B are reliable defaults. They consistently land in the best tier across disparate geometries.

A numeric gate keeps decisions objective. The normalized K provides an immediate, unit-invariant criterion to accept or reject a seven-row selection before committing to it in a pipeline.

V. RESULTS

This section reports quantitative behavior for the 36 seven-row selections on two synthetic three-point geometries. In both cases the target coordinates were generated by a non-trivial similarity transform and additive Gaussian noise (Case A: σ = 5 mm, Case B: σ = 8 mm). Each selection is evaluated by the unit-invariant conditioning K (condition number of the scaled Jacobian, Sec. IV.A), by RMS residuals on the seven used rows (RMS7) and on all nine rows (RMS9), and by standard rotation diagnostics (detR, orthonormality error as onErr). Bands used. Stable $K \le 10$; usable $10 < K \le 30$; unstable $K > 30$.

Case A: near-planar geometry

Table I lists all 36 selections sorted by K. A clear separation emerges: selections consistent with Rule A (two full points + one twist-aware component) and Rule B (one full point + matched 2D pairs that include the local normal) occupy the top with $K \approx 4-6$ and sub-millimetric RMS7/RMS9 (e.g., the first five rows show $K = 4.0$, RMS9 ≈ 1.1×10^{-3} m). Mid-tier choices jump to $K \approx 90-400$, while the worst cases explode to $K \approx 4.1\times10^{3}$ and 4.3×10^{3}, confirming the geometry-dependence predicted in Section III.

TABLE I. CASE A — 36 SEVEN-ROW SELECTIONS RANKED BY THE NORMALIZED CONDITIONING K. LOW-K ROWS COINCIDE WITH RULE A/B FAMILIES; HIGH-K ROWS OMIT THE EFFECTIVE NORMAL OR ADD A TWIST-PARALLEL COMPONENT.

Subset	s	Om	Phi	Kap	Tx	Ty	Tz	detR	onErr	RMS7	RMS9	K
AX AY AZ BX BY BZ CZ	1.0025	1.5052	-2.3035	3.7001	300.005	700.002	1200.004	1.0e+00	2.6e-17	7.3e-13	1.13e-03	4.0
AX AY AZ BZ CX CY CZ	1.0025	1.5052	-2.3035	3.6982	300.005	700.002	1200.004	1.0e+00	2.4e-17	6.2e-13	1.13e-03	4.0
AX AY AZ BX BZ CX CZ	1.0025	1.5052	-2.3035	3.6981	300.005	700.002	1200.004	1.0e+00	1.1e-16	5.2e-13	1.13e-03	4.0
AX AY AZ BY BZ CY CZ	1.0025	1.5052	-2.3035	3.7001	300.005	700.002	1200.004	1.0e+00	2.2e-16	5.9e-13	1.13e-03	4.0
AZ BX BY BZ CX CY CZ	1.0025	1.5052	-2.3034	3.6990	300.003	700.000	1200.004	1.0e+00	2.2e-16	2.3e-13	7.99e-04	4.0
AX AZ BX BY BZ CX CZ	1.0025	1.5052	-2.3035	3.6981	300.005	699.999	1200.004	1.0e+00	1.1e-16	6.3e-13	1.56e-03	4.9
AX AZ BX BZ CX CY CZ	1.0025	1.5052	-2.3035	3.6981	300.005	700.002	1200.004	1.0e+00	1.1e-16	1.9e-13	1.08e-03	4.9
AY AZ BY BZ CX CY CZ	1.0025	1.5052	-2.3035	3.7001	300.002	700.002	1200.004	1.0e+00	2.5e-16	3.3e-13	1.63e-03	4.9
AY AZ BX BY BZ CY CZ	1.0025	1.5052	-2.3035	3.7001	300.005	700.002	1200.004	1.0e+00	1.1e-16	2.8e-13	1.18e-03	4.9
AX AZ BX BY BZ CY CZ	1.0025	1.5052	-2.3035	3.7000	300.005	700.002	1200.004	1.0e+00	1.9e-17	7.4e-13	1.08e-03	5.9
AY AZ BX BZ CX CY CZ	1.0025	1.5052	-2.3035	3.6981	300.005	700.002	1200.004	1.0e+00	2.9e-17	4.3e-13	1.18e-03	5.9
AX AZ BY BZ CX CY CZ	1.0025	1.5052	-2.3034	3.6980	300.005	699.999	1200.004	1.0e+00	1.1e-16	3.8e-13	1.57e-03	5.9
AY AZ BX BY BZ CX CZ	1.0025	1.5052	-2.3034	3.7000	300.002	700.002	1200.004	1.0e+00	5.2e-18	4.0e-13	1.63e-03	5.9
AX AY AZ BX BY CX CY	1.0025	1.5409	-2.3218	3.7001	300.005	700.002	1200.004	1.0e+00	1.2e-16	2.9e-13	2.34e-02	90.0
AY AZ BX BY BZ CX CY	1.0025	1.5359	-2.3034	3.7000	300.004	700.002	1200.004	1.0e+00	2.2e-16	2.3e-13	1.79e-02	93.8
AX AZ BX BY CX CY CZ	1.0025	1.5052	-2.3317	3.6988	300.005	700.000	1200.004	1.0e+00	1.2e-16	2.9e-13	1.65e-02	93.9
AX AY AZ BX BY BZ CX	1.0025	1.5515	-2.3035	3.7001	300.005	700.002	1200.004	1.0e+00	3.1e-17	7.8e-13	2.70e-02	104.6
AX AY AZ BY BZ CX CY	1.0025	1.5515	-2.3034	3.7000	300.005	700.002	1200.004	1.0e+00	1.1e-16	8.8e-13	2.70e-02	104.6
AX AY AZ BX CX CY CZ	1.0025	1.5052	-2.3016	3.6981	300.005	700.002	1200.004	1.0e+00	1.1e-17	8.2e-13	1.60e-03	104.8
AX AY AZ BX BY CY CZ	1.0025	1.5052	-2.3015	3.7001	300.005	700.002	1200.004	1.0e+00	2.2e-16	8.4e-13	1.62e-03	104.8
AX AY BX BY BZ CX CZ	1.0025	1.5346	-2.3328	3.7001	300.005	700.002	1200.055	1.0e+00	2.5e-16	2.0e-13	1.71e-02	109.7
AX AY BY BZ CX CY CZ	1.0025	1.5346	-2.3328	3.7001	300.005	700.002	1200.055	1.0e+00	2.5e-17	9.6e-14	1.71e-02	109.7
AX AY BX BY BZ CX CY	1.0025	1.5409	-2.3218	3.7001	300.005	700.002	1200.036	1.0e+00	3.1e-16	1.8e-13	1.47e-02	120.0
AX AY BX BY CX CY CZ	1.0025	1.5409	-2.3218	3.7001	300.005	700.002	1200.066	1.0e+00	1.1e-16	1.6e-13	2.31e-02	120.0
AX BX BY BZ CX CY CZ	1.0025	1.5273	-2.3255	3.6996	300.005	700.001	1200.043	1.0e+00	2.2e-16	1.1e-13	1.29e-02	122.1
AX AY AZ BX BY BZ CY	1.0025	1.5087	-2.3035	3.7001	300.005	700.002	1200.004	1.0e+00	2.8e-17	4.7e-13	2.27e-03	184.8
AX AY AZ BX BZ CX CY	1.0025	1.5086	-2.3035	3.6983	300.005	700.002	1200.004	1.0e+00	1.1e-16	3.0e-13	2.23e-03	185.0
AX AY AZ BY CX CY CZ	1.0025	1.5053	-2.3855	3.7003	300.005	700.002	1200.004	1.0e+00	1.2e-17	4.5e-13	4.78e-02	186.0
AX AY AZ BX BY CX CZ	1.0025	1.5052	-2.3855	3.7001	300.005	700.002	1200.004	1.0e+00	2.2e-16	7.0e-13	4.79e-02	186.1
AY BX BY BZ CX CY CZ	1.0025	1.5473	-2.3455	3.7001	300.006	700.002	1200.078	1.0e+00	1.6e-16	2.0e-13	2.45e-02	212.1
AX AZ BX BY BZ CX CY	1.0025	1.6123	-2.3035	3.7026	300.005	700.006	1200.004	1.0e+00	3.1e-16	1.6e-13	6.24e-02	376.2
AY AZ BX BY CX CY CZ	1.0025	1.5051	-2.1851	3.6998	299.997	700.002	1200.004	1.0e+00	2.2e-16	2.3e-13	6.90e-02	388.5
AX AY BX BZ CX CY CZ	1.0025	1.5008	-2.2991	3.6979	300.005	700.002	1199.996	1.0e+00	1.1e-16	3.9e-13	2.87e-03	400.8
AX AY BX BY BZ CY CZ	1.0025	1.5007	-2.2990	3.7001	300.005	700.002	1199.996	1.0e+00	1.1e-16	7.7e-14	2.93e-03	402.4
AX AY AZ BY BZ CX CZ	0.9694	1.5567	-2.3821	3.8268	300.005	700.002	1200.004	1.0e+00	2.5e-16	2.1e-13	1.56e+00	4058.2
AX AY AZ BX BZ CY CZ	1.0024	1.5053	-2.3037	3.6176	300.005	700.002	1200.004	1.0e+00	1.5e-17	1.8e-13	6.73e-02	4330.6

Case B: general 3D geometry

Table II is more tolerant: many selections fall in the stable/usable bands (e.g., $K = 3.9-30$), and only a minority exceed $K \approx 35-150$. This reflects that the effective normal is not aligned with a global axis, so more combinations incidentally capture sufficient out-of-plane sensitivity. Yet the Rule A/B families remain among the best (small RMS9, consistent rotations).

TABLE II. CASE B — 36 SEVEN-ROW SELECTIONS RANKED BY K. MORE COMBINATIONS ARE ACCEPTABLE THAN IN CASE A, BUT RULE A/B OPTIONS STILL DOMINATE THE TOP TIER.

Subset	s	Om	Phi	Kap	Tx	Ty	Tz	detR	onErr	RMS7	RMS9	K
AX AY AZ BY BZ CY CZ	1.0007	-0.8004	1.9058	-2.6022	500.509	999.303	1502.094	1.0e+00	1.1e-16	5.0e-13	2.84e-03	3.9
AX AY AZ BX BZ CX CZ	1.0007	-0.8014	1.9056	-2.6048	500.509	999.304	1502.095	1.0e+00	1.1e-16	5.8e-13	2.24e-03	4.3
AX AY AZ BZ CX CY CZ	1.0007	-0.8006	1.9058	-2.6051	500.509	999.304	1502.094	1.0e+00	1.1e-16	8.3e-13	2.33e-03	4.3
AX AY AZ BX BY BZ CZ	1.0007	-0.8016	1.9056	-2.6024	500.509	999.304	1502.095	1.0e+00	1.1e-16	8.1e-13	2.68e-03	4.3
AY AZ BX BY BZ CY CZ	1.0007	-0.8004	1.9058	-2.6022	500.503	999.303	1502.094	1.0e+00	1.1e-17	6.3e-13	1.89e-03	4.4
AZ BX BY BZ CX CY CZ	1.0007	-0.8003	1.9058	-2.6024	500.503	999.303	1502.094	1.0e+00	2.5e-16	5.1e-13	1.99e-03	4.9
AX AZ BX BY BZ CX CZ	1.0007	-0.8014	1.9056	-2.6048	500.509	999.299	1502.095	1.0e+00	1.6e-17	3.6e-13	3.56e-03	5.2
AY AZ BY BZ CX CY CZ	1.0007	-0.8004	1.9058	-2.6022	500.502	999.303	1502.094	1.0e+00	1.6e-16	8.6e-13	2.15e-03	5.3
AX AZ BX BZ CX CY CZ	1.0007	-0.8014	1.9056	-2.6048	500.509	999.309	1502.095	1.0e+00	1.1e-16	4.2e-13	3.53e-03	6.0
AX AZ BY BZ CX CY CZ	1.0008	-0.7994	1.9060	-2.6057	500.509	999.295	1502.094	1.0e+00	2.2e-16	2.7e-13	4.17e-03	7.9
AY AZ BX BZ CX CY CZ	1.0007	-0.8005	1.9058	-2.6027	500.504	999.303	1502.094	1.0e+00	2.2e-16	3.2e-13	1.80e-03	8.5
AY AZ BX BY BZ CX CZ	1.0008	-0.8002	1.9059	-2.6021	500.502	999.303	1502.094	1.0e+00	9.2e-18	5.1e-13	2.28e-03	9.4
AX AY AZ BX BY CX CZ	1.0007	-0.8003	1.9109	-2.6026	500.510	999.304	1502.095	1.0e+00	1.1e-16	1.8e-12	4.14e-03	10.9
AX AY AZ BX BY CX CY	1.0007	-0.7934	1.9098	-2.6031	500.510	999.302	1502.096	1.0e+00	1.1e-16	4.3e-13	5.51e-03	10.9
AX AY AZ BY CX CY CZ	1.0007	-0.7988	1.9128	-2.6024	500.511	999.303	1502.095	1.0e+00	1.1e-16	9.2e-13	5.03e-03	11.2
AX AY AZ BX BY BZ CY	1.0007	-0.7940	1.9062	-2.6031	500.509	999.302	1502.096	1.0e+00	1.1e-16	5.3e-13	5.25e-03	11.3
AX AZ BX BY BZ CY CZ	1.0007	-0.8018	1.9055	-2.5975	500.508	999.315	1502.095	1.0e+00	1.1e-16	3.1e-13	6.05e-03	11.5
AX AZ BX BY BZ CX CY	1.0007	-0.7919	1.9064	-2.6046	500.509	999.299	1502.096	1.0e+00	1.6e-16	4.6e-13	6.59e-03	11.8
AX AY AZ BX BZ CX CY	1.0007	-0.7954	1.9061	-2.6046	500.509	999.302	1502.096	1.0e+00	1.6e-16	1.1e-12	4.26e-03	13.1
AY AZ BX BY CX CY CZ	1.0007	-0.8002	1.9067	-2.6022	500.503	999.303	1502.094	1.0e+00	1.1e-16	5.6e-13	1.96e-03	13.7
AY BX BY BZ CX CY CZ	1.0007	-0.8008	1.9064	-2.6021	500.503	999.304	1502.093	1.0e+00	4.7e-18	3.7e-13	2.07e-03	14.2
AX AY AZ BX BZ CY CZ	1.0007	-0.8011	1.9057	-2.6108	500.510	999.304	1502.095	1.0e+00	1.1e-16	4.4e-13	7.10e-03	14.2
AX AY BX BY BZ CX CY	1.0007	-0.7934	1.9098	-2.6031	500.510	999.302	1502.089	1.0e+00	6.4e-18	3.5e-13	7.76e-03	14.4
AX AY BX BZ CX CY CZ	1.0007	-0.7980	1.9015	-2.6065	500.508	999.303	1502.105	1.0e+00	1.1e-16	1.6e-13	4.03e-03	15.6
AX AY BX BY CX CY CZ	1.0007	-0.7934	1.9098	-2.6031	500.510	999.302	1502.111	1.0e+00	2.5e-16	4.2e-13	8.88e-03	15.8
AX AY AZ BY BZ CX CZ	1.0006	-0.8049	1.9049	-2.6032	500.509	999.306	1502.096	1.0e+00	2.5e-16	3.9e-13	1.02e-02	18.8
AX AY BX BY BZ CY CZ	1.0007	-0.7953	1.8980	-2.6029	500.507	999.302	1502.113	1.0e+00	1.1e-16	2.9e-13	6.61e-03	19.4
AX AY BX BY BZ CX CZ	1.0007	-0.8068	1.9120	-2.6020	500.510	999.306	1502.080	1.0e+00	1.1e-16	2.7e-13	6.17e-03	21.2
AX AZ BX BY CX CY CZ	1.0007	-0.7982	1.9203	-2.5987	500.512	999.312	1502.095	1.0e+00	1.1e-16	2.6e-13	1.04e-02	21.4
AX AY AZ BY BZ CX CY	1.0006	-0.7877	1.9066	-2.6039	500.509	999.301	1502.098	1.0e+00	9.4e-18	5.2e-13	1.03e-02	21.8
AX AY BY BZ CX CY CZ	1.0008	-0.8084	1.9181	-2.6010	500.512	999.305	1502.065	1.0e+00	1.1e-17	5.1e-13	1.00e-02	30.6
AY AZ BX BY BZ CX CY	1.0008	-0.8025	1.9057	-2.6019	500.501	999.304	1502.094	1.0e+00	1.1e-16	1.2e-12	2.99e-03	35.1
AX AY AZ BX CX CY CZ	1.0007	-0.8052	1.8884	-2.6119	500.506	999.304	1502.094	1.0e+00	1.1e-16	6.9e-13	1.36e-02	44.4
AX AY AZ BX BY BZ CX	1.0007	-0.7792	1.9075	-2.6043	500.510	999.298	1502.099	1.0e+00	2.5e-16	3.7e-13	1.57e-02	44.9
AX BX BY BZ CX CY CZ	1.0007	-0.7801	1.8804	-2.6156	500.504	999.273	1502.155	1.0e+00	1.1e-16	2.4e-13	2.04e-02	57.5
AX AY AZ BX BY CY CZ	1.0006	-0.7795	1.9960	-2.6047	500.533	999.300	1502.097	1.0e+00	1.1e-16	1.1e-13	6.62e-02	146.6

VI. PRACTICAL IMPLICATIONS AND WORKFLOW

Default recipe (three points):

- Prefer Rule A (two full 3D points + one twist-aware component).
- In near-planar layouts with known normal, Rule B (one full 3D point + matched 2D pairs including the normal) is equally reliable.

Quality gate:

- Compute K at the solution. Accept if $K \le 10$; treat as usable if $10 < K \le 30$; reject otherwise and switch to a Rule A/B selection. Always check detR≈+1, small $\|R^T R - I\|_F$, and compare RMS9 vs. RMS7 (a large gap suggests overfitting the restricted subset).

Minimal-data workflow.

1. Enumerate the 36 seven-row selections.
2. Initialize (e.g., Umeyama on all nine rows), then refine on each selection.
3. Record K, RMS7, RMS9, detR, orthonormality error onErr.
4. Pick the smallest- K selection (tie-break by RMS9).
5. If no selection meets $10 < K \leq 30$, acquire more information (extra coordinates/points) rather than relying on a fragile minimal fit.

VII. Conclusions

We revisited minimal seven-equation selections for the seven-parameter 3D similarity (Helmert) transformation based on three common points. The key finding is geometry-dependence: stability is not a property of a fixed set of nine combinations. Two simple geometry-aware rules

- (Rule A) two full 3D points plus one twist-aware component, and
- (Rule B) one full 3D point plus matched 2D pairs including the normal

consistently identify well-conditioned selections. A normalized, unit-invariant conditioning measure K provides an objective accept/reject criterion.

Experiments on a near-planar and a general 3D layout confirm this picture: in Case A, Rule A/B families dominate with $K \approx 4–6$, while geometry-inconsistent selections deteriorate to $K \gg 30$ (even $>10^3$); in Case B, more combinations are acceptable, yet Rule A/B remain among the best.

In this sense, our selection-and-conditioning perspective complements exact algebraic treatments of the seven-parameter problem [3], combinatorial Gauss–Jacobi strategies [5], and the broader algebraic viewpoint advocated in geodesy [6], while also relating to Procrustean extensions beyond seven parameters [7].

This work complements [4], where multiple seven-parameter solutions are combined via a Jacobi-type weighted average. Our results supply a principled pre-selection and weighting criterion: compute K for each seven-row solution, discard high-K subsets (e.g., $K > K_{max} \approx 30$), and weight the retained ones inversely with K^2 (optionally with RMS7). In our tests this improves the stability and accuracy of the final averaged parameters without altering the core numerical machinery of [4].

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