DOI: 10.12700/AIS.2025.024

Proving Olympiad Inequalities by Using Some Unusual Ideas

Borbély József
Óbudai Egyetem, Székesfehérvár, Hungary
borbely.jozsef@uni-obuda.hu

***Abstract*– There are a lot of challenging problems concerning inequalities in mathematical olympiads which have an elegant or unorthodox elementary solution. In this paper we will present some of these.**

I. Introduction

In this paper we will prove six olympiad level inequalities using some unusual techniques. Every proof will be completely elementary.

II. The Problems and Their Solutions

First we will prove an inequality, which was proposed in 2010 by the author in the Hungarian competition NMMV.

Inequality 1: *Let* n *be a positive integer number, and let* c *be a positive constant. If the nonnegative numbers*

$x_1, x_2, \ldots, x_n$ *satisfy the condition*

$\sum_{i=1}^{n}(x_i^2 + ix_i)=c$, *then the inequality*

$\sqrt{c+\frac{n^2}{4}}-\frac{n}{2} \le \sum_{i=1}^{n} x_i$ *holds.*

Proof:

The most important question is how we can use the condition that the numbers are nonnegative. After some experimentation we can come up with the (maybe surprising) idea that the nonnegativity of the given numbers implies the inequality

$\sum_{i=1}^{n}(x_i^2 + ix_i) \le (\sum_{i=1}^{n} x_i)^2 + n\cdot\sum_{i=1}^{n} x_i$.

This is comfortable for us because the right hand side of the inequality is a second degree polynomial of $(\sum_{i=1}^{n} x_i)$.
Let us denote $\sum_{i=1}^{n} x_i$ by $\sum_{i=1}^{n} x_i = z$. Using this notation we have to check for which values of z we have

$c \le z^2 + n\cdot z$.

This inequality is equivalent to

$0 \le z^2 + n\cdot z - c$.

The roots of the polynomial $p(z) = z^2 + n\cdot z - c$ are

$\frac{-n+\sqrt{n^2+4c}}{2} = \sqrt{c+\frac{n^2}{4}}-\frac{n}{2}$

and $\frac{-n-\sqrt{n^2+4c}}{2} = -\sqrt{c+\frac{n^2}{4}}-\frac{n}{2}$.

Thus $p(z) = z^2 + n\cdot z - c =$

$= [z-(\sqrt{c+\frac{n^2}{4}}-\frac{n}{2})]\cdot[z-(-\sqrt{c+\frac{n^2}{4}}-\frac{n}{2})]$

Here the factor $[z-(-\sqrt{c+\frac{n^2}{4}}-\frac{n}{2})]$ must be nonnegative, because the value of $z = \sum_{i=1}^{n} x_i$ is nonnegative.
Thus $[z-(\sqrt{c+\frac{n^2}{4}}-\frac{n}{2})]$ must be nonnegative as well which implies the desired result. ∎

Our second inequality is also a problem which was invented by the author and it was posed in the Hungarian competition OKTV in 2012.

Inequality 2: *Let* $1 \le k \le n$ *be integer numbers, and let*

$x_1, x_2, \ldots, x_n$ *be nonnegative numbers with sum*

$\sum_{i=1}^{n} x_i = 1$.

With these conditions the maximal possible value of

$x_1x_2\cdot\ldots\cdot x_k + x_2x_3\cdot\ldots\cdot x_{k+1}+\ldots+x_{n-k+1}x_{n-k+2}\cdot\ldots\cdot x_n$

is $\frac{1}{k^k}$.

Proof:

Here we will use a similar technique to the one we have seen in the solution of the first problem, namely we will smuggle some terms into the expression to be able to tackle the problem more easily.
If we consider the expression

$x_1x_2\cdot\ldots\cdot x_k + x_2x_3\cdot\ldots\cdot x_{k+1}+\ldots+x_{n-k+1}x_{n-k+2}\cdot\ldots\cdot x_n$,

then we can note that in every term the k indices cover all possible remainders if we divide them by k. This observation can us inspire to introduce the following partial sums: for every $0 \le r \le k-1$, let S_r be the sum of the numbers x_i for which the index i has the remainder r if we divide it by k.
Using this notation we have

$\sum_{i=1}^{n} x_i = \sum_{r=0}^{k-1} S_r = 1$, and

$$x_1x_2 \cdot \ldots \cdot x_k + x_2x_3 \cdot \ldots \cdot x_{k+1} + \ldots + x_{n-k+1}x_{n-k+2} \cdot \ldots \cdot x_n \le$$

$$\le \prod_{r=0}^{k-1} S_r.$$

Using the well known inequality between the arithmetic and geometric means we conclude that the value $\prod_{r=0}^{k-1} S_r$ will be maximal if $S_0 = S_1 = \ldots = S_{k-1} = \frac{1}{k}$, thus the value of

$$x_1x_2 \cdot \ldots \cdot x_k + x_2x_3 \cdot \ldots \cdot x_{k+1} + \ldots + x_{n-k+1}x_{n-k+2} \cdot \ldots \cdot x_n$$

cannot be greater than $\frac{1}{k^k}$.

The value $\frac{1}{k^k}$ can be achieved for example by choosing $x_1 = x_2 = \ldots = x_k = \frac{1}{k}$, and $x_{k+1} = x_{k+2} = \ldots = x_n = 0$. ■

Inequality 3 will be a problem of the Hungarian competition OKTV from the year 2013. Here one can come up with a surprising and elegant solution.

Inequality 3:

For real numbers

$a_1 \le a_2 \le \ldots \le a_n \le b_1 \le b_2 \le \ldots \le b_n$ *we have*

$$\left(\sum_{i=1}^{n} a_i + \sum_{j=1}^{n} b_j\right)^2 \ge 4n \cdot \sum_{k=1}^{n} a_k b_k .$$

Proof:

Obviously it is enough to prove the inequality

$$\left(\sum_{i=1}^{n} a_i + \sum_{j=1}^{n} b_j\right)^2 - 4n \cdot \sum_{k=1}^{n} a_k b_k \ge 0.$$

The key observation is that the expression on the left hand side is similar to the discriminant of a quadratic polynomial. Let us namely define the polynomial

$$p(z) = \sum_{k=1}^{n} (z - a_k) \cdot (z - b_k).$$

It is easy to check that its discriminant is

$$\left(\sum_{i=1}^{n} a_i + \sum_{j=1}^{n} b_j\right)^2 - 4n \cdot \sum_{k=1}^{n} a_k b_k .$$

Due to the arrangement of our numbers, the value of $p(a_n)$ cannot be greater than 0, and the value of $p(b_n)$ cannot be less than 0. Thus the polynomial p(z) has a real root, hence its discriminant is nonnegative, and this gives the desired result. ■

The following problem was also posed in the Hungarian competition OKTV in the year 2008. By using a clever substitution we can give a short and elementary solution.

Inequality 4:

If the sum of the nonnegative real numbers t_1, t_2, t_3 *is*

$t_1+t_2+t_3 = 2\pi$, *then*

$$2\cos t_1 + 6\cos t_2 + 3\cos t_3 \ge -7.$$

Proof:

The trick is to interprete the problem geometrically. Let us define the vectors v_1, v_2, v_3 in such a way that the angle between v_1 and v_2 is t_1, the angle between v_2 and v_3 is t_2, the angle between v_3 and v_1 is t_3, and the length of v_1 is 1, the length of v_2 is 2, the length of v_3 is 3.
We will use the dot products of these vectors. Obviously

$$v_1v_2 = 2\cos t_1,\ v_2v_3 = 6\cos t_2,\ v_3v_1 = 3\cos t_3.$$

Thus we have

$$0 \le (v_1 + v_2 + v_3)^2 =$$

$$= v_1^2 + v_2^2 + v_3^2 + 2\cdot(v_1v_2 + v_2v_3 + v_3v_1) =$$

$$= 1+4+9+2\cdot(2\cos t_1 + 6\cos t_2 + 3\cos t_3).$$

This implies that

$$-7 \le 2\cos t_1 + 6\cos t_2 + 3\cos t_3.$$

The value (-7) can be achieved by choosing for example

$t_1 = t_2 = \pi$ and $t_3 = 0$. ■

Inequality 5 is a problem from the IMO (International Mathematical Olympiad) team selection test held in Hungary from the year 2005. No solution was published for this problem, the following one is the proof of the author.

Inequality 5:

If a, b, c, d *are nonnegative real numbers such that*

$a^2-ab+b^2 = c^2-cd+d^2$, *then* $(a+b)\cdot(c+d) \ge 2\cdot(ab+cd)$.

Proof:

Since a, b, c, d are nonnegative, they can be considered as lengths of some segments.
Consider a triangle with one side of length „a", and another side of length „b" with an enclosed angle of 60 degrees.
Similarly, consider a triangle with one side of length „c", another side of length „d", with an enclosed angle of 60

degrees. The lengths of the third sides in the two triangles are the same due to the law of cosines.
Therefore, due to the generalized law of sines, the radii of the circumcircles of the two triangles are the same. Let the remaining two angles in the first triangle be $(60° - \varphi)$ and $(60° + \varphi)$, respectively, and in the other one $(60° - \mu)$ and $(60° + \mu)$.

Since the radii of the circumscribed circles are equal, by normalizing the original inequality using the law of sines, we should prove the inequality

$[\sin(60° - \varphi) + \sin(60° + \varphi)]\cdot[\sin(60° - \mu) + \sin 60° + \mu \geq$

$\geq 2\cdot \sin(60° - \varphi)\cdot \sin(60° + \varphi) +$
$+ 2\cdot \sin(60° - \mu)\cdot \sin(60° + \mu).$

By using the well known identities

$\sin(x - y) + \sin(x + y) = 2\sin x\cdot \cos y$ and

$\cos(x - y) - \cos(x + y) = 2\sin x\cdot \sin y$

the inequality is equivalent to

$2\cdot\sin 60°\cdot\cos\varphi \cdot 2\cdot\sin 60°\cdot\cos\mu \geq$

$\geq \cos(2\varphi) - \cos 120° + \cos(2\mu) - \cos 120°.$

This is equivalent to

$3\cdot \cos\varphi\cdot\cos\mu \geq \cos(2\varphi) + \cos(2\mu) + 1,$

which can be written as

$0 \geq 2\cdot\cos^2(\varphi) + 2\cdot\cos^2(\mu) - 1 - 3\cdot \cos\varphi\cdot\cos\mu.$

Both φ and μ lie between 0° and 60°, thus the values of $\cos\varphi$ and $\cos\mu$ lie both between $\frac{1}{2}$ and 1. Thus for appropriate real numbers v and w we have

$\cos\varphi = \frac{1}{2}\cdot(1 + \sin v),\ \cos\mu = \frac{1}{2}\cdot(1 + \sin w).$

We substitute these ones:

$2\cdot\cos^2(\varphi) + 2\cdot\cos^2(\mu) - 1 - 3\cdot \cos\varphi\cdot\cos\mu =$
$= 2\cdot[\frac{1}{2}\cdot(1 + \sin v)]^2 + 2\cdot[\frac{1}{2}\cdot(1 + \sin w)]^2 - 1 -$

$-3\cdot[\frac{1}{2}\cdot(1 + \sin v)]\cdot[\frac{1}{2}\cdot(1 + \sin w)] =$

$= \frac{1}{2} + \frac{\sin^2(v)}{2} + \sin v + \frac{1}{2} + \frac{\sin^2(w)}{2} + \sin w - 1 - \frac{3}{4} -$

$-\frac{3}{4}\cdot \sin v\cdot \sin w - \frac{3}{4}\cdot \sin v - \frac{3}{4}\cdot \sin w =$

$= \frac{\sin^2(v)}{2} + \sin v + \frac{\sin^2(w)}{2} + \sin w - \frac{3}{4} - \frac{3}{4}\cdot \sin v\cdot \sin w -$

$-\frac{3}{4}\cdot \sin v - \frac{3}{4}\cdot \sin w.$

Using that $\sin^2(v) \leq \sin v$ and $\sin^2(w) \leq \sin w$ we have the following upper estimate:

$\frac{\sin^2(v)}{2} + \sin v + \frac{\sin^2(w)}{2} + \sin w - \frac{3}{4} - \frac{3}{4}\cdot \sin v\cdot \sin w -$

$-\frac{3}{4}\cdot \sin v - \frac{3}{4}\cdot \sin w \leq$

$\leq \frac{\sin v}{2} + \sin v + \frac{\sin w}{2} + \sin w - \frac{3}{4} - \frac{3}{4}\cdot \sin v\cdot \sin w -$

$-\frac{3}{4}\cdot \sin v - \frac{3}{4}\cdot \sin w =$

$= -\frac{3}{4} - \frac{3}{4}\cdot \sin v\cdot \sin w + \frac{3}{4}\cdot \sin v + \frac{3}{4}\cdot \sin w =$

$= \frac{3}{4}\cdot(\sin v - 1)\cdot(1 - \sin w).$

Here $\sin v - 1 \leq 0$ and $1 - \sin w \geq 0$, thus we proved the statement of the problem. ∎

Inequality 6 was published in the journal "Matematika tanítása" in 2011. This is a generalization of a problem posed in the competition USAMO in 1980. Here we present the solution of the author of this paper, which differs from the official one.

Inequality 6:

If $r_1, r_2, \ldots, r_n$ *are real numbers and* $S=\sum_{i=1}^{n} \sin^2 r_i$, *then*

$\sum_{i=1}^{n} \frac{\sin^2 r_i}{S+\cos^2 r_i} + \prod_{i=1}^{n} \cos^2 r_i \leq 1.$

Proof:

For every index $1 \leq i \leq n$, let x_i be $x_i = \sin^2 r_i$. Using this notation we have to prove that

$\sum_{i=1}^{n} \frac{x_i}{1+S-x_i} + \prod_{i=1}^{n} (1 - x_i) \leq 1.$

The role of the numbers x_i is symmetrical, thus we can assume that

$0 \leq x_1 \leq x_2 \leq \ldots \leq x_n.$

Now for every index $1 \leq i \leq n$ we have

$\frac{x_i}{1+S-x_i} \leq \frac{x_i}{1+x_1+x_2+\cdots+x_{n-1}}.$

Summing up these inequalities we get

$\sum_{i=1}^{n} \frac{x_i}{1+S-x_i} + \prod_{i=1}^{n} (1 - x_i) \leq$

$\leq \sum_{i=1}^{n} \frac{x_i}{1+x_1+x_2+\cdots+x_{n-1}} + \prod_{i=1}^{n} (1 - x_i) =$

$$= 1 - \frac{1-x_n}{1+x_1+x_2+\cdots+x_{n-1}} + (1-x_n)\cdot\prod_{i=1}^{n-1}(1-x_i) =$$

$$= 1 - (1-x_n)\cdot\left(\frac{1}{1+x_1+x_2+\cdots+x_{n-1}} - \prod_{i=1}^{n-1}(1-x_i)\right).$$

Since $1 - x_n \geq 0$, it would be enough to prove that

$\frac{1}{1+x_1+x_2+\cdots+x_{n-1}} - \prod_{i=1}^{n-1}(1-x_i) \geq 0$, which is equivalent to

$$(1 + x_1 + x_2 + \cdots + x_{n-1})\cdot\prod_{i=1}^{n-1}(1-x_i) \leq 1.$$

Due to the nonnegativity of the numbers x_i we have

$$(1 + x_1 + x_2 + \cdots + x_{n-1}) \leq \prod_{i=1}^{n-1}(1+x_i).$$

Using this observation we get

$$(1 + x_1 + x_2 + \cdots + x_{n-1})\cdot\prod_{i=1}^{n-1}(1-x_i) \leq$$
$$\leq \prod_{i=1}^{n-1}(1+x_i)\cdot\prod_{i=1}^{n-1}(1-x_i) =$$
$$= \prod_{i=1}^{n-1}(1-x_i^2) \leq \prod_{i=1}^{n-1} 1 = 1,$$

which is the desired result. ∎

REFERENCES

[1] Dušan Djukić, Vladimir Janković, Ivan Matić, Nikola Petrović, The IMO Compendium: A Collection of Problems Suggested for The International Mathematical Olympiads: 1959-2009 Second Edition, Problem Books in Mathematics, 2nd ed., 2011

[2] A matematika tanítása, Mozaik kiadó, 2011/1. szám, 419. feladat

Online sources:

[3] http://benoke98.f.fazekas.hu/olimpia/feladatok/valogato2005.pdf

[4] https://nmmv.berzsenyi.hu/feladatok/2010

[5] https://www.oktatas.hu/kozneveles/tanulmanyi_versenyek_/oktv_kereteben/versenyfeladatok_javitasi_utmutatok