ÓBUDAI EGYETEM
ÓBUDA UNIVERSITY

NEUMANN JÁNOS
FACULTY OF INFORMATICS

Annex 5

DEGREE PROJECT

THESIS

Investigating Pendulum-based Non-Linear Problems

OE-NIK	Student's name:	**Sovetova, Kymbat**
2022	Student's registration number:	**T/007194/FI12904/N**

Óbuda University
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DIPLOMA THESIS
TOPIC DESCRIPTION

Student name: **Kymbat Sovetova**
Registration number: T/007194/FI12904/N
Neptun's code: [redacted]

Title of diploma thesis:

Investigating Pendulum-based Non-Linear Problems
Nemlineáris Inga alapú problémák vizsgálata

Internal supervisor: Dr. Krisztián Kósi
External supervisor:

Deadline: 15th of May, 2021.

Subjects of the final exam: Engineering computational methods
Optimization
System and control theory
Stochastics

Topic description

Pendulum-based models are highly used to solve some problems, which are similar to various type of pendulums. Pendulums are highly non-linear problems, which are represented by non-linear differential equations. Some of the pendulum problems are the chaotical system. In this thesis was shown the different type of pendulum problems and shown some application of the Pendulum problems. Investigation of those problems will be done with mathematic based simulation methods.

The diploma thesis should cover:

- discuss some Pendulum models,
- discuss the used mathematical methods,
- show applications of Pendulum models,
- create simulations of the non-linear problems,
- visualize and interpret the results.

Ph.

....
Dr. György Eigner
acting head of institute

Term of limitation: **15th of May, 2023.**

The diploma thesis is adequate and can be submitted:

....
external supervisor

....
internal supervisor

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Annex 7

STUDENT'S DECLARATION

I the undersigned student hereby declare that this degree project / thesis is the result of my own work; I have disclosed the references and tools used in an identifiable manner. The results indicated in my degree project / thesis completed may be used by the university and the institution announcing the task for their own purposes free of charge.

Dated: Budapest, 27th of April 2022

....
student's signature

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John von Neumann Faculty of Informatics

Log of consultations

Name of the student: Sovetova Kymbat

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Title of the thesis work:

Investigating Pendulum-based Non-Linear Problems

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Please use BLOCK CAPITAL LETTERS to fill in this form!

Ocas-sion	Date	Topics discussed	Signature
1.	20.11	Discuss the toolset	Kósi Kriszt.
2.	27.11	Discuss the Litriture	Kósi Kri.
3.	03.12	Discuss the Mathematical tools	Kósi Kri.
4.	10.12	Discuss the Mathematical Models	Kósi Kr

This log should be signed on each consultation (by a supervisor), 4 times altogether.

The student fulfilled the requirements of the Thesis work subject, I allow his/her participation on the presentation/defence[1].

Kósi Kr.
....

Internal supervisor

Budapest, 2020. Dec. 11.

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[1] Underline the appropriate one.

CONSULTATION LOG

Student's name: Kymbat Sovetova

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Degree project / thesis[1] title in Hungarian:

Nemlineáris Inga alapú problémák vizsgálata.

Degree project / thesis[2] title in English:

Investigating Pendulum-based Non-Linear Problems

Institutional supervisor: Dr Kósi Krisztián

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Occ.	Date	Content	Signature
1.	14.02.2022	Simulation code for Single-Inverted Pendulum	Kósi Kriszt
2.	14.03.2022	Simulation code for Double-Inverted Pendulum	Kósi Kriszt
3.	28.03.2022	Simulation code for Simple Triple Pendulum	Kósi Kriszt
4.	04.04.2022	Applications for simulated models	Kósi Kriszt

The Consultation log is required to be countersigned by either supervisor on a total of 4 occasions of consultation.

The student has complied with the requirements of the subject titled Degree project I / Degree project II (BSc) or Thesis 1 / Thesis 2 / Thesis 3 / Thesis 4[3], and is allowed to proceed for reporting / defense[4].

Dated: Budapest, 27th of April 2022

Kósi Kriszt
....
institutional supervisor

[1] Underline as appropriate
[2] Underline as appropriate
[3] Underline as appropriate
[4] Underline as appropriate

Abstract

Modeling of pendulums can be applied in control theory. Moreover, for example, today, the pendulum modelling is implemented in various spheres, like engineering and control theory. The brightest example is a Segway because it implies self-balancing technology [1]. It is known that self-balancing construction bases on single inverted pendulum construction and its motion, which is going to be one of the objects which I have chosen for simulation.

Pendulums by their nature have quite simple construction which makes them perfect object for research. In brief simple pendulums can be under stabilization via oscillation. Numerical simulations are practiced in this thesis, as well as consideration stable and unstable domains which are crucial for studying of chaotic motion for example of double pendulum. For future research it would be fruitful to consider The capability of widening the spectrum of magnitude and periodicity of the single pendulum setup would be profitable for future study.

Contents

Chapter 1

Introduction

To start with, current work is mainly concentrated on modeling and simulating of triple inverted pendulum. Firstly, researches which were done on this topic were considered through making qualitive investigation. As examples of covered researches single and double pendulums' modeling were provided in this work. As method Robust Fixed Points Transformations was taken. For more clear illustration of given approach corresponding theorems will be provided. In this work models of single simple, single inverted, double simple, double inverted and triple pendulums are presented. The main purpose of this research is to model and simulate inverted pendulums for further application of received results in areas like control algorithms, control theory, dynamics and transportation gadgets. Current work also includes background data concerning methods which are going to be actively used in practical part. I have just gathered main definitions and formulas without proves, however proves can be found in sources which are shown in bibliography. Theoretical parts will cover topics essential for control and modelling in practical part. The inverted pendulum is a common control theory and kinematics problem that is often used to evaluate control strategies. The inverted pendulum model is regarded to be the richest among the typical robot benchmarks, despite its extremely basic structure, which is suited for designing a real-time implementation. It not only can discuss a wide range of technological issues, but also certain biological examples. The inverted pendulum has a number of mathematical simulations and experimental designs that make it a useful teaching and research tool. To present a full picture of historical, current, and difficult advancements based on the idea of inverted pendulum stabilization, a total of 300 references in the literature dating back to Galileo's initial tests in 1602 have been collated. To start with, current work is mainly concentrated on modeling and simulating of triple inverted pendulum. Firstly, researches which weredone on this topic were considered through making qualitive investigation. As examples of covered researches single and double pendulum's modeling were provided in this work. As method Robust Fixed Points Transformations was taken. For more clear illustration of given approach corresponding theorems were provided. In this workmodels of single, double and triple pendulums are presented. The main purpose of this research is to model and simulate inverted pendulums forfurther application of received results in areas like control algorithms,control theory, dynamics and transportation gadgets. Wei Chen in her paper studied the application of fuzzy control method to investigate balance issue of triple inverted pendulum. The outcome researcher got, illustrated used technique to be useful in reaching stability of chosen system of triple inverted pendulum. In that paper the linear model was used, so basing on feedback parameters the control parameters were projected. Outcomes of simulation in that paper demonstrated usability of that method for triple inverted pendulum, moreover it can be employed for different variables and nonlinear control cases [2]. In my thesis, I used the formula of a simple triple pendulum as a model. Another research in this field was done by X.Xiong and Z. Wan, where new technique was practiced successfully for simulation of inverted pendulum via implementation of LDR controller. So, in details they used swarm optimization technique and outcome of simulation illustrated that applied technique for matrixes Q and R gave satisfactory outcomes [3]. In practical part of current thesis, from my side I took as a model single inverted pendulum and its application in the system of unicycle. Next paper was devoted for simulation of inverted pendulum on the cart. Four controllers (PID, TC, state feedback, and fuzzy) were figuring in this practice and experiments demonstrated that they had satisfactory productivity. There also was emphasized satisfactory robustness in condition that angle was 0.3 radians [4]. In my paper for comparison, I considered double inverted pendulum's simulation on the movable vehicle. The last but the least work I consider as background research is work of N. Muskinja and B. Researchers wanted to stabilize inverted pendulum, after moving it from initial position with participation of cart. For such simulation various ways of solving of given issues were under consideration and different trials were practiced as well. Again, as in Wei Chen's research, that paper also proved that fuzzy control is quite beneficial in nonlinear practices [5]. The main motivation of my research is making control and simulation of different representatives of single pendulum construction, in order to implement taken results for future research about constructions, which include in their systems pendulum structure. Therefore, in paper I am willing to demonstrate simulation of simple and inverted versions of pendulum, moreover the constructions which are going to be considered include single, double and triple pendulums to be simulated. But control part is going to be devoted only for double pendulum as an example. My scope of research has several stages which I am going through. The main issue was to find appropriate model, which would clearly define equations of motions, so I can simulate their movement and receive good graphical representation of it. However, after theoretical investigation I could choose models, which will be applied later in the practical part of current research. Definitely, the most essential part is demonstration of applications, the areas where such simulations could take a part in purpose of further developments of those areas. In applications part, you are able to find out the data covering constructions including different kinds of pendulums which were considered in this work. From this work, you can see how the system functions and some theoretical data also will be presented in first chapters, because they take a part in simulation and control process. So, such descriptions provided in this work, were possible to be illustrated after certain calculations in simulation part which included derivations of equations. So, I hope that received outputs will be useful in outer environment. After simulation process the relevant comparisons are planned as well. It is obvious that some initial conditions change due to complexness of represented constructions. But, the primary purpose of this work Is still simulation, where I calculate mathematically all data and predict via practical part of this work movements of them in case of represented initial values.

Chapter 2

Methods

2.1 Ordinary Differential Equations

Differential equation is an equation with derivative more than one or one dependent variables, so with considerations of those dependent variables it is called differential equation. Differential equations can be divided due to their linearity, type or order. For instance, there is an ordinary differential equation with derivatives regarding to one independent variable, while partial differential equation has partial derivative regarding numerous independent variables. It is known, that if we get highest derivative and solve it for differential equation with m-th order and collect others to another part of equation, we will obtain the normal form of equation. Also, there are linear and nonlinear equations. We call ODE nonlinear if it is not linear in its derivatives, while it is called linear in the event that a linear function exists F in $y, y', y'', ..., y^{(n)}$. In this thesis I am willing to apply nonlinear ones [6].

In case if variables rank within equation, we call it functional. Replacements can be considered as a key for solving such equation if it is elementary. There is a method to solve several kinds of functional equations, which is defining special values and $f(n)$ for $n \in N_0$ and usage of density. Ordinary Differential Equations (ODE) came from functional equations, which is a special case [7].

ODE has two main branches, which are linear and nonlinear. If equation can be illustrated as , which I took from source [8]

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y + p_0(x)y = f(x), \tag{2.1}$$

$$p_n(x) \neq 0, p_j(x), j = 1, 2, ..., n. \tag{2.2}$$

There are homogenous and nonhomogeneous equations. Basically, identification of the fact whether it is homogeneous or not depends on $f(x)$. If it is 0 the equation is homogeneous, if not then it is nonhomogeneous. Another significant moment is an order of differential equations, which can be identified via largest derivative's order of dependent variable.

We have initial value with conditions where m order followed with $y(x_0) = y_0, y'(x_0) = y_1, ..., y^{(m-1)}(x_0) = y_{m-1}$. We can call differential equation autonomous if it is possible to write it as $y^{(n)} = f(y, y', ..., y^{(n-1)})$, which was given in the source[8]. There is a way to solve autonomous first order equation through drawing supposed solution via calculus. But it is not always solvable using analytical way of solving [8].

2.2 Euler-Lagrange equation

First, one of the basic notions required to be aware about is ability to solve minimization exercises. In this passage I am going to discuss the classical method of analytical approach, but there are also contemporary direct ways. There is a variety of variational cases, where the quantity is continuously differentiable scalar function which is the easiest one. In such case we have dependency of function from its first derivative. In order to undermine the objective function, it is necessary to build an acceptable $y = u(x) \in C^1[a, b]$ function, which I obtained from the source [9] and modified somewhat.

$$J[u] = \int_a^b \mathcal{L}(x, u, u')dx \tag{2.3}$$

Here, Langrangian is a function that has to be integrated for variational exercise. It was called because of Joseph-Louis Lagrange, who was Italian astronomer and mathematician. Scalars x, p and u illustrate u' derivative. It is also believed that Langrangian is a plain function. To remarkably indicate this minimization function there is a need to stick appropriate boundary conditions at interval's endpoints. In the beginning, we center our attention on values of the functions which have to be fixed, which was given in source [9]

$$u(a) = \alpha, u(b) = \beta \tag{2.4}$$

at these points.

Next, critical points have to be interpreted by definition. Points to consider originally, they were defined as topical minimizers of an appropriate objective function found on a finite-dimensional vector space, where v represents for variation and u stands for function. Variational derivative is a gradient operator and can be signed as δJ. Turning back to our fixed point, as a next step finding derivative of function is considered.

Which was given in the source[9]

In brief, Euler-Lagrange equation assists in decreasing the line distance. State of critical function is proven by fulfilment of Euler-Lagrange equation and boundary conditions.

However, there is no assurance that it is a minimizer. Critical point theoretically maximizes the variational problem. Straight line cannot maximize distance and plays role of minimizer for minimum distance exercise.

However, there are certain limitations regarding one tangle agent which is infinite-dimensional function space's convergence is significantly more appropriate fine compared with Euclidean space. In common, answers for Euler–Lagrange boundary value problems are critical functions[10].

Let us derive the equation in (2.7):

$$J[u]=\int_a^b \mathcal{L}(x,u,u')dx \tag{2.5}$$

is stationary Boundary conditions: $y(a)=y_1$,$y(b)=y_2$. Suppose $u(x)$ makes J stationary and satisfies the above boundary conditions. Let us introduce a function $\eta(x)$, $\eta(a)=\eta(b)=0$. We define $\overline{u}(x)=u(x)+\varepsilon\eta(x)$ satisfies same boundary conditions as u. $\overline{u}$ represents a family of curves.

We need to find the particular $\overline{u}(x)$ which makes $J[u]=\int_a^b \mathcal{L}(x,\overline{u},\overline{u}')$ stationary. Since J depends only on ε, to make J stationary, set:

$$\begin{gathered}\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}=0,\text{when}\frac{dJ}{d\varepsilon}_{\varepsilon=0}=0=\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\int_a^b \mathcal{L}(x,\overline{u},\overline{u}')dx=0\Rightarrow\\ \Rightarrow\int_a^b\left.\frac{\partial}{\partial\varepsilon}\mathcal{L}(x,\overline{u},\overline{u}')\right|_{\varepsilon=0}\Rightarrow\int_a^b\left.\left[\frac{\partial\mathcal{L}}{\partial\overline{u}}\frac{\partial\overline{u}}{\partial\varepsilon}+\frac{\partial\mathcal{L}}{\partial\overline{u}'}\frac{\partial\overline{u}'}{\partial\varepsilon}\right]\right|_{\varepsilon=0}dx=0\Rightarrow\\ \Rightarrow\int_a^b\left.\left[\frac{\partial\mathcal{L}}{\partial u}\eta+\frac{\partial\mathcal{L}}{\partial\overline{\overline{u}'}}\frac{\partial\overline{u}'}{\partial\varepsilon}\right]\right|_{\varepsilon=0}dx=0\Rightarrow\int_a^b\left.\left[\frac{\partial\mathcal{L}}{\partial\overline{u}}\eta+\frac{\partial\mathcal{L}}{\partial\overline{u}'}\eta'\right]\right|_{\varepsilon=0}dx=0\end{gathered} \tag{2.6}$$

Integrating by parts:

$$\begin{gathered}\int_a^b\frac{\partial\mathcal{L}}{\partial\overline{u}'}\eta dx=\frac{\partial\mathcal{L}}{\partial\overline{u}'}\int_a^b\eta' dx-\int_a^b\left(\int\eta'\right)\frac{d}{dx}\left[\frac{\partial\mathcal{L}}{\partial\overline{u}'}\right]dx=\\ =\frac{\partial\mathcal{L}}{\partial\overline{u}'}[\eta]_a^b-\int_a^b\eta\frac{d}{dx}\left[\frac{\partial\mathcal{L}}{\partial\overline{u}'}\right]dx\Rightarrow\int_a^b\left.\left[\frac{\partial\mathcal{L}}{\overline{u}'}\eta-\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\overline{u}'}\eta\right)\right]\right|_{\varepsilon=0}dx=0\Rightarrow\\ \Rightarrow\int_a^b\left[\frac{\partial\mathcal{L}}{\partial\overline{u}}-\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial u'}\right)\right]\eta dx=0\end{gathered} \tag{2.7}$$

Thus, $\frac{\partial\mathcal{L}}{\partial u}-\frac{d}{dx}(\frac{\partial\mathcal{L}}{\partial u'})=0$. If $u(x)$ is an external of J, then $u(x)$ must satisfy this differential equation.

If the functional J in (2.7) reaches an extremum on some function, then the ordinary differential equation:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial u'}\right)-\frac{\partial\mathcal{L}}{\partial u}=Q \tag{2.8}$$

which is called the Euler-Lagrange equation, must be satisfied for it. This type of form of this equation is commonly used in mathematical modeling systems. Q is some external force that can be equal to zero.

2.3 Euler integral

In represented thesis I am willing to practice the numerical method for solving ODE problems which is Euler's integral which was given in source[9]

$$\frac{dy}{dx} = f(x,y), y(0) = y_0 \tag{2.9}$$

However, we can apply it for first ODE here, certainly there a are exact ways to use it while solving higher order ODEs. But let's discuss how to represent equation in shown state via consideration of example, which was given in the source[9]

Example 1

Rewrite

$$\frac{dy}{dx} + 4y = 1.6e^{-x}, y(0) = 9dx \tag{2.10}$$

in

$$\frac{dy}{dx} = f(x,y), y(0) = y_0 \tag{2.11}$$

form.

Solution

$$\frac{dy}{dx} + 4y = 1.6e^{-x}, y(0) = 9dx \tag{2.12}$$

$$\frac{dy}{dx} = 1.6e^{-x} - 4y, y(0) = 9dx \tag{2.13}$$

$$f(x,y) = 1.6e^{-x} - 4y \tag{2.14}$$

Next step is derivation, so if we are given $x = 0$ of $y = y_0, x = 0 = x_0$

On this stage y slope is known, regarding x, $f(x,y)$, at $x = x_0$, slope = $f(x_0, y_0)$. x_0 and y_0 are $y(x_0) = y_0$ from initial conditions.

As it is illustrated in Figure 2.1, the slope at $x = x_0$ is $\frac{Rise}{Run} = \frac{y_1 - y_0}{x_1 - x_0} = +f(x_0, y_0)$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) \tag{2.15}$$

which was given in the source [9]

$x_1 - x_0$ is step size h, $y_1 = y_0 + f(x_0, y_0)$, h is received

Now we apply y_1 value for finding y_2, so it is foretold value at x_2, we know

$$\begin{aligned} y_2 &= y_1 + f(x_1, y_1)h \\ x_2 &= x_1 + h \end{aligned} \tag{2.16}$$

Looking at demonstrated equations, in case if we have $y = y_i$ at x_i value, next as in the source[9]

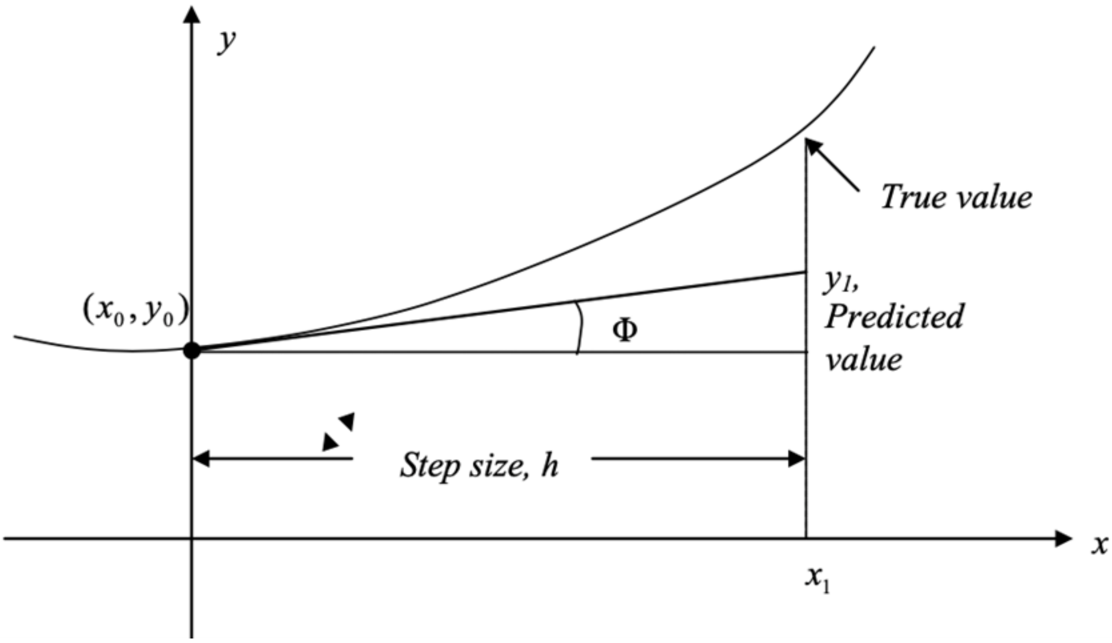

Figure 2.1: First step of Euler's approach is plotted in the graph [9]

$$y_{i+1} = y_i + f(x_i y_i)h \tag{2.17}$$

Figure 2.2 depicts the Euler's method

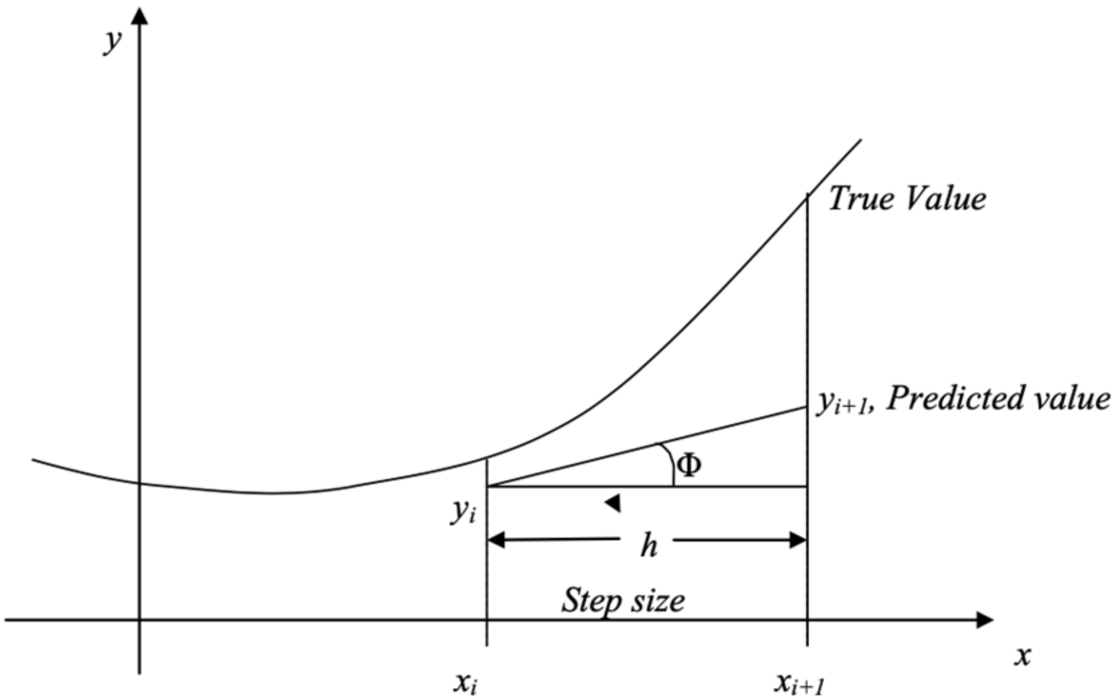

Figure 2.2: Euler's approach is depicted visually [9]

Here is another issue, whether it is possible to solve definite integral via this method. If there is a need to solve integral of $f(x)$ function, which was given in the source[9].

$$I = \int_a^b f(x)dx \tag{2.18}$$

If we want to find solution via solving it as ODE, two basic theorems would definitely be applied. The first claims that if f is a continuous function in $[a, b]$, so F is its antiderivative, which leads to

$$\int_a^b f(x)dx = F(b) - F(a) \tag{2.19}$$

In this situatuation the theorem asserts that, in the situation of f - a continuous function inside the open interval D, a position within the interval D, and in that case, which was described in the source[9], the following result holds

$$F(x) = \int_a^b f(t)dt \tag{2.20}$$

Next, at each position in D, $F'(x)$ equals $f(x)$

$F'(x) = f(x)$ at every point in D

It is also feasible to formulate $\int_a^b f(x)dx$ as a solution to ODE in the following order: $\frac{dy}{dx} = f(x), y(a) = 0$, through which we obtain the value obtained from [9].

$$\int_a^b f(x)dx \tag{2.21}$$

2.4 Metric Spaces

There is a basic notion regarding real analysis on R is possible to be implemented to A set after determining of $d(x, y)$ for $x, y \in A$. R is appropriate example of metrics space together with length function $d(x, y) = |x - y|$. Lets turn to shorth definition to define metric space. First of all let's imagine we have non-empty set and call it N, so there is function which fulfil certain statements in order to be called metric on N. First property explains $d(x, y) = 0$ if and only if $x = y$, following property is $d(x, y) = d(x, y)$ for $x, y \in N$, next definition says $d(x, y) \leq d(x, y)$ for $x, y, z \in N$. In brief we call metric space the couple $(N, d), d$ is metric on N[11].If $A = R$ length will be $d(x, y) = |x - y|$.

2.4.1 Open and closed balls

So we have decided that (N, d) is metric space and length is determined by d_M. With its center at $a \in N$and r radius we have an open ball which is the set:

$$B(a, r) = \{x \in X | d(x, a) < r\} \tag{2.22}$$

Closed ball is

$$B[a, r] = \{x \in X | d(x, a) \leq r\} \tag{2.23}$$

with center at N and r radius[12].

2.4.2 Open and closed sets

As it is known from calculus that open set is the set that did not involve its boundaries. There is a main idea that once we choose a unit $x \in A$, there is a need to define an open, as a result, it is totally implicated on the same set A, allowing the set to be opened.

The term "open balls" refers to an acceptable case of the term "open set," and thus A set $A \subseteq X$ is open $\iff$ $A = A$. Assume there is $N = (X, d)$, $x \in X, \forall r > 0$ metric space, so there is a set $A = B(x, r)$ which is open set, after there would be closed set. We have a closure set; for example, once there is $N = (X, d)$ and $A = X$, then there would be a closure dot dot $xXofA$ that follows the following rule: When r is greater than zero, we obtain $\forall r > 0$ we have $B(x, r) \cap A \neq \emptyset$. It does not really signify how near you are to x; there will always be a dot in A that would be close to x no matter where you are. As a consequence, A's closure refers to the collection of every closer point of A.

$A \subseteq X$ is closed $\iff$ $A = A$, so A for any set is betwixt closure and interior[13].

2.4.3 Convergence

To start with lets define metric space as (X, d) we have a sequence in X. Given a sequence, we say it converges to $x \in X$, in case if there is a $y \in X$ so that $y_n \to y$. Also we say that subset A is closed in case if each convergent sequence dot in A converges to dot in A[14].

In case if terms of sequence get near to one another as sequence evolves, we call it as Cauchy sequence. In short $\{a_n\}_{n=0}^{\infty}$ is called Cauchy sequence only in case if for each $\epsilon > 0$, we have $N > 0$, so $n, m > N$ with $|a_n - a_m| < \epsilon$. In brief Cauchy sequences are helpful due to the fact that they increase the concept about complete field, and in such field each Cauchy converges.

So we discussed the definition which defines the Cauchy sequence. However it is more cunning to show that the sequence is not Cauchy sequence. Fore this purpose we need $N > 0$ so for every $\epsilon > 0$, we have $m, n > N$ with $|a_n - a_m| < \epsilon$. It does not matter how close or far the terms in sequence, we do not have any security that they will be near.

If there is a need to define if the sequence is Cauchy, there is a sense to implement intuition of the conditions increasing near together for making decision about the fact is it Cauchy or not, after confirm it via definition. To sum up we call sequence Cauchy if for each $\epsilon > 0$, we have $N > 0$, so $m, n > n \to d(a_m, a_n) < \epsilon$.

Convergent sequence and Cauchy sequence are closely connected. Each convergent sequence is also Cauchy, due to if $a_n \to x$, after $|a_n - a_m| \leq |a_m - x| + |x - a_n|$ which was given in the source[15].

2.5 Contraction mapping

If we consider (L, d) to be a metric space, map $f : K \to K$ can be referred to as contraction mapping of (L, d) in the case of R $0 \leq a < 1$, also known as the construction's constant. If any $x, y \in K$ then $d(f(x), f(y)) \leq ad(x, y)$ which I took from the source[16]. It is feasible to have a loosely defined contraction mapping that is uniformly continuous[16].

Normed space

Lets define a pair $(V, \| \cdot \|)$, and determine V as vector space, while the norm is $\| \cdot \| : V \to R$ on V. Moreover certain terms have to be fulfilled to call this pair normed space. Terms:

1. $\|x\| > 0$ if $x \neq 0$,
2. $\|x\| = 0$ if and only if $x = 0$,
3. $\|\alpha x\| = \|\alpha\| \cdot \|x\|$ for every $\alpha \in R$.
4. $\|x + y\| \leq \|x\| + \|y\|$. which I took from the source[16].

Any $(V, \| \cdot \|)$ is referenced to as a metric space, and the metric $d(x, y) = x - y$ on V is denoted by the symbol $d(x, y)$[17].

2.6 Banach space

Due to simple definition of what Banach space is, every mathematician student knows how to interpret this concept. Firstly, there is a normed linear space where we have a function $\| \cdot \|$, which fulfils, which was given in the source[18]

$$\|x + y\| \leq \|x\| + \|y\|, \tag{2.24}$$

$$|\lambda x\| = |\lambda| \|x\|, \tag{2.25}$$

$$\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0, \tag{2.26}$$

There is unconstrained metric in a normed space which was given in the source [18]

$$d(x, y) = \|x - y\|. \tag{2.27}$$

In case of receiving of metric space through such option it can be considered as complete. Every Cauchy sequence now has to converge in order to identify a normed space as a Banach space. In brief, Banach spaces is a topic which is one of the most essential section of university program for mathematicians. There are theorems such as uniform boundedness principle and open mapping principle which are applied in proving theorems in classical analysis. Lets consider one of theorems. For instance,

Theorem. Each infinite dimensional Banach space which is possible to dissociate is homeomorphic to the Hilbert space l_2[19]. Basically, there are four various methods of how we can consider a Banach space. First one is a metric space, which took a part in my investigation and is special case of normed space. Others are uniform, linear topological and a Banach space. The primary object of Banach space topic is its division into isometries and isomorphism. It is known that isomorphic cases are more complex, because of isometric invariants are known more about. Also we cannot say for sure how to illustrate Banach space from the point of its prime factors. However, each fractional response is great move in direction of a structure theory [18].Because it is complete and has Cauchy convergence, we have concluded that X is a Banach space. Another requirement for recognizing X as Banach space is that the $f_{n n\in N}$ is Cauchy sequence in $X \implies \exists X$ is Cauchy in such that $f_n \to$ the Banach space is complete and Cauchy convergence, as indicated in[18]. It is now necessary for every Cauchy sequence to converge in order to recognize a normed space as a Banach space.

Lets imagine we have a certain sequence x_i with elements in metric space (X, d), so $\varepsilon > 0$ and we have a number N, so $d(x_n, x_m) < \varepsilon, \forall m, n \geq N$.

Also, there is such notion as Cauchy condition, so every time Cauchy sequence is obligatorily a convergent sequence, but convergence is not obligatorily genuine. If it is said that there is limit for Cauchy sequence for metric space property it is called complete.

Convergence of series $\sum_{k=1}^{\infty}, x_k \in X$ is considered as convergent in case if there is a limit S of $\lim\limits_{n\to\infty} \|S - \sum_{k=1}^{n} x_k\| = 0$. It is convergent in case if $\sum_{k=1}^{\infty} \|x_k\| < \infty$, it is also can be considered as convergent unconditionally, in case of convergence in terms when conditions are swifty reformed. Absolute unconditional convergences are equal if series is in condition of finite-dimensional space.

In convex Banach space the unconditional convergence of $\sum x_k$ means $\sum_{k=1}^{\infty} \delta(\|x_k\|) < \infty$

$\sum x_k$ is faintly Cauchy in case $\sum |f(x_k)|$ converges for every $f \in X^*$. If there is no any isomorphic subspace to c_0(which is the space of numerical sequences which converge to zero with the norm $\|x\| = \max_n |\xi_n|$), then every weak Cauchy series in X converges[18].

2.7 Banach Fixed Point Iteration

For metric spaces there is a need to ensure originality of a fixed point for definite class of mapping. Banach's Fixed Point Theorem has functional way for identifying this point. In

1992 mathematician from Poland, Stefan Banach set given demand.

If (X, d) is a non-empty complete metric space. $L : X \to X$ is contraction mapping to X which is a number, which was given in source [20] $0 \leq a < 1$, so $d(L_x, L_y) \leq ad(x, y)$

For every x, y from X. After there is precisely single fixed point x^* in map T from X . a is frequently cited as compression ratio. In case if $a = 1$ means not constructing mapping, which is against the theorem. In common, in order to demonstrate subsistence and originality of answers for specific boundary value problem's classes Banach's theorem is applied in theory of differential equations. Another application of this theory can be found in theory of integral equation, for illustration of presence and originality of answer to second kind inhomogeneous linear Fredholm integral equation and several kinds of nonlinear integral equations. Also, numerical methods broadly practice this theorem, for example Newton's and Jacobi methods can be examined from position of Banach's theorem[20].

2.8 Robust Fixed-Point Transformation

2.8.1 Single Variable Systems

Robust fixed point transformation in this chapter is going to considered for single and multiple variable systems. First of all lets consider unique function given for single extent freedom systems, following equation were taken from the source [21].

$$r_{n+1} = (K_c + r_n)\left[1 + B_c tanh\left(A_c\left(f(r_n) - r_{n+1}^{Des}\right)\right)\right] - K_c \tag{2.28}$$

Also there is calculation regarding tracking error relaxation which were kinetically preassigned. It can be found in the estimation r_{n+1}^{Des}. There are control parameters which describe mentioned relaxation, which are A_c, K_c, B_c. There is a resolution for an adaptive control problem in form of $r_n = r_*$, $f(r_*) = r_{n+1}^{Des}$ outputs $r_{n+1} = r_*$, so it is also our fixed point. So we can insure $\left|\frac{\partial r_{n+1}}{\partial r_n}\right| < 1$, by establishing A_c condition as a parameter, which is offered by equation taken from [21]:

$$\left.\frac{\partial r_{n+1}}{\partial r_n}\right|_{r_*} \approx 1 + K_c B_c \left.\frac{dtanh(x)}{dx}\right|_0 A_c \left.\frac{df}{dr}\right|_{r_*} \tag{2.29}$$

$\left.\frac{x}{x}\right|_{x=0}$ So here, it is tolerable state, which we get in case of implementation of smple integral estimation. So we do it for single variable differentiable function which I got from [21]:

$$f(b) - f(a) = \int_a^b \frac{df}{d\xi} d\xi \tag{2.30}$$

therefore

$$|f(b) - f(a)| \leq \int_a^b \frac{df}{d\xi} d\xi \leq K\,|b-a| \tag{2.31}$$

if $\left|\frac{df}{d\xi}\right| \leq K$ where $0 \leq K \leq 1$

So I case of providing of fallowness of mentioned request will result to and contractive map and convergence.

2.8.2 Multiple Variable Systems

Now lets turn to multiple variable systems (MIMO), so for high degree freedom we need conversion via applying of differentiable function $F : \mathbb{R} \to \mathbb{R}$, with $F(x_*) = x_*$ fixed point. There is a convergence for second order scheme while control stage, which was given in [21].

$$\begin{aligned} r(i) &\stackrel{\text{def}}{=} \ddot{q}^{Def}(i) \in \mathbb{R}^n \\ r(i+1) &\stackrel{\text{def}}{=} \left[F(A_c||h(i)|| + x_*) - x_*\right] e(i) + r(i) \end{aligned} \tag{2.32}$$

There is return error determined as $h(i) \stackrel{\text{def}}{=} f(r(i)) - r^{Des}(i+1)$

Frobenius norm vector is $e(i) \stackrel{\text{def}}{=} \frac{h(i)}{||h(i)||}$. Also we have adaptive control parameter,which is $A_c \in \mathbb{R}$. Resolution of control problem is $r(i) = r_*$ where $h(i) = 0$ and $r(i+1)r_*$. And it is considered as a fixed point for 14.4. In neighborhood of solution for transformation Taylor's 1st order series approximation of $F(\cdot)$ at x for moderate $A_c||h(i)||$ leads to, which was taken from source [21].

$$F(A_c||h(i)|| + x_*) - x_* \approx F(x_*) + F'(x_*)A_c||h(i)|| - x_* = F'(x_*)A_c||h(i)|| \tag{2.33}$$

On $f(\cdot)$ at r_* is, which was taken from [21]

$$\begin{aligned} f(r) - r^{Des} = f(r_* + r - r_*) - r^{Des} \approx f(r_*) + \left(\frac{\partial f}{\partial r}\right)(r - r_*) - r^{Des} \approx \\ \approx \left(\frac{\partial f}{\partial r}\right)(r - r_*) \end{aligned} \tag{2.34}$$

So in pair they results in approximation such as,(equation was taken from [21])

$$r(i+1) - r_* \approx \left[I + A_c F'(x_*) \left(\frac{\partial f}{\partial r}\right)_{r_*}\right](r(i) - r_*) \tag{2.35}$$

Iteration transforms to fixed point in case of contractiveness of the matrix. So if assign matrix is contractive, then as equation 14.7 illustrates that Cauchy sequence is convergent. Control's phase space is an initial derivative of the state condition, more commonly saying y is the derivative and x is position. We have velocity in form of first derivative, while acceleration is the second derivative. So as control's signal we have u which is the result of the Inverse Model[21].

2.8.3 Control Loop

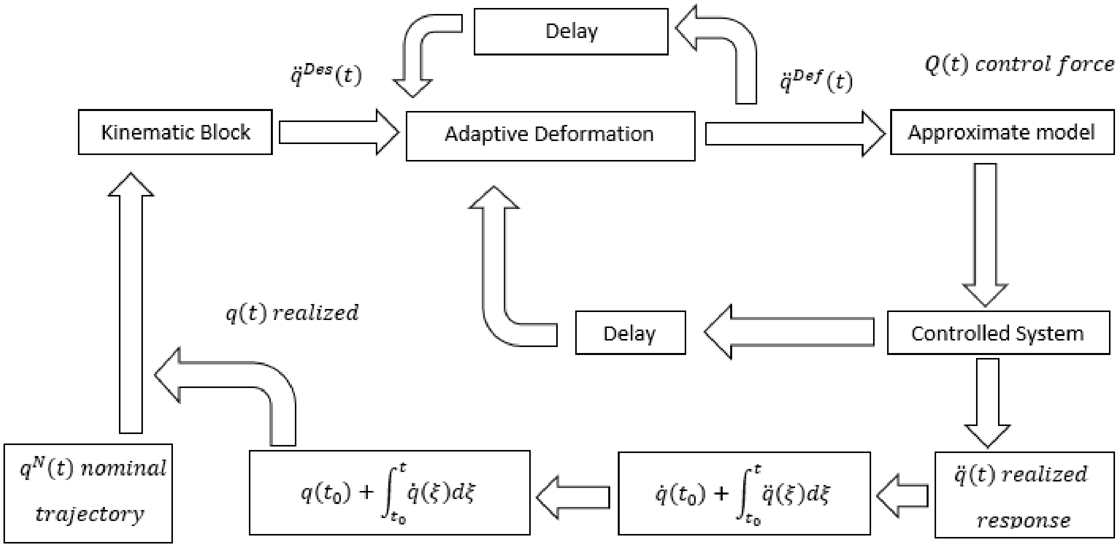

Figure 2.3: Structure of RFPT

Figure (2.3) depicts the approach proposed for converting the control job it in to a Fixed Point Problem, which is detailed in further detail below. The preceding is true for the nth order system, and this order was employed in the creation of the Kinematic Block, among other things. A 2nd order dynamical physical system, where the 2nd order time-derivative of a state variable, $\ddot{q}^{Des}$, may be instantly transformed by the control force Q, is described as follows: Using the "Kinematic Block" it is possible to design a desired q on a purely kinematic basis that will reduce tracking error to zero (i.e. $q^N(t) - q(t) \to 0$ as $t \to \infty$) provided that it is effectively came to the realization $q^N(t)$ denotes the "nominal trajectory" to be tracked, and $q(t)$ denotes the realized one.

Using a deformation function in the block "Adaptive Deformation" relying on the distorted signal of the controller in the prior control cycle $\ddot{q}^{Def}(n-1)$, on the predicted values for it $\ddot{q}^{Def}(n-1)$, and on the genuinely desired response $\ddot{q}^{Des}(n)$, the actually deformed value $\ddot{q}^{Def}(n)$ is introduced into the obtainable estimated dynamic model of the controlled system to achieve the real control force $Q(n)$. Because of this, the "Realized Response" for $Q(n)$ is defined by the "Response Function," which is written as $\ddot{q}(n) = f(\ddot{q}^{Def}(n), ...)$ where "..." indicates that the state variables q and $\dot{q}$ are present, and because they cannot vary as quickly as the spontaneously variable q, they are approximated as "parameters" in the following considerations. The "Response Function" The controller observes from the system's behavior during the previous control cycle, if this scheme converges, and this information is used to generate the real control force. On the basis of "Banach's Fixed Point Theorem" it is possible to obtain convergence.

In the stage of kinematic block, we just calculate it with $(\Lambda + \frac{d}{dt})^{n+1} e_{int}$ for PID.

The error is defined as e, and if we find difference of realized and nominal trajectory, our error can be found. After u should be conveyed, so we find control signal in that way. In control loop, as the next step we find response of the system, after which Euler integral is implemented[21].

Chapter 3

Models

3.1 Single-Inverted Pendulum

The single pendulum is a construction with solid rod and hesitant kernel . Rod is able to move loosely about mentioned kernel as it can be seen in Figure 3.1. The mass of the black component is m while the length of the pivot is l. To derivate the equation which represents the movement, Lagrangian is supposed to be applied. In currents work the model of single inverted pendulum is chosen, because the application of it planned to be illustrated in the system of unicycle. First of all lets consider the model, where we have to derive equations, for which knowledge regarding partial derivatives will be essential[22]. As everybody knows, there is initial form of Lagrangian, which was given in source[22]

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = Q \tag{3.1}$$

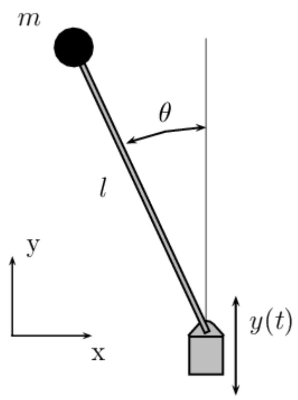

Figure 3.1: Schematic representation of a perfected pendulum in its inverted state [22]

For derivation process we can implement Euler-Lagrange equation, which was given in the source[22].In most cases Q external force is equal to 0 and that leads to:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \tag{3.2}$$

In common, our pendulum is a shaft which has length L, and on the other side there is a stud which gives maintenance to it. There is a motion of mentioned stud which is defined as s(t) = A sinωt [22].

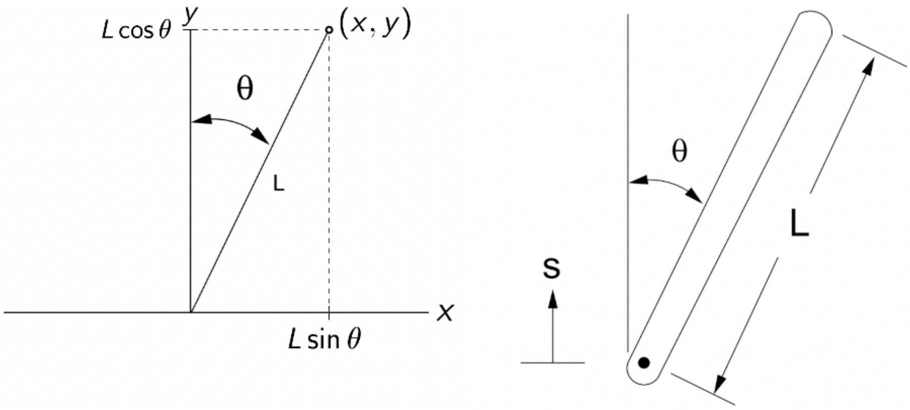

Figure 3.2: Single Inverted Pendulum variables [23]

Analyzing illustrated diagrams, we have, as was done in the source[23]:

$$x = Lsin\theta, \dot{x} = Lcos(\theta)\dot{\theta} \tag{3.3}$$

$$y = Lcos\theta, \dot{y} = -Lsin(\theta)\dot{\theta} \tag{3.4}$$

Due to Langrange determination

$$\mathcal{L} = E_k - E_p \tag{3.5}$$

$$\mathcal{L} = \frac{1}{2}mv^2 - mgy \tag{3.6}$$

In our construction various switches of position exist, which are shown as a vector through velocity, therefore

$$\begin{aligned} v^2 &= \dot{x}^2 + \dot{y}^2 \\ &= L^2\dot{\theta}^2cos^2\theta + L^2\dot{\theta}^2sin^2\theta \\ &= L^2\dot{\theta}^2(cos^2\theta + sin^2\theta) \\ &= L^2\dot{\theta}^2 \end{aligned} \tag{3.7}$$

Interchanging into the equation (3.6) gives us

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 - mgLcos\theta \tag{3.8}$$

Reconsidering equation (3.2)

On this stage we need to find out $\ddot{\theta}$, by calculating two sides of equation (3.8)

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 + mgLsin\theta = mgLsin\theta \tag{3.9}$$

There are two stages:

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\dot{\theta} - 0 = mL^2\dot{\theta} \tag{3.10}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta} \tag{3.11}$$

Finally, we can conclude the solution with consideration of $\ddot{\theta}$, because Euler-Lagrange took a place in two sides of equation (3.2)

$$mL^2\ddot{\theta} = mgLsin\theta \tag{3.12}$$

$$\ddot{\theta} = \frac{g}{L}sin\theta \tag{3.13}$$

Next, changes regarding y are applied after addition of oscillator

$$x = Lsin\theta, \dot{x} = Lcos(\theta)\dot{\theta} \tag{3.14}$$

$$y = Lcos\theta + Asin\omega t, \dot{y} = -Lsin(\theta)\dot{\theta} + A\omega cos\omega t \tag{3.15}$$

Here, we apply Lagrange determination again like in (3.5) and (3.6)

State variation is shown as a vector, so

$$\begin{aligned} &v^2 = \dot{x}^2 + \dot{y}^2 \\ &= L^2\dot{\theta}^2cos^2\theta + L^2\dot{\theta}^2sin^2\theta - 2AL\omega sin\theta\ cos(\omega t)\dot{\theta} + A^2\omega^2cos^2(\omega t) \\ &= L^2\dot{\theta}^2(cos^2\theta + sin^2\theta) - 2AL\omega sin\theta cos(\omega t)\dot{\theta} + A^2\omega^2cos^2(\omega t) \\ &= L^2\dot{\theta}^2 - 2AL\omega sin\theta cos(\omega t)\dot{\theta} + A^2\omega^2cos^2(\omega t) \end{aligned} \tag{3.16}$$

Through replacement for Lagrangian in (3.6), there is

$$\mathcal{L} = \frac{1}{2}mL^2\dot{\theta}^2 - mAL\omega sin\theta cos(\omega t)\dot{\theta} + \frac{1}{2}mA^2\omega^2cos^2(\omega t) - mgLcos\theta - mgAsin(\omega t) \tag{3.17}$$

$$\begin{aligned} &\tfrac{\partial \mathcal{L}}{\partial \theta} = 0 - mAL\omega cos\theta cos(\omega t)\dot{\theta} + 0 + mgLsin\theta - 0 = \\ &= -mAL\omega cos\theta cos(\omega t)\dot{\theta} + mgLsin\theta \end{aligned} \tag{3.18}$$

$$\tfrac{d}{dt}\tfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = mL^2\ddot{\theta} - mAL\omega cos\theta cos(\omega t)\dot{\theta} + mAL\omega^2 sin\theta sin(\omega t) \tag{3.19}$$

$$\begin{aligned}&\mathcal{L} = \tfrac{1}{2}mL^2\dot{\theta^2} - mAL\omega sin\theta cos(\omega t)\dot{\theta}\\&+\tfrac{1}{2}mA^2\omega^2cos^2(\omega t) - mgLcos\theta - mgAsin(\omega t)\end{aligned} \tag{3.20}$$

There are two further moves, we compute in two steps:

$$\begin{aligned}&\tfrac{\partial\mathcal{L}}{\partial\dot{\theta}} = mL^2\dot{\theta} - mAL\omega sin\theta cos(\omega t) + 0 - 0 - 0\\&= mL^2\dot{\theta} - mAL\omega sin\theta cos(\omega t)\end{aligned} \tag{3.21}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = mL^2\ddot{\theta} - mAL\omega cos\theta cos(\omega t)\dot{\theta} + mAL\omega^2 sin\theta sin(\omega t) \tag{3.22}$$

We look for answer with consideration of $\ddot{\theta}$

$$L\ddot{\theta} + A\omega^2 sin\theta sin(\omega t) = gsin\theta \tag{3.23}$$

$$L\ddot{\theta} = gsin\theta - A\omega^2 sin\theta sin(\omega t) \tag{3.24}$$

$$\ddot{\theta} = \frac{1}{L}(g - A\omega^2 sin(\omega t))sin\theta \tag{3.25}$$

Now we calculate for θ

$$\begin{aligned}&mL^2\ddot{\theta} - mAL\omega cos\theta cos(\omega t)\dot{\theta} + mAL\omega^2 sin\theta sin(\omega t)\\&-mAL\omega cos\theta cos(\omega t)\dot{\theta} + mgLsin\theta\end{aligned} \tag{3.26}$$

$$L\ddot{\theta} + A\omega^2 sin\theta sin(\omega t) = gsin\theta \tag{3.27}$$

$$L\ddot{\theta} = gsin\theta - A\omega^2 sin\theta sin(\omega t) \tag{3.28}$$

$$\ddot{\theta} = \frac{1}{L}(g - A\omega^2 sin(\omega t))sin\theta[23] \tag{3.29}$$

Further, I am willing to demonstrate the model for application of single inverted pendulum in the system of unicycle [23].

To start with unicycle is a construction which has one wheel and can be directed bu person who controls it via pushing pedals and rotation motors. If to compare its working scheme, it is different from the one which bike has. Tenet of acting of unicycle reminds inverted pendulum, so controller pushes the pedal and wheel torque steadies the whole unicycle. Here we have linear shifting power which influences to equalizing of the construction.

First of all in our construction there is not only inverted pendulum by itself but running stage under it as well. To illustrate it we apply Lagrangian and y-axis is a vertical route

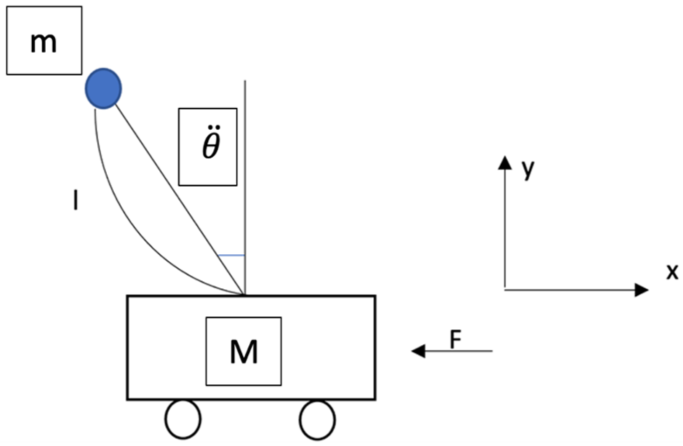

Figure 3.3: Inverted Pendulum variables [24]

and to show horizontal there is a x-axis, l is for length, while angle is taken as $\theta(t)$. Later we will see these denotations in time function [24]. There are two kinetic energies, the one which belongs to pendulum $1/2m{v_2}^2$ and another one devoted for mobile platform $1/2Mv_1^2$, while potential energy is $mglcos\theta$. Here is the Lagrangian of our pendulum taken from the source [24]

$$\mathcal{L} = T - U = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mglcos\theta \tag{3.30}$$

There is x element for v_1, and x, y elements for v_2 . v_2 is described via x and θ
After replacement, we get

$$v_1^2 = x'^2 \tag{3.31}$$

$$v_2^2 = \left(\frac{d}{dt}\left(x + lsin\theta\right)\right)^2 + \left(\frac{d}{dt}(l\cos\theta)\right)^2 = x'^2 + 2x'l\theta'cos\theta + l^2\theta'^2 \tag{3.32}$$

$$\mathcal{L} = \frac{1}{2}(M+m)x'^2 + mlcos\theta x'\theta' + \frac{1}{2}ml^2\theta'^2 - mglcos\theta \tag{3.33}$$

Next equations are possible to be derived. After F's shifting power, there is responsive movement of mobile stage, after which it steadies the movement of pendulum, which was given in the source [24]

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial x'}\right) - \frac{\partial\mathcal{L}}{\partial x} = (M+m)x'' + ml\theta''cos\theta - ml\theta'^2sin\theta = F \tag{3.34}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\theta'}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = ml(-gsin\theta + x''cos\theta + l\theta'') = N = 0 \tag{3.35}$$

That is why, simplistic pendulum equation can be obtained.

$$l\left[1-\left(\frac{m}{M+m}\right)cos^2\theta\right]\theta''+1\left(\frac{m}{M+m}\right)sin\theta cos\theta\theta'^2-gsin\theta=-F/(M+m) \quad (3.36)$$

In case of known θ'', θ', and θ, respond power, $F(\theta'', \theta', \theta)$, gives opportunity to pendulum in balanced condition. If $F(\theta'', \theta', \theta) = (1+\epsilon)mlcos\theta^2\theta'' - mlsin\theta cos\theta\theta'^2$, we get plain mold like this:

$$l\left[1+\varepsilon\left(\frac{m}{M+m}\right)cos^2\theta\right]\theta''-gsin\theta=0 \quad (3.37)$$

We have $\theta \sim 0, cos\theta \sim 1, sin\theta \sim 0$ and $F(\theta'', \theta', \theta) \sim A\cdot\theta'' - B\cdot\theta\cdot\theta'^2$, A and B are invariables[24]. Because θ' and θ got bigger, respond powers' amplitude F must be larger as well to steady the pendulum.

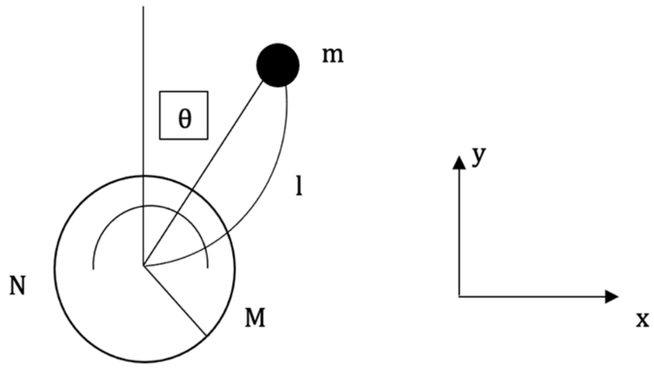

Figure 3.4: Unicycle

In the Figure to, there is a body mass m, frame l, wheel which has M mass, radius a. Here the movement in the wheel plane is considered. Also there is a shifting energy which is $\frac{1}{2}Mv_1^2$ and spinning energy $\frac{1}{2}I\omega_1^2$ for wheel only. Kinetic power of body is of flesh is $\frac{1}{2}mv_2^2$ .While potential power is $mglcos\theta$. Below Lagrangian of simplistic unicycle is shown and which was given in the source [24]

$$\mathcal{L}=T-U=\frac{1}{2}Mv_1^2+\frac{1}{2}I\omega_1^2+\frac{1}{2}mv_2^2-mglcos\theta \quad (3.38)$$

Inertia momenta- $I = Ma^2$

After replacement we get

$$v_1 = x' = a\omega_1 = a\theta' \quad (3.39)$$

$$v_2^2=\left[\frac{d}{dt}(x+lsin\theta)\right]^2+\left[\frac{d}{dt}(lcos\theta)\right]^2 \quad (3.40)$$

$$\mathcal{L} = \frac{1}{2}(2M+m)x'^2 + mlx'\theta' cos\theta + \frac{1}{2}ml^2\theta'^2 - mglcos\theta \tag{3.41}$$

After Lagrangian there is a derivation of following equations. N induces motion response of wheel, so it leads to settling of unicycle.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial x'}\right) - \frac{\partial \mathcal{L}}{\partial x} = (2M+m)x'' + ml\frac{\partial^2}{\partial t^2}(sin\theta) = F = 0 \tag{3.42}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = ml(-gsin\theta + x''cos\theta + l\theta'') = N \tag{3.43}$$

That is why there is simplistic equation

$$\left[ml^2 - \left(\frac{m^2l^2}{2M+m}\right)cos^2\theta\right]\theta'' - \left(\frac{m^2l^2}{2M+m}\right)sin\theta cos\theta\theta'^2 - mglsin\theta = N \tag{3.44}$$

Next due to reaching stabilization we get reduction in form of

$$\left(l + \varepsilon\left(\frac{ml}{2M+m}\right)cos\theta^2\right)\theta'' - gsin\theta = 0 \tag{3.45}$$

In brief, practice of simple models of inverted single pendulum applied in unicycle was illustrated. Commonly saying we get linear power from linear motion of movable platform from electric motor or outer forces. Wheel motion is triggered by pushes of pedals by controller or from rotational motors [24].

3.2 Double Pendulum

Simulation of double pendulum has to consider additional factors. But in brief mass of attached ball and length of the kernel, as well as initial point can be chosen and changed. There can be moderate angles and larger ones. For smaller ones, the pendulum acts as linear double spring, while for larger angles we have non-linear pendulum.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{\partial \mathcal{L}}{\partial \theta} \tag{3.46}$$

We have various particles connected to each other and terms such as acceleration, velocity and location.

x = horizontal position , y = vertical position, θ = angle of pendulum, L = length of kernel In this case y is grows upwards, so the first upper pendulum is denoted as 1, second one as 2. Firstly we show positions of $x1, y1, x2, y2$ in regarding angles θ_1, θ_2, as it was given in source [25].

$$x_1 = L_1 sin\theta_1 \tag{3.47}$$

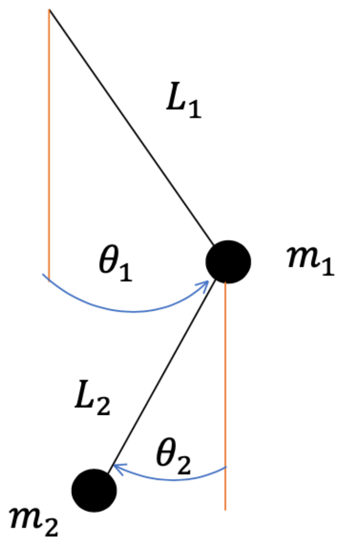

Figure 3.5: Double Pendulum[25]

$$y_1 = -L_1 cos\theta_1 \tag{3.48}$$

$$x_2 = x_1 + L_2 sin\theta_2 \tag{3.49}$$

$$y_2 = y_2 - L_2 cos\theta_2 \tag{3.50}$$

Time of location is taken into account while representing velocity as in the source[25]

$$x_1' = \theta_1' l L_1 cos\theta_1 \tag{3.51}$$

$$y_1' = \theta_1' l L_1 sin\theta_1 \tag{3.52}$$

$$x_2' = x_1' + \theta_2' L_2 cos\theta_2 \tag{3.53}$$

$$y_2' = y_1' + \theta_2' L_2 sin\theta_2 \tag{3.54}$$

Here is the next derivative

$$x_1'' = -\theta_1'^2 L_1 sin\theta_1 + \theta_1'' L_1 cos\theta_1 \tag{3.55}$$

$$y_2'' = \theta_1'^2 L_1 cos\theta_1 + \theta_1'' L_1 sin\theta_1 \tag{3.56}$$

$$x_2'' = x_1'' - \theta_2'^2 L_2 sin\theta_2 + \theta_2'' L_2 cos\theta_2 \tag{3.57}$$

$$y_2'' = y_1'' + \theta_2'^2 L_2 cos\theta_2 + \theta_2'' L_2 sin\theta_2 \tag{3.58}$$

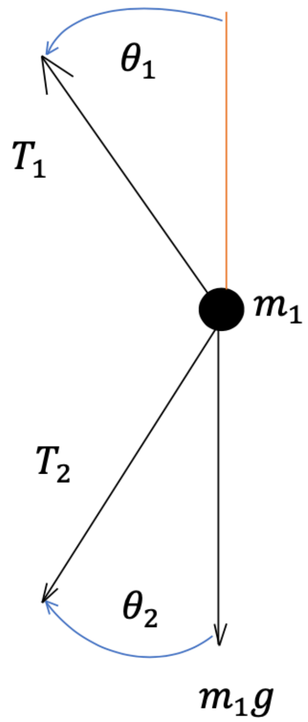

Figure 3.6: Upper mass[25]

We have an illustration with frame scheme to show overhead mass and below variables are determined as: T = tension m = mass g = gravitation invariable

Tensions T1 and T2 are taken for defining upper and lower kernels relatively, while gravity is simply $m1g$. There are vertical power and horizontal, and each has its own equation. Because they are referable without corresponding to each other. Below via application of Newton's law the pure force is shown [24]. $F = ma$

$$m_1 x_1'' = -T_1 sin\theta_1 + T_2 \sin\theta_2 \tag{3.59}$$

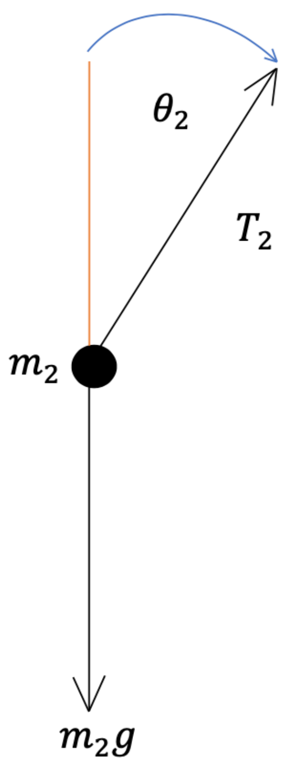

Figure 3.7: Lower mass[25]

$$m_1y_1'' = T_1cos\theta_1 - T_2\cos\theta_2 - m_1g \tag{3.60}$$

Lower pendulum is as it was given in [25]:

$$m_2x_2'' = -T_2sin\theta_2 \tag{3.61}$$

$$m_2y_2'' = T_2cos\theta_2 - m_2g \tag{3.62}$$

θ_1 is positive and θ_2 is negative, due to concord which says anticlockwise angle to be assertive.

Now, the equation of motion part starts.

This part is devoted for detection of for θ_1'', θ_2'' in terms of $\theta_1, \theta_1', \theta_2, \theta_2'$. Firstly equations (3.60), (3.61) are calculated for $T_2sin\theta_2$ and $T_2cos\theta_2$ after which replacement in (3.58) and (3.59) happens, as it was given in [25].

$$m_1x_1'' = -T_1sin\theta_1 - m_2x_2'' \tag{3.63}$$

$$m_1y_1'' = T_1cos\theta_1 - m_2y_2'' - m_2g - m_1g \tag{3.64}$$

Next, there is multiplication of equation (3.62) to $cos\theta_1$, while equation (3.63) after is under same action but to $sin\theta_1$

$$T_1 sin\theta_1 cos\theta_1 = -cos\theta_1(m_1x_1'' + m_2x_2'') \tag{3.65}$$

$$T_1 sin\theta_1 cos\theta_1 = sin\theta_1(m_1y_1'' + m_2y_2'' + m_2g + m_1g) \tag{3.66}$$

$$sin\theta_1(m_1y_1'' + m_2y_2'' + m_2g + m_1g) = -cos\theta_1(m_1x_1'' + m_2x_2'') \tag{3.67}$$

Now equations (3.60) and (3.61) are multiplied to $cos\theta_2$ and $sin\theta_2$ respectively

$$T_2 sin\theta_2 cos\theta_2 = -cos\theta_2(m_2x_2'') \tag{3.68}$$

$$T_2 sin\theta_2 cos\theta_2 = sin\theta_2(m_2y_2'' + m_2g) \tag{3.69}$$

$$sin\theta_2(m_2y_2'' + m_2g) = -cos\theta_2(m_2x_2'') \tag{3.70}$$

$$\theta_1'' = \frac{-g(2m_1 + m_2)sin\theta_1 - m_2 g sin(\theta_1 - 2\theta_2) - 2sin(\theta_1 - \theta_2)m_2(\theta_2'^2 L_2 + \theta_1'^2 L_1 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2cos(2\theta_1 - 2\theta_2))} \tag{3.71}$$

which I took from [25].

$$\theta_2'' = \frac{2sin(\theta_1 - \theta_2)(\theta_1'^2 L_1(m_1 + m_2) + g(m_1 + m_2)cos\theta_1 + \theta_2'^2 L_2 m_2 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2cos(2\theta_1 - 2\theta_2))} \tag{3.72}$$

which I took from [25].

In the end equations of motions were received, as in [25]. To obtain, however, numerical solution Runge-Kutta method is required. For mentioned purpose, equations' order has to be decreased from 2nd to 1st [25].

• ω_1 - angular velocity of top rod

• ω_2 - angular velocity of bottom rod

1st order equation

$$\theta_1' = \omega_1 \tag{3.73}$$

$$\theta_1' = \omega_2 \tag{3.74}$$

$$\omega_1' = \frac{-g(2m_1 + m_2)sin\theta_1 - m_2 g sin(\theta_1 - 2\theta_2) - 2sin(\theta_1 - \theta_2)m_2(\omega_2^2 L_2 + \omega_1^2 L_1 cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2cos(2\theta_1 - 2\theta_2))} \tag{3.75}$$

$$\omega_2' = \frac{2sin(\theta_1 - \theta_2)(\omega_1^2 L_1(m_1 + m_2) + g(m_1 + m_2)cos\theta_1 + \omega_2^2 L_2 m_2 cos(\theta_1 - \theta_2)))}{L_1(2m_1 + m_2 - m_2 cos(2\theta_1 - 2\theta_2))}[25] \tag{3.76}$$

which I took from [25].

Here, we have mold to use Runge-Kutta[25]. Next, inverted version of double pendulum going to be considered.

3.3 Double Inverted Pendulum

The construction of double-inverted pendulum is just upside-down version of simple double pendulum. As a result, in this part of this thesis, differential equations systems are used as a framework for simulation of a double-inverted oscillation.

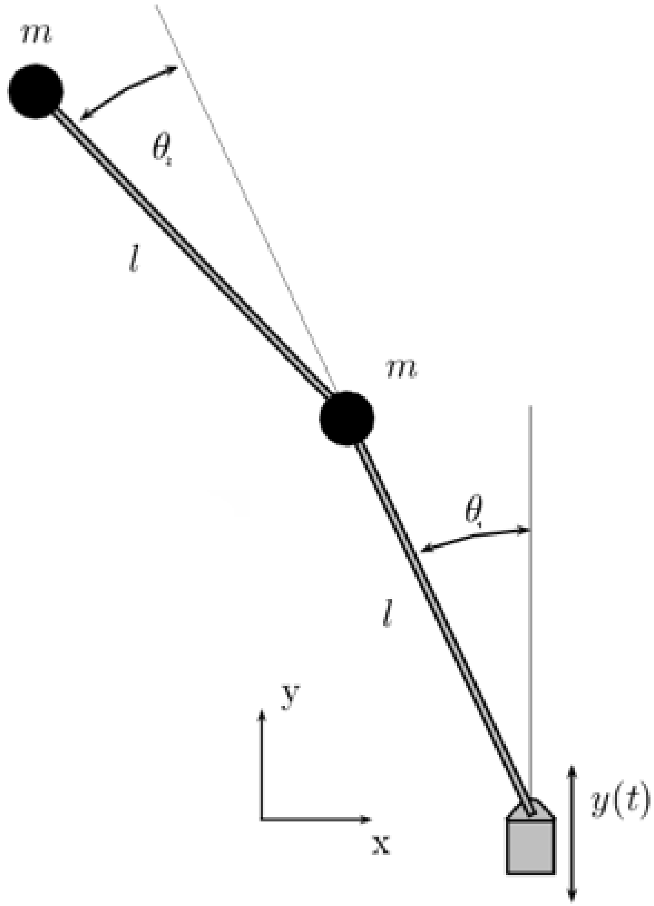

Figure 3.8: Drawing of a Double Pendulum in Inverted Position[22]

There are m1 and which are masses, and l_1 and l_2 which are lengths of relative pendulums, while θ_1 and θ_2, in Figure 3.8 are the angles, and the system is quite unstable. Stabilization, nevertheless, is possible via fluctuation of basement at appropriate amplitudes.

In such construction we can expect indigested acting, because of sporadic trajectory. The illustration below demonstrates inverted double pendulum on cart[22].

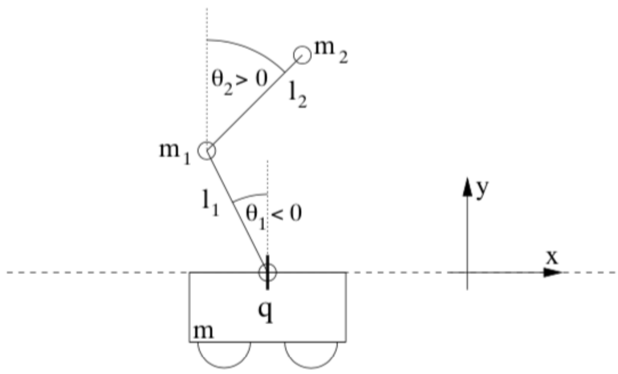

Figure 3.9: Double Inverted Pendulum on a cart [26]

First and second kernels' lengths are l_1 and l_2 realtivly, and cart mass is m, while we give no mass for upper kernel m_1, and m_2 is for second kernel's mass. Masses are headed at single dot. Deviation of kernels are depicted via $\theta_1 = \theta_1(t)$ and $\theta_2 = \theta_2(t)$ at $t \in R.q = q(t)$ is a cart's horizontal location. Time concerning derivatives, which was given in the source[26]:

$$\frac{d}{dt}q(t) = \dot{q}, \quad \frac{d}{dt}\theta_1(t) = \dot{\theta}_1, \quad \frac{d}{dt}\theta_2(t) = \dot{\theta}_2. \tag{3.77}$$

Mainly there is a need to receive stabilization. $u = u(t)$ is force which affects the cart. We have outer factors are w_1, w_2, w_3 as power on q, θ_1, θ_2, damping coefficients d_1, d_2, d_3 masses m_1, m_2, m_3 and friction forces - $d_1 q, d_2\theta_1, d_3\theta_2$.

$$q_0 := \begin{bmatrix} q \\ 0 \end{bmatrix}, \quad q_1 := \begin{bmatrix} q + i_1 \sin\theta_1 \\ l_1 \cos\theta_1 \end{bmatrix}, \tag{3.78}$$

and

$$q_2 := \begin{bmatrix} q - l_1 \sin\theta_1 + l_2 \sin\theta_2 \\ l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{bmatrix} \tag{3.79}$$

source[26] respectively.

Kinetic energy:

$$\begin{aligned} K =& \frac{1}{2}\left\{m\left\|\dot{q}_0\right\|^2 + m_1\left\|\dot{q}_1\right\|^2 + m_2\left\|\dot{q}_2\right\|^2\right\} \\ =& \frac{1}{2}\left\{m\dot{q}^2 + m_1\left[\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1\right)^2\right] + \right. \\ & \left. m_2\left[\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)^2\right]\right\} \end{aligned} \tag{3.80}$$

Potential energy, which was given in [26]

$$P = g\left\{m_1 l_1 \cos\theta_1 + m_2\left(l_1 \cos\theta_1 + l_2 \cos\theta_2\right)\right\} \tag{3.81}$$

We determine Lagrangian $\mathcal{L} := K - P$ motion's equation, due to mechanics principle

$$\begin{aligned}
u + w_1 - d_1\dot{q} = & \frac{d}{dt}\left\{\frac{\partial\mathcal{L}}{\partial\dot{q}}\right\} - \left\{\frac{\partial\mathcal{L}}{\partial q}\right\} \\
= & \frac{d}{dt}\left\{m\dot{q} + m_1\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1\right) + m_2\left(\dot{q} + l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)\right\} - \{0\} \\
= & \left(m + m_1 + m_2\right)\ddot{q} + l_1\left(m_1 + m_2\right)\ddot{\theta}_1\cos\theta_1 - l_1\left(m_1 + m_2\right)\left(\dot{\theta}_1\right)^2\sin\theta_1 \\
& + m_2 l_2\ddot{\theta}_2\cos\theta_2 - m_2 l_2\left(\dot{\theta}_2\right)^2\sin\theta_2 \\
= & \left(m + m_1 + m_2\right)\ddot{q} + l_1\left(m_1 + m_2\right)\ddot{\theta}_1\cos\theta_1 + m_2 l_2\ddot{\theta}_2\cos\theta_2 \\
& - l_1\left(m_1 + m_2\right)\left(\dot{\theta}_1\right)^2\sin\theta_1 - m_2 l_2\left(\dot{\theta}_2\right)^2\sin\theta_2
\end{aligned} \tag{3.82}$$

$$\begin{aligned}
w_2 - d_2\dot{\theta}_1 = & \frac{d}{dt}\left\{\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right\} - \left\{\frac{\partial\mathcal{L}}{\partial\theta_1}\right\} \\
= \ldots = & \left\{l_1\left(m_1 + m_2\right)\dot{\theta}\dot{\theta}_1\sin\theta_1 + l_1 l_2 m_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1 - \theta_2\right) - g\left(m_1 + m_2\right)l_1\sin\theta_1\right\} \\
& + \frac{d}{dt}\left\{l_1\left(m_1 + m_2\right)\dot{q}\cos\theta_1 + l_1^2\left(m_1 + m_2\right)\dot{\theta}_1 + l_1 l_2 m_2\dot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right\} \\
= & \left\{l_1\left(m_1 + m_2\right)\dot{q}\dot{\theta}_1\sin\theta_1 + l_1 l_2 m_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1 - \theta_2\right) - g\left(m_1 + m_2\right)l_1\sin\theta_1\right\} \\
& + \left\{l_1\left(m_1 + m_2\right)\ddot{q}\cos\theta_1 + l_1^2\left(m_1 + m_2\right)\ddot{\theta}_1 + l_1 l_2 m_2\ddot{\theta}_2\cos\left(\theta_1 - \theta_2\right)\right. \\
& \left. - l_1\left(m_1 + m_2\right)\dot{q}\dot{\theta}_1\sin\theta_1 - l_1 l_2 m_2\dot{\theta}_2\left(\dot{\theta}_1 - \dot{\theta}_2\right)\sin\left(\theta_1 - \theta_2\right)\right\} \\
= & l_1\left(m_1 + m_2\right)\ddot{q}\cos\theta_1 + l_1^2\left(m_1 + m_2\right)\ddot{\theta}_1 + l_1 l_2 m_2\ddot{\theta}_2\cos\left(\theta_1 - \theta_2\right) \\
& + l_1 l_2 m_2\left(\dot{\theta}_2\right)^2\sin\left(\theta_1 - \theta_2\right) - g\left(m_1 + m_2\right)l_1\sin\theta_1
\end{aligned} \tag{3.83}$$

taken from[26]

$$
\begin{aligned}
w_3 - d_3\dot{\theta}_2 =& \frac{d}{dt}\left\{\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right\} - \left\{\frac{\partial \mathcal{L}}{\partial \theta_2}\right\} \\
&= \ldots = \left\{-l_2 m_2 g \sin\theta_2 + l_2 m_2 \dot{q}\dot{\theta}_2 \sin\theta_2 - l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\right\} \\
&+ \frac{d}{dt}\left\{l_2^2 m_2 \dot{\theta}_2 + l_2 m_2 \dot{q}\cos\theta_2 + l_1 l_2 m_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right\} \\
=& \left\{-l_2 m_2 g \sin\theta_2 + l_2 m_2 \dot{q}\dot{\theta}_2 \sin\theta_2 - l_1 l_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)\right\} \\
&+ \left\{l_2^2 m_2 \ddot{\theta}_2 + l_2 m_2 \ddot{q}\cos\theta_2 + l_1 l_2 m_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)\right. \\
&\left. - l_2 m_2 \dot{q}\dot{\theta}_2 \sin\theta_2 - l_1 l_2 m_2 \dot{\theta}_1 \left(\dot{\theta}_1 - \dot{\theta}_2\right) \sin(\theta_1 - \theta_2)\right\} \\
=& l_2 m_2 \ddot{q}\cos\theta_2 + l_1 l_2 m_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + l_2^2 m_2 \ddot{\theta}_2 \\
&- l_1 l_2 m_2 \left(\dot{\theta}_1\right)^2 \sin(\theta_1 - \theta_2) - l_2 m_2 g \sin\theta_2,
\end{aligned}
\tag{3.84}
$$

$$
\underbrace{\begin{bmatrix} m + m_1 + m_2 & l_1(m_1 + m_2)\cos\theta_1 & m_2 l_2 \cos\theta_2 \\ l_1(m_1 + m_2)\cos\theta_1 & l_1^2(m_1 + m_2) & l_1 l_2 m_2 \cos(\theta_1 - \theta_2) \\ l_2 m_2 \cos\theta_2 & l_1 l_2 m_2 \cos(\theta_1 - \theta_2) & l_2^2 m_2 \end{bmatrix}}_{=:M(y)} \underbrace{\begin{bmatrix} \ddot{q} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}_{=\ddot{y}} =
$$

$$
= \underbrace{\begin{bmatrix} l_1(m_1 + m_2)\left(\dot{\theta}_1\right)^2 \sin\theta_1 + m_2 l_2 \left(\dot{\theta}_2\right)^2 \sin\theta_2 \\ -l_1 l_2 m_2 \left(\dot{\theta}_2\right)^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2) l_1 \sin\theta_1 \\ l_1 l_2 m_2 \left(\dot{\theta}_1\right)^2 \sin(\theta_1 - \theta_2) + g l_2 m_2 \sin\theta_2 \end{bmatrix} - \begin{bmatrix} d_1 \dot{q} \\ d_2 \theta_1 \\ d_3 \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{=:w}}_{=:f(y,\dot{y},u,w)}
\tag{3.85}
$$

or

$$
M(y)\ddot{y} = f(y, \dot{y}, u, w) \tag{3.86}
$$

Determinate of M(y):

$$
\det M(y) = \ldots = l_1^2 l_2^2 m_2 (\underbrace{m m_1}_{>0} + \underbrace{m_1^2 \sin^2\theta_1 + m_1 m_2 \sin^2\theta_1 + m m_2 \sin^2(\theta_1 - \theta_2)}_{\geq 0}) > 0
\tag{3.87}
$$

for all $y \in R^3$, so M(y) is invertible.

$$
\ddot{y} = M^{-1}(y) f(y, \dot{y}, u, w) \tag{3.88}
$$

$$
\begin{bmatrix} y \\ \dot{y} \end{bmatrix} \tag{3.89}
$$

Order reducing:

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{y} \\ M^{-1}(y)f(y,\dot{y},u,w) \end{bmatrix}}_{=:F(x,u,w)} \tag{3.90}$$

ODE system[26].

3.4 Triple pendulum

The last, but not least model taken into consideration is triple pendulum system.

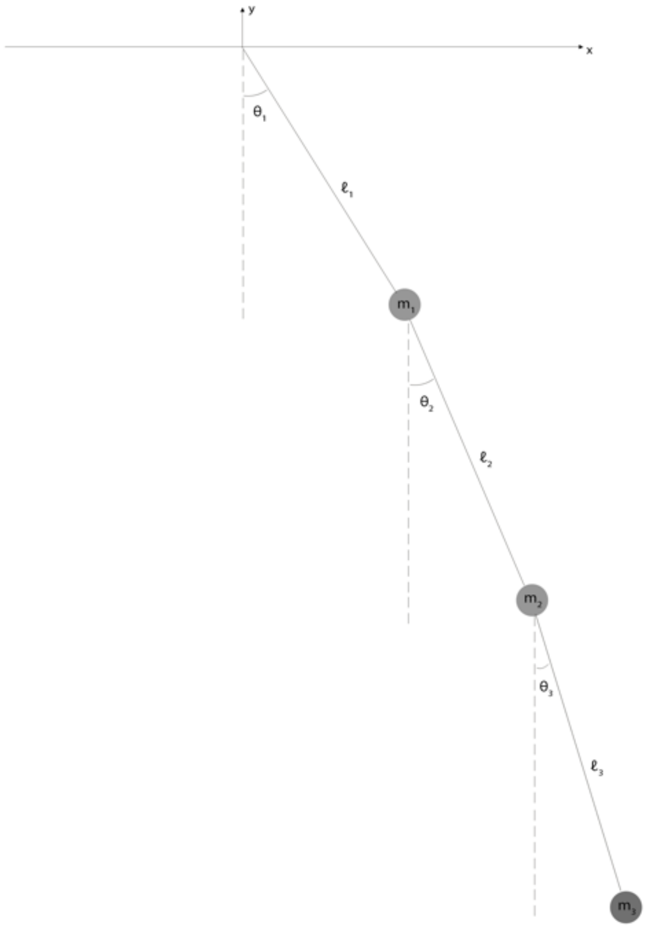

Figure 3.10: A Triple Pendulum [27]

Lagrangians of triple pendulum:$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$, which was given in [27].

Motion equation of triple pendulum with using Euler-Lagrange concerning θ_1, θ_2 and θ_3, which I took from source [27].

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)-\frac{\partial \mathcal{L}}{\partial q}=0 \tag{3.91}$$

For θ_1,

$$\begin{aligned}
&gl_1\left(m_1\sin\left(\theta_1\right)+m_2\sin\left(\theta_1\right)+m_3\sin\left(\theta_1\right)\right)+m_2l_1l_2\sin\left(\theta_1-\theta_2\right)\dot{\theta}_1\dot{\theta}_2\\
&+m_3l_1l_3\sin\left(\theta_1-\theta_3\right)\dot{\theta}_1\dot{\theta}_3+m_3l_1l_2\sin\left(\theta_1-\theta_2\right)\dot{\theta}_1\dot{\theta}_2+l_1^2\ddot{\theta}_1\left(m_1+m_2+m_3\right)\\
&+m_2l_1l_2\left[\sin\left(\theta_2-\theta_1\right)\left(\dot{\theta}_1-\dot{\theta}_2\right)\dot{\theta}_2+\cos\left(\theta_1-\theta_2\right)\ddot{\theta}_2\right]\\
&+m_3l_1l_2\left[\sin\left(\theta_2-\theta_1\right)\left(\dot{\theta}_1-\dot{\theta}_2\right)\dot{\theta}_2+\cos\left(\theta_1-\theta_2\right)\ddot{\theta}_2\right]\\
&+m_3l_1l_3\left[\sin\left(\theta_3-\theta_1\right)\left(\dot{\theta}_1-\dot{\theta}_3\right)\dot{\theta}_3+\cos\left(\theta_1-\theta_3\right)\ddot{\theta}_3\right]=0
\end{aligned} \tag{3.92}$$

For θ_2,

$$\begin{aligned}
&gl_2\left(m_2\sin\left(\theta_2\right)+m_3\sin\left(\theta_2\right)\right)+\dot{\theta}_1\dot{\theta}_2l_1l_2\sin\left(\theta_2-\theta_1\right)\left[m_2+m_3\right]\\
&+m_3l_2l_3\sin\left(\theta_2-\theta_3\right)\dot{\theta}_2\dot{\theta}_3+l_2^2\ddot{\theta}_2\left(m_2+m_3\right)\\
&+\left(m_2+m_3\right)l_1l_2\left[\sin\left(\theta_2-\theta_1\right)\left(\dot{\theta}_1-\dot{\theta}_2\right)\dot{\theta}_1+\cos\left(\theta_2-\theta_1\right)\ddot{\theta}_1\right]\\
&+m_3l_2l_3\left[\sin\left(\theta_3-\theta_2\right)\left(\dot{\theta}_2-\dot{\theta}_3\right)\dot{\theta}_3+\cos\left(\theta_2-\theta_3\right)\ddot{\theta}_3\right]=0
\end{aligned} \tag{3.93}$$

For θ_3,

$$\begin{aligned}
&m_3gl_3\sin\left(\theta_3\right)-m_3l_2l_3\sin\left(\theta_2-\theta_3\right)\dot{\theta}_2\dot{\theta}_3-m_3l_1l_3\sin\left(\theta_1-\theta_3\right)\dot{\theta}_1\dot{\theta}_3\\
&+m_3l_1l_3\left[\sin\left(\theta_3-\theta_1\right)\left(\dot{\theta}_1-\dot{\theta}_3\right)\dot{\theta}_1+\cos\left(\theta_1-\theta_3\right)\ddot{\theta}_1\right]\\
&+m_3l_2l_3\left[\sin\left(\theta_3-\theta_2\right)\left(\dot{\theta}_2-\dot{\theta}_3\right)\dot{\theta}_2+\cos\left(\theta_2-\theta_3\right)\ddot{\theta}_2\right]+m_3l_3^2\ddot{\theta}_3=0
\end{aligned} \tag{3.94}$$

Here, $g = 9,8m/s^2, m$-mass,l-length.Next, we group equation terms. From this, we may analyze the motion of a triple pendulum, and $gl_jsin(\theta_j)mk$ is the same for the equations provided, but the mass changes. Coordinates are signified by $\theta_j, j = 1, 2, 3$. So we make a model via describing this function σ_{jk} [27].

$$\sigma_{jk}=\left\{\begin{array}{ll}0 & j>k\\ 1 & j\leq k\end{array}\right., \qquad \sum_{k=1}^{n=3} gl_j\sin\left(\theta_j\right)m_k\sigma_{jk} \tag{3.95}$$

Here we indicated conditions for n=3 of triple pendulum.

$m_3l_3^2\ddot{\theta}_3$ part can be noticed in all three equations. Next σ_{jk} is applied

$$\sum_{k=1}^{n=3} m_kl_j^2\ddot{\theta}_j\sigma_{jk} \tag{3.96}$$

2 various kinds of conditions are illustrated by the end.

$$\sum_{k=1}^{n=3}\left(\sum_{q\geq k}^{n=3} m_q\sigma_{jq}\right) l_j l_k \sin\left(\theta_j-\theta_k\right)\dot{\theta}_j\dot{\theta}_k \tag{3.97}$$

Order of arguments are changed and replaced for sin function, and we handle negative signs, which was given in [27].

$$\phi_{jk}=\begin{cases} 0 & j=k \\ 1 & j\neq k \end{cases}$$

Then,

$$\sum_{k=1}^{n=3}\left(\sum_{q\geq k}^{n=3} m_q\sigma_{jq}\right) l_j l_k \left[\sin\left(\theta_k-\theta_j\right)\left[\dot{\theta}_j-\dot{\theta}_k\right]\dot{\theta}_k+\phi_{jk}\cos\left(\theta_j-\theta_k\right)\ddot{\theta}_k\right] \tag{3.98}$$

Finally, we get following expression [27]:

$$\begin{gathered}\textstyle\sum_{k=1}^{n=3}\left(gl_j\sin\left(\theta_j\right)m_k\sigma_{jk}+m_k l_j^2\ddot{\theta}_j\sigma_{jk}+\left(\sum_{q\geq k}^{n=3} m_q\sigma_{jq}\right)l_j l_k\sin\left(\theta_j-\theta_k\right)\dot{\theta}_j\dot{\theta}_k\right.\\ \left.+\left(\textstyle\sum_{q\geq k}^{n=3} m_q\sigma_{jq}\right)l_j l_k\left[\sin\left(\theta_k-\theta_j\right)\left[\dot{\theta}_j-\dot{\theta}_k\right]\dot{\theta}_k+\phi_{jk}\cos\left(\theta_j-\theta_k\right)\ddot{\theta}_k\right]\right)=0\end{gathered} \tag{3.99}$$

Equation of motion will participate in further simulation process and results are going to be analyzed in Results part of current work [27].

Chapter 4

Simulation

4.1 Single Inverted Pendulum and Unicycle

Generally, the main parts of current thesis work are modelling and control parts. All previously covered materials were applied in practical process. The Euler-Lagrange equation came from variation theory and the equations were derived from there. Afterwards, it produces ODE systems. In current work, several pendulum models had to be derived, so for this purpose I Euler-Lagrange equation helped in purposes of application of it in practical part. Simulation by itself consisted of from solving ODE and rearranging it. I needed to use numerical solving for which I used Euler integral.

The basic idea of simulation part is illustration of systems from real world, in my case I took unicycle and human arm via computer modelling. It can be said that in general simulation of some models, is an appropriate way to implement in case of absence of closed shape analytical resolutions. Since we cannot always solve completely dynamic exercises via mathematical equations, therefore simulation through computer is potent way to implement while solving complex systems. System which are chosen, must emulate real systems over given time. It means we can create synthetic story for the system to be researched. Consequently, there is a great opportunity to make certain conclusion regarding system distinctives. In brief, simulation is the method for researching systems from reality via mimicking their conducts through application of models applied digitally. Moreover, we can even find solutions for probability models via numerical method. For example, if there is a complex model with probability factor, we can apply partial ODE, similarly as we would have done for complex function usage of numerical method. There are certain concepts which are required to be followed such as events, model itself and variables. Experimental studies give us a great amount of data concerning checking different assumptions we make regarding chosen models, so in the same way modeling is also practiced dissecting complicated practical cases to forecast possible consequences if to apply certain actions and initial values. If we talk about variables, these are just assembly of required data for detection of events occurring in the system at appropriate stage and time. It is one of the tasks of research process to choose certain variables, because not always the same variables would be good for two different systems with identical physical system. During modelling different lapses can be found and corrected. Certainly, there is a need to give certain titles to our variables. There are also dependent and independent variables. For instance, in simulation of ball's movement other (except mass and several others) variables directly connected with time, which are then considered as dependents variables. If it is needed, we can decrease complexity of the model via making some variables constant. Except naming variables, we must describe all involved units like hours, minutes, seconds, or days. Next, we switch to making diagrams of our models, where we demonstrate relations between variables. To make model easier to consider, we can imagine that certain complex processes are simpler. Then, after stating relations among variables we state functions for them. We must evaluate different formulas, weather they are applied to certain model or no. Majority of models have ODEs. After these stages we need to solve models. In my case I used Euler-Lagrange equation to derive the pendulum models and made simulation. From mathematical standpoint simulation process mainly consisted of solving ODEs. I needed to solve numerically ODE by applying Euler integral.

For given simulations I have applied Euler numerical integration method, which was briefly discussed in the first chapter. From simulation results we can see how inverted pendulum swings with consideration of time and angle. There is a positive tendency in graph, which shows that θ angle degree increases when with t time. Simulation parameters: length=1; g = 9.8; n points = 2000; dt = 0.004

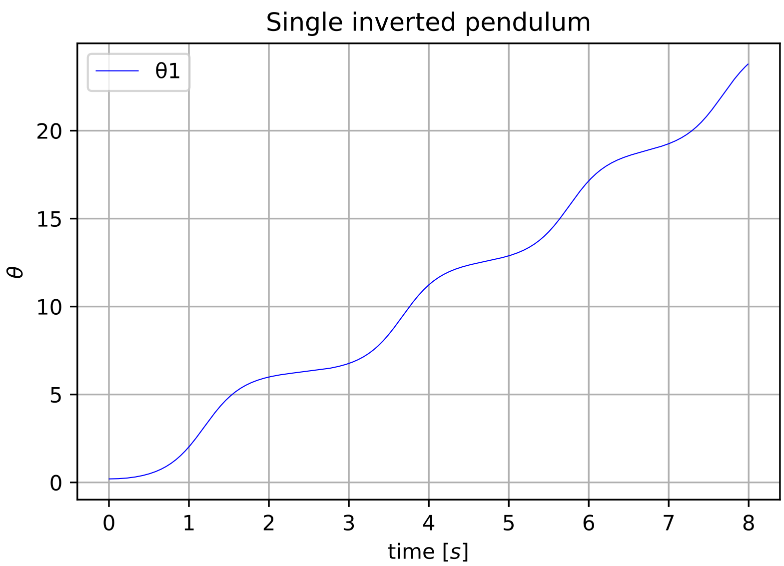

Figure 4.1: Simulation of Single-inverted pendulum

Next graphs are results of unicycle control where unicycle movement, position, velocity, and acceleration are represented. Initial conditions are constant for 4 received graphs:$m = 1; l = 1; N \backsim -A \cdot \theta'' - B \cdot \theta \cdot \theta'^2$, where A and B are constants,$M = 1$; A=0.005; B=0.005; θ_1 =0.524. Unicycle graph shows sharp rise between t=0.5 and t=1.5 after which dramatic and continuous fall can be seen since time continues to increase, until it reaches t=3.5 where it shows rise again.

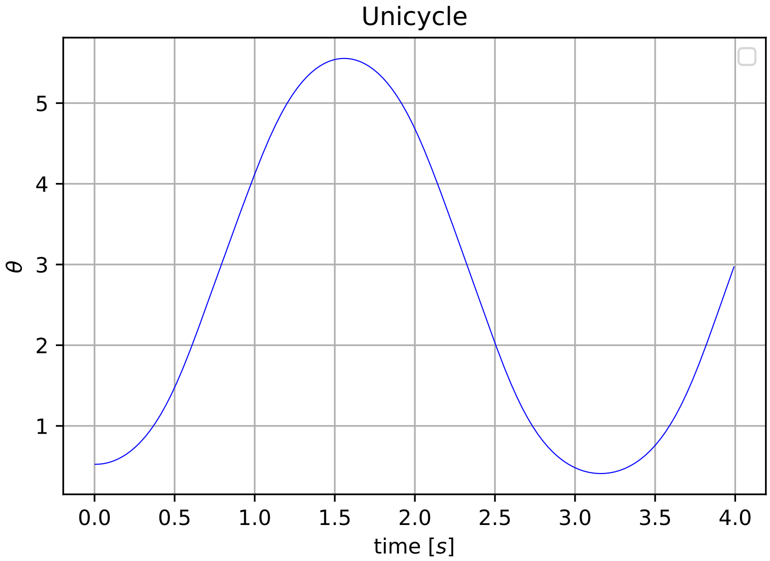

Figure 4.2: Unicycle motion

Position graph illustrates intensive fluctuation. Index reaches its highest points at t=0.0 and t=3.2(approximately), while lowest points are at t=0.8(approximately) and t=2.3(approximately)

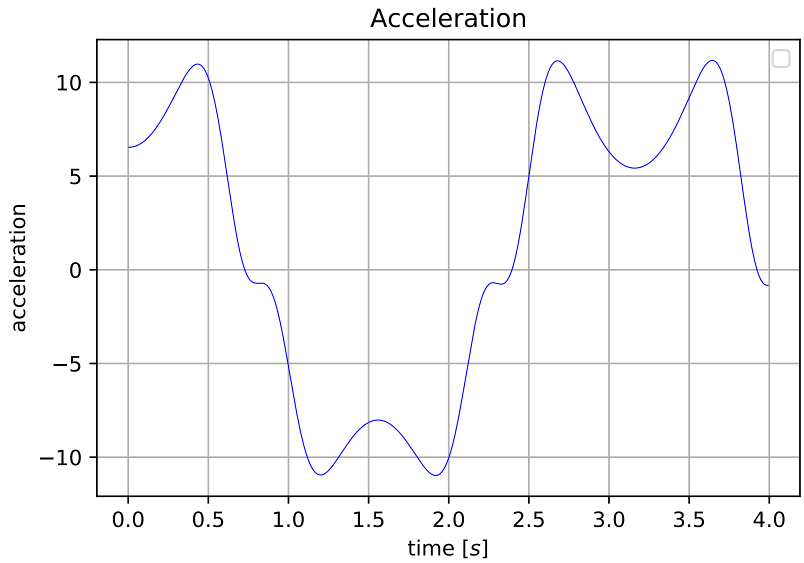

Figure 4.3: Acceleration

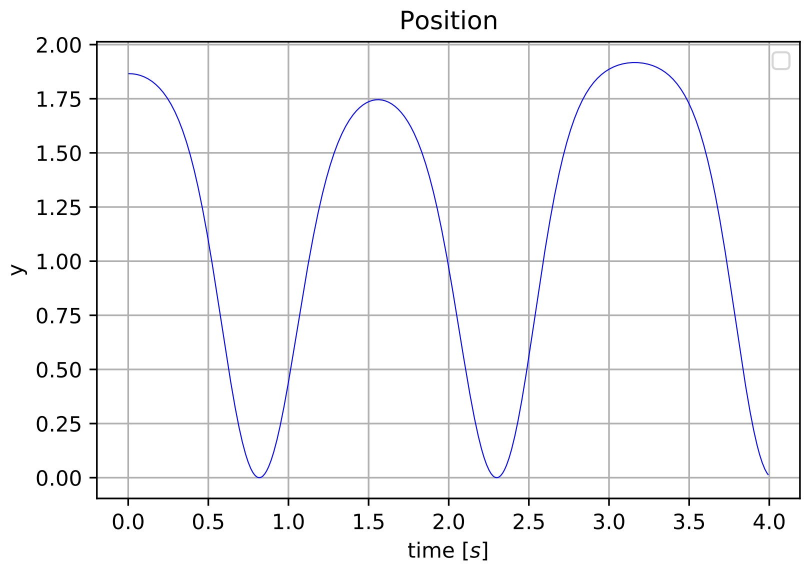

Figure 4.4: Position

Velocity graph shows fluctuations with identical differences. For example fall from t=0.6 until t=2.4 has the same value as the rise from t=2.5 to t=3.9.

Graph of acceleration taken as a result of control shows quite unstable result with negative oscillation from t=0.5 to t=1.2, from t=2.6 to t=3.2 from t=3.7 to t=4.0 and rise from t=1.9 to t=2.6.

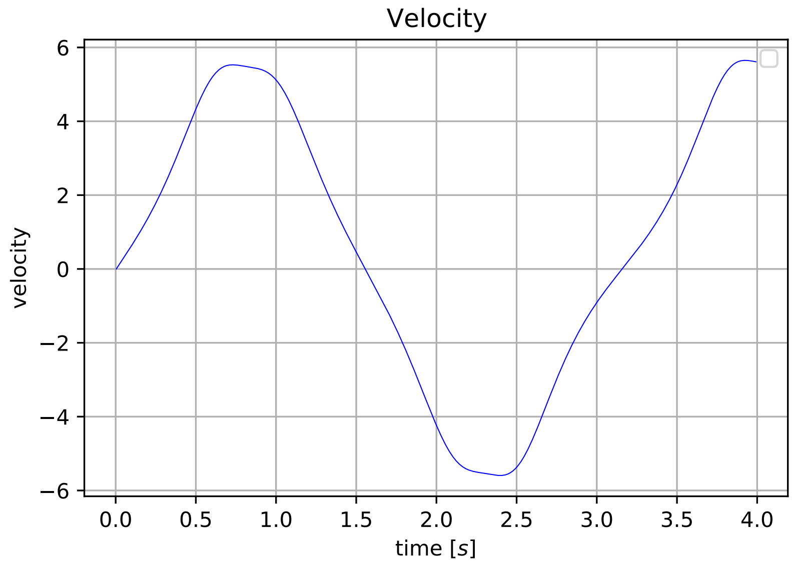

Figure 4.5: Velocity

4.2 Double Inverted Pendulum

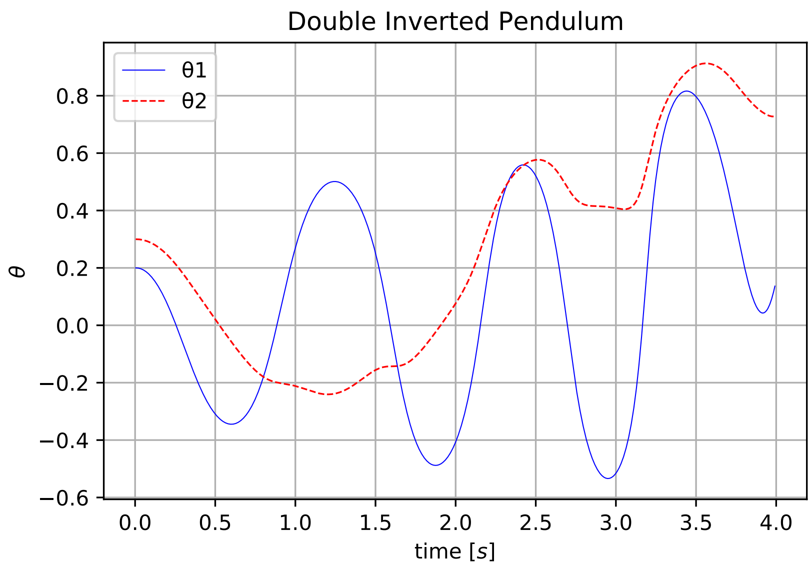

Figure 4.6: Simulation of Double Inverted Pendulum

Now we add second pendulum two our inverted construction and simulate it in Python. There is an obvious difference in θ of Figure 3.6 and θ_1 Figure 3.7. It happens because of force and mass influence of attached pendulum. Curves of θ_1 increase gradually with time. However, we can see different behavior of attached pendulum which is θ_2. θ_2 continuously decreases until time reaches 1.5 seconds. From that time fluctuations with positive tendency can be emphasized. After 3 rd. second θ_2 illustrates wider angles compared with θ_1. Simulation parameters: length= 1; l_1 = l_2 = 1; m_1=m_2=1; g = 9.8; n points=1000; dt = 0.004

4.3 Triple Pendulum

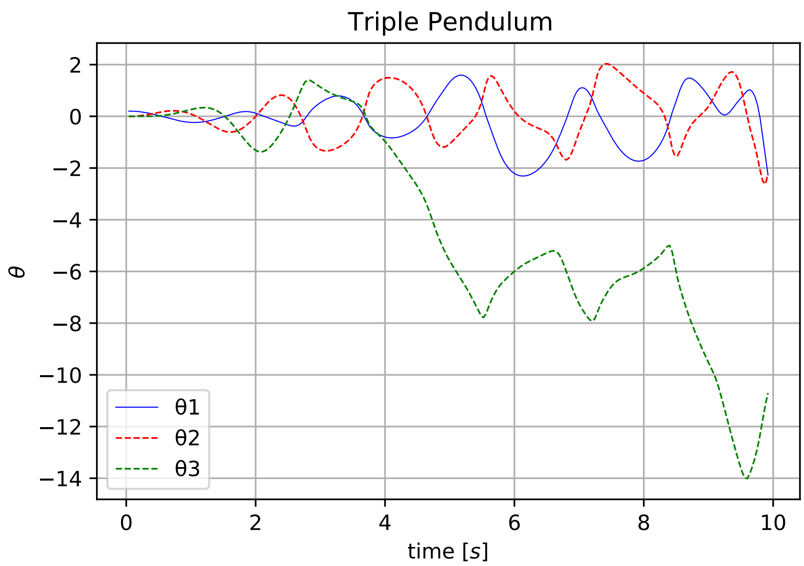

Figure 4.7: Simulation of Triple pendulum

In final graph we add third pendulum and turn it down, so it is not inverted in this case. θ_3 is third pendulum attached to double pendulum. Graph describes how swings of θ_3 is enormously deferent compared wit θ_1 Λ θ_2. Behavior of three different parts of one construction is approximately the same until 4 seconds, after which angles of θ_3 change dramatically. Simulation parameters: length= 1; l_1 = l_2 = l_3 = 1; m_1=m_2=m_3=1; g = 9.8; n points=250; dt = 0.04

Chapter 5

Control

5.1 Control of Double Pendulum

For the control of double pendulum I used RFPT MIMO system. Initial conditions for θ_1 and θ_2 were taken as $\theta_1(0)$=$\theta_2(0)$=0.2. Parameters of exact model like mass and length were defined as: m_1=m_2=1, l_1=l_2=1. I needed to choose the control force for the equations. In practice one can replace any function from the model's equations with "u" control force if it is based on physical reasoning behind it. Or another option is to add the "u" control force to the equations. I take "u" as an external force(might be a robotic arm pushing the pendulum).

It follows that I add "u" force to the equations of the model in (3.71-3.72):

$$\theta_1'' = \frac{-g(2m_1+m_2)sin\theta_1 - m_2 gsin(\theta_1-2\theta_2) - 2sin(\theta_1-\theta_2)m_2(\theta_2'^2 L_2 + \theta_1'^2 L_1 cos(\theta_1-\theta_2))}{L_1(2m_1+m_2-m_2cos(2\theta_1-2\theta_2))} + u \tag{5.1}$$

$$\theta_2'' = \frac{2sin(\theta_1-\theta_2)(\theta_1'^2 L_1(m_1+m_2) + g(m_1+m_2)cos\theta_1 + \theta_2'^2 L_2 m_2 cos(\theta_1-\theta_2))}{L_1(2m_1+m_2-m_2cos(2\theta_1-2\theta_2))} + u \tag{5.2}$$

In this case inverse model of control looks like:

$$u = \theta_1'' - \frac{-g(2m_1+m_2)sin\theta_1 - m_2 gsin(\theta_1-2\theta_2) - 2sin(\theta_1-\theta_2)m_2(\theta_2'^2 L_2 + \theta_1'^2 L_1 cos(\theta_1-\theta_2))}{L_1(2m_1+m_2-m_2cos(2\theta_1-2\theta_2))} \tag{5.3}$$

$$u = \theta_2'' - \frac{2sin(\theta_1-\theta_2)(\theta_1'^2 L_1(m_1+m_2) + g(m_1+m_2)cos\theta_1 + \theta_2'^2 L_2 m_2 cos(\theta_1-\theta_2))}{L_1(2m_1+m_2-m_2cos(2\theta_1-2\theta_2))} \tag{5.4}$$

Nominal trajectory was chosen as $Asin(\omega t)$, where A is an amplitude which I took as 1.2 and 1.3 for θ_1 and θ_2 nominal trajectories. And ω parameter is equal to 6.5 and 6.75 respectively.

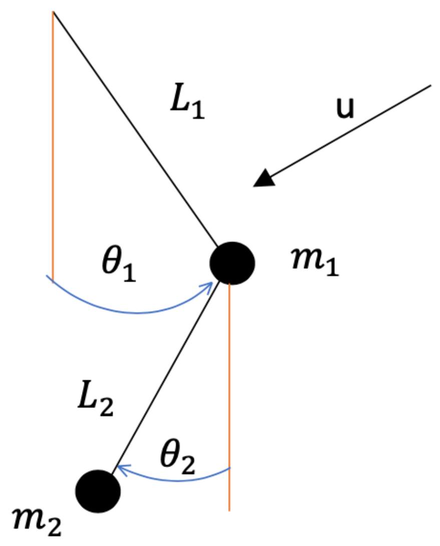

Figure 5.1: External force

And kinematic block follows as:

$$\lambda^3 err_{int} + 3\lambda^2 err + 3\lambda err' + err'' \tag{5.5}$$

For the approximate model the same parameters look like:m_1=m_2=0.98, l_1=l_2=0.98. g is equal to 9.8 as always. Control parameters were tuned multiple times during the experiment in order to get the good result and finished being like this: A=1.97e-5, K=1000, B=-1, λ=35.

5.2 Results

Following illustrations are devoted to describe motion of double pendulum where nominal and realized path of theta 1. It can be seen that oscillation appears between -1.0m and 1.3m. The highest point is reached at t=0.3s, 1.2s, 2.2s, 3.2s and lowest ones at t=0.4s, 1.7s, 2.7s, 3.7s.

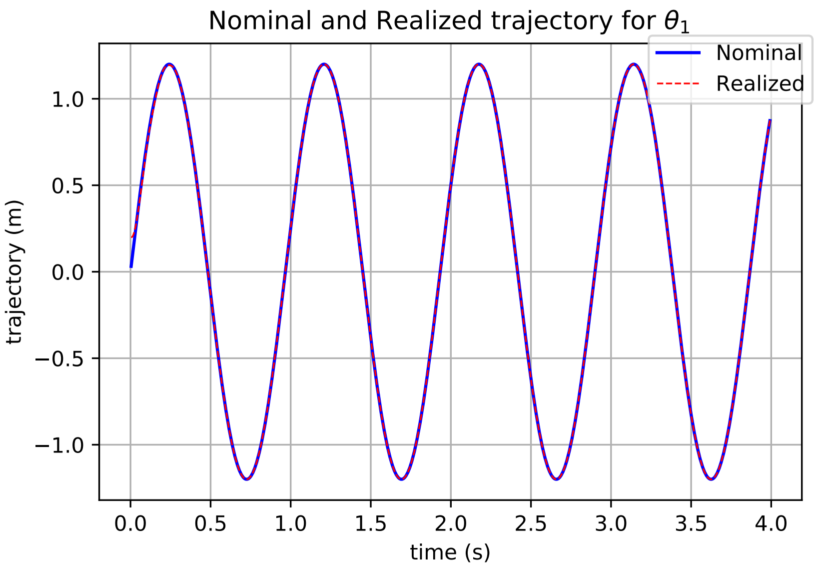

Figure 5.2: Control of Double Pendulum, trajectory for θ_1

Figure (5.3) describes θ_2 trajectory, where fluctuation is between -0.2m and 1.3m. Again the graph looks acceptable as well as graph for θ_1. Error between nominal and realized trajectory seems to be small. Realized trajectory both for θ_1 and θ_2 has a sinusoidal shape as was planned.

And the graphs that show the tracking error(figures 5.4-5.5) can confirm that error between nominal and realized trajectories is close to zero almost during the whole experiment time both for θ_1 and θ_2.

Considering figures (5.6) and (5.7), usually, one checks whether the calculated signal is in allowed domain. If it is not, and if the curve of the control signal might lie in the not allowed domain, then the result should be considered as not satisfactory. Sometimes one can set the domain not to be less than 0, but it is not our case. In our case the fact that curve goes below 0 is appropriate enough. Overall control signal is valid.

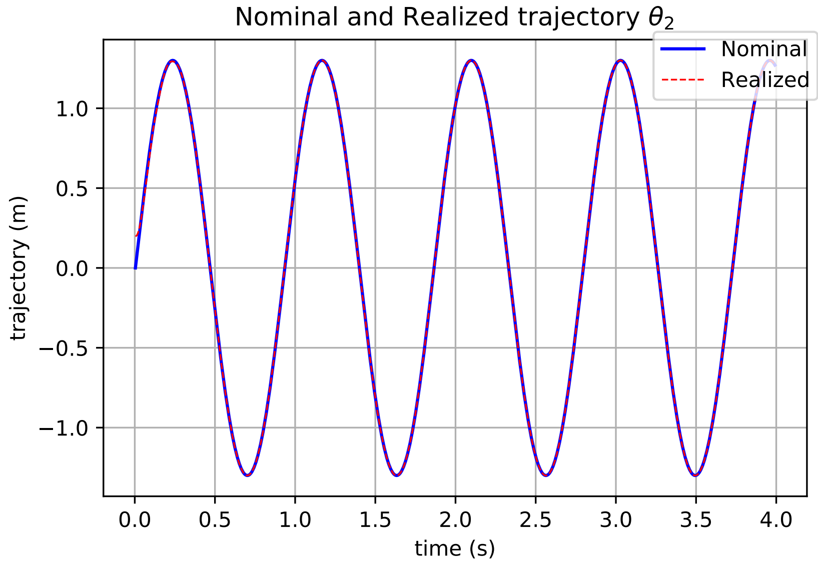

Figure 5.3: Control of Double Pendulum,trajectory for θ_2

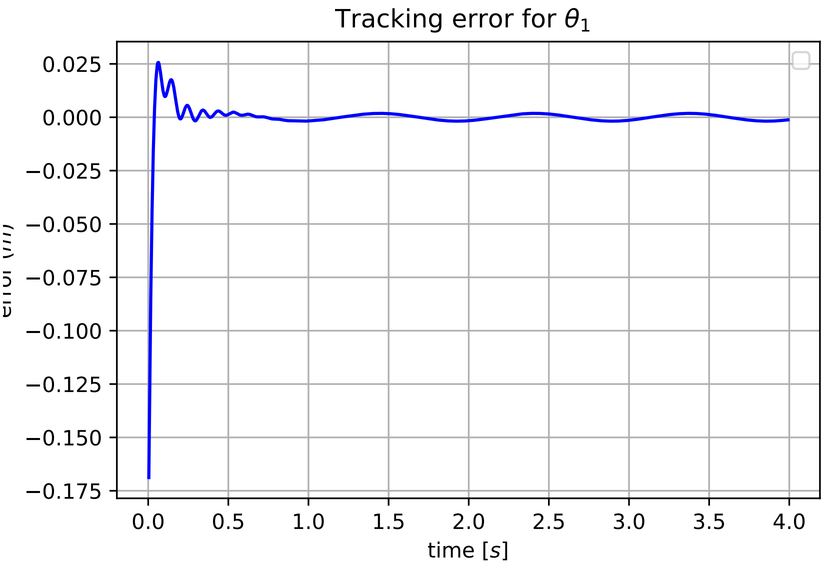

Figure 5.4: Tracking error for θ_1

In case of acceleration results(figures 5.8-5.9) nominal and desired curves should follow each other. So the graph is valid for both θ_1 and θ_2. It can be seen that Desired trajectory of Acceleration of θ_1 is stable in the acceleration frame between 50 and -50, during rising time from 0.5s and 4.0s. The highest acceleration was reached at 0 seconds in the beginning of the graph, while the lowest one is reached at time -90. Realized and Nominal trajectories illustrate identical path of acceleration during represented period time. Desired acceleration of θ_2 also has stable acceleration but in a bit different accel-

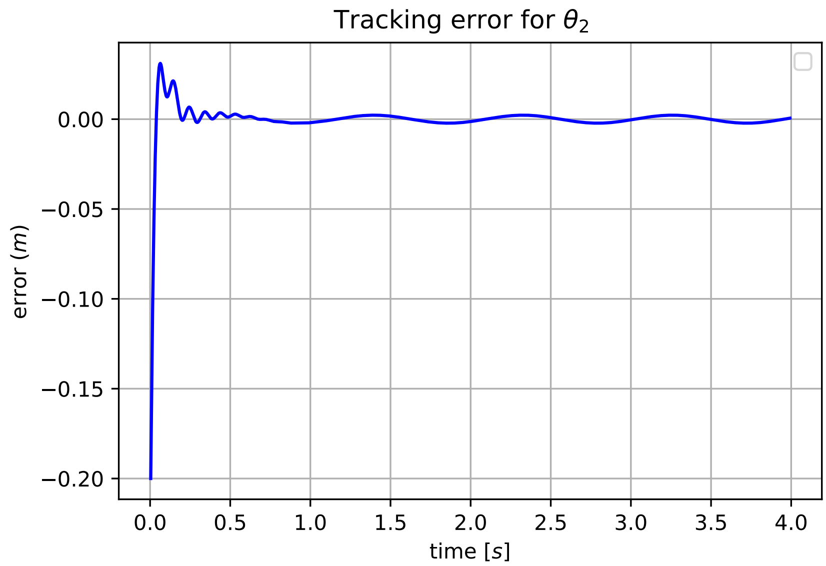

Figure 5.5: Tracking error for θ_2

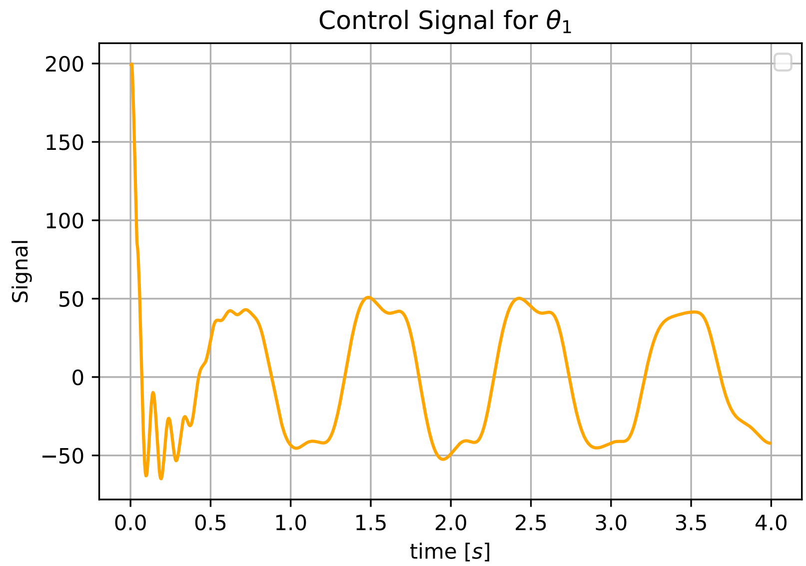

Figure 5.6: Control signal for θ_1

eration frame between -48 and 52 from 0.5s to 4.0s. Nominal and Realized trajectories fluctuate around for $theta_1$ and θ_2 as well.

Viewing figures (5.10-5.11), the same, nominal and realized should follow each other and have small error in case of phase space result. In our case, we have sinus and cosines colliding together, which results to something which looks like a circle and that is what we have in our graph. So this graph is valid as well.

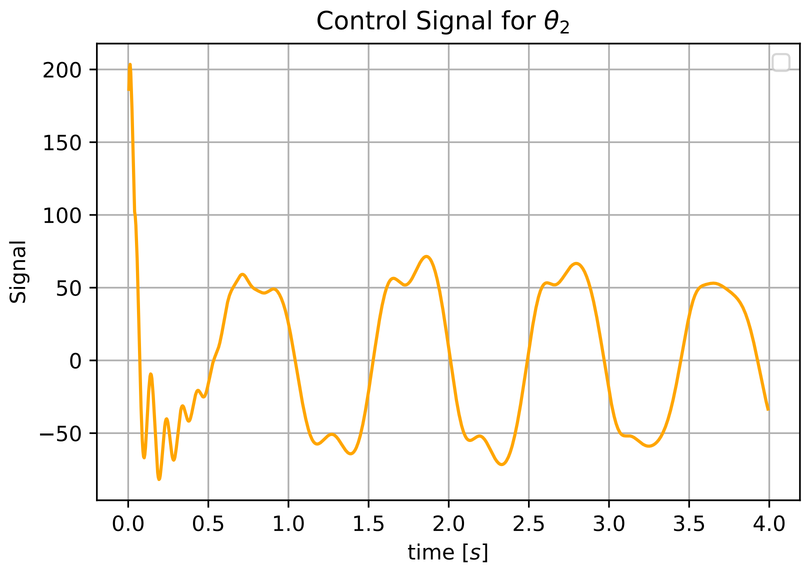

Figure 5.7: Control signal for θ_2

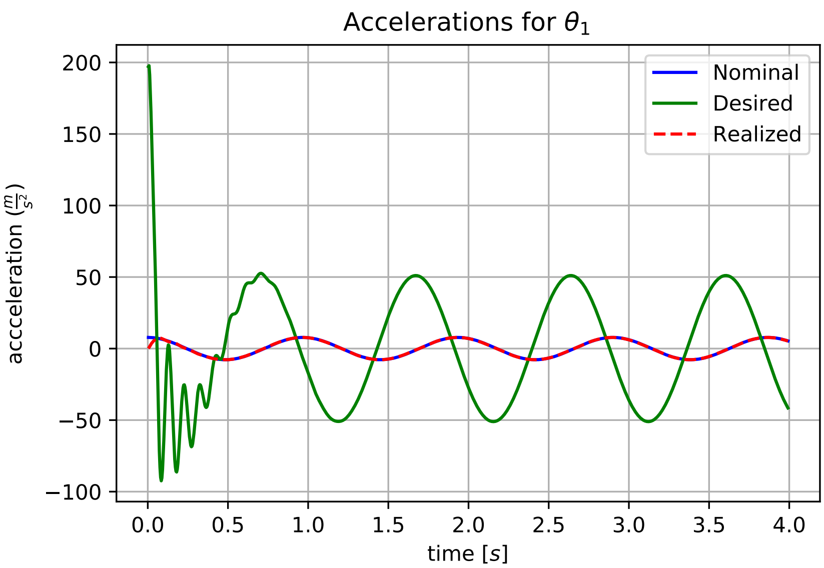

Figure 5.8: Acceleration for θ_1

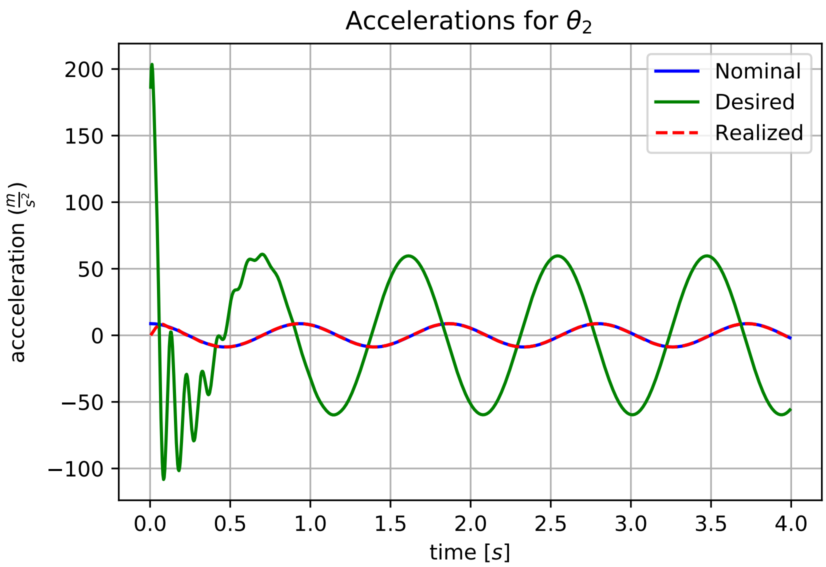

Figure 5.9: Acceleration for θ_2

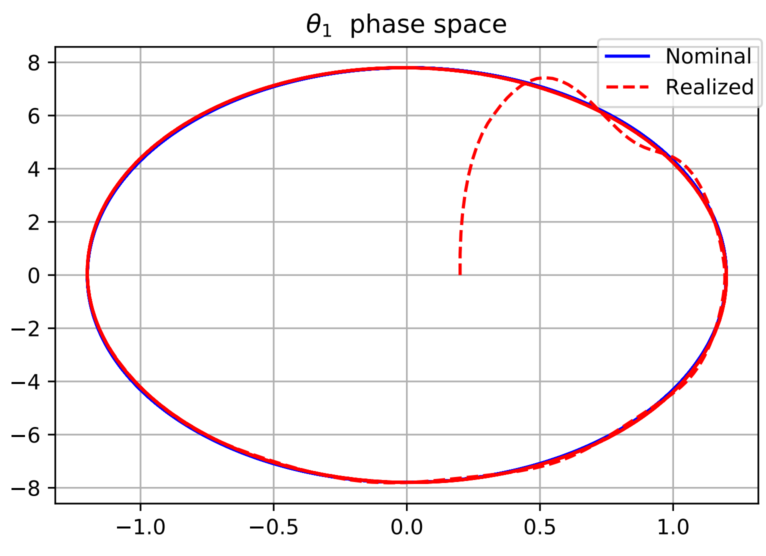

Figure 5.10: Phase space for θ_1

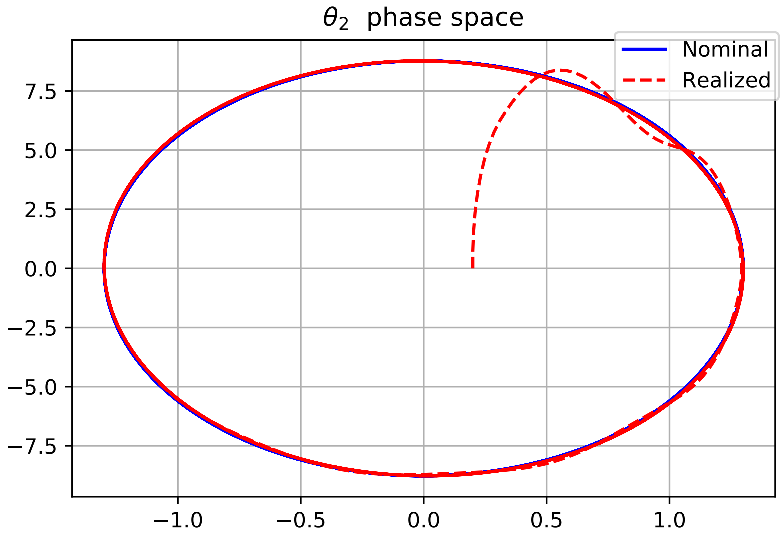

Figure 5.11: Phase space for θ_2

Chapter 6

Applications

6.1 Single Inverted Pendulum

Nowadays robots are integral parts of our everyday life, therefore people always work on their research in this field to make it more productive and cheaper. For instance, self-balancing robots' actions are quite flexible. They are made of a couple of wheels and frame to hold whole construction. The way how it works, reminds a human in balance. There are certain pluses in such constructions such as, good resistance and stability while facing with inclines. Robot's actions are not plane and motor which creates all movement is in its body [28]. From made investigation it is known that such robots are possible to be bought in the market freely [29, 30]. The second example is a SFIT robot that integrates radio control(Swiss Federal Institute of technology) [31, 32]. Next example of such robot is Nrobot which considers for points which are: position, speed, wheel speed and angular position. In comparison with mentioned types, this one is able climb and use stairs[33]. Stephen Hassenplag made legway which used kit and tilt angles where determined by using two electro-optical proximity sensors (EOPD). Another instance is Equibot by Dan Piponi which used infrared diapason finder to determine length among ground and robot, so controller can count the title angle[34]. More detailed example is self-balancing construction. It was made via modeling through differential equations and stabilization was made via linearization and controller. The main part certainly was identifying system's equations. Researchers who made this experiment made coding it in laboratory and some loop operations were involved in controlling [35].

Electric unicycle is a construction which balances by itself and is able to carry a human on it, in the way that person can change the speed by feet movement. There is a gyroscope which was used by Segway as well. It is one the most exciting issues in controlling to make create independent balancing of unicycle. In other words, it is just a nonlinear control system. We ca see one-dimensional control is shown in wheel movement. In the market unicycles devoted for sales can only be balanced in the way person moves

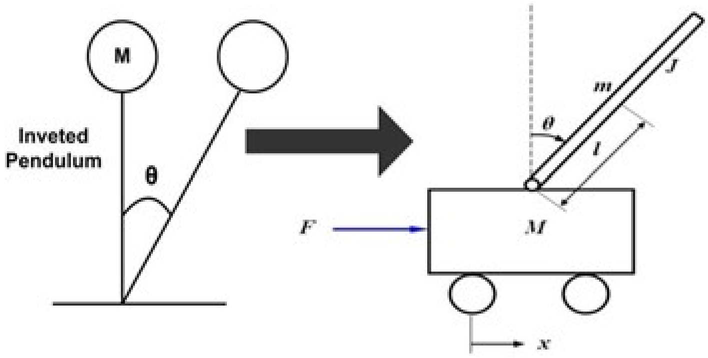

Figure 6.1: Microcontroller-based fully independent a double-wheel self-balancing robot [35]

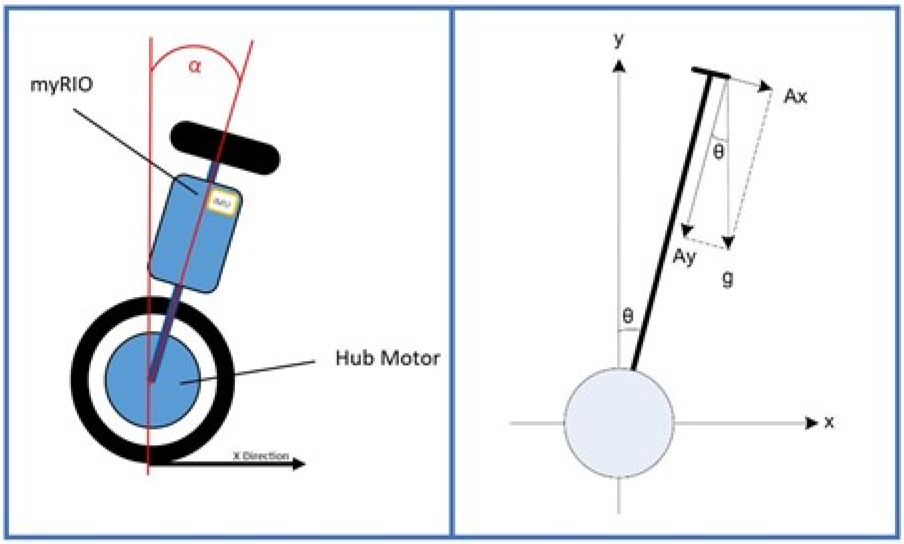

Figure 6.2: The Self-Balancing electric unicycle [36]

on it. Certainly, without lateral stability it would not be possible. Even if for biaxial unicycles there is a need of side balancing function. There are to control structures. They implement gyroscopes, wheels for support, also auxiliary scales which can be considered as an inverted single pendulum. Definitely, all theory explained here is supported by real constructed unicycles from past experience. Inverted pendulum is not as easy to design as it seems to be in unicycle system. Moreover, there is a couple of systems to consider that show various control exercises to solve. If we talk about inverted pendulums there are common types of them, which are single arm, sled and double inverted pendulum which is also modeled in this thesis and going to be discussed further [36]. Principle of single inverted pendulum repeats the motion of unicycle, which is the reason I got it as an application option and made simulation in order to demonstrate its movement.

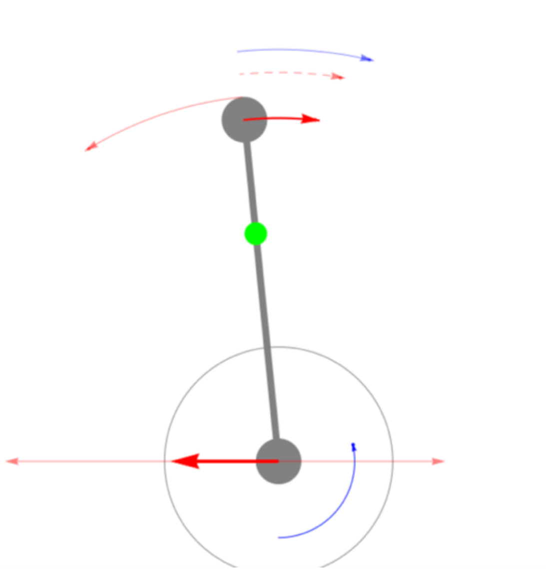

Figure 6.3: The inverted pendulum benchmark [36]

6.2 Double Pendulum

For double pendulum I am going to describe device invented for entertainment purposes and was extremely popular a couple of years ago, which is a spinner. However, construction of it is little bit more complicated than real one because of addition second hand for single pendulum, so we could get double inverted pendulum from it [46]. The main difference is existence of two centers and a couple of arms, compared with initial option where there was only one bearing which allowed individual to hold it and spin. Below the approximate construction is illustrated:

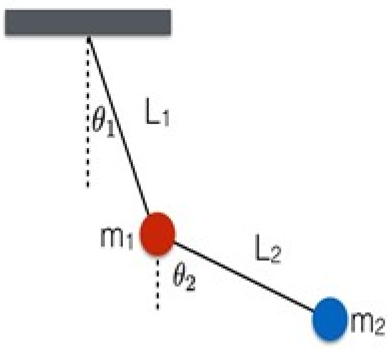

Figure 6.4: Double pendulum diagram [37]

In brief double is chaotic, because for instance if we take single pendulum and implement slight changes to original conditions, we do not notice dramatic changes in its movement. However in case of double constructions such changes lead to considerable modification of its movement. Consequently, dependence is associated in our case with chaotic system. It is possible to forecast the chaotic system' actions, but it is becoming harder and harder to do, in case if original conditions are precise. For double pendulum, when we do the modelling via two variables it is possible to illustrate orientation of construction fully. Lets imagine we have same lengths for arms, subsequently there are two angles θ_1 and θ_2. There are 2 certain issues which makes this construction complicated. Arms cause forces to attached masses, which are not invariable. They alter in course and sizes. It is not enough to apply equation of forces because they will act as it is needed in order to carry mass in a certain way. Next trouble is with angle under second arm which is counted from vertical branch which cannot describe entire motion. In short, good solution is to apply Langrangian to get equation of motion. Via these methods we can get angular acceleration, so it works for angles and their velocities. Unfortunately, we do not have easy answer for the movement of a couple of masses, so it is significant to count everything numerically to define its motion, what I did in practical part of this thesis [37].

Figure 6.5: Spinner based on double pendulum structure [37]

Double inverted pendulum is the main idea of constructed double spinner, and the control of it which I made in practical part of current thesis work demonstrates its movement and can be applied in construction of similar structures.

6.3 Double Inverted pendulum

Construction chosen as an application for simulation of double inverted pendulum is an orchards spray tower. There are 3 degrees of latitude in model I have chosen. External force in our model is an accidental act. However, in this exact case we have to take into account soil variabilities. For such purpose parametric probability technique. Through maximum entropy rule distribution of random variables was used. Such sprayers are used for spraying operation. There are a couple of fans and vibrative machinery which functions nonlinearly.

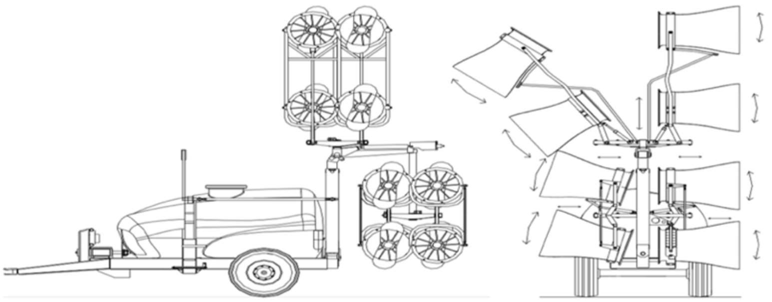

Figure 6.6: A spray tower in an orchard is mathematically modelled and dynamically investigated [38]

Such pulverizer attached to a car from a backside as it can be seen in the picture above. It has nonlinear dynamics, so as it has been mentioned before, soil abnormalities are considered in the system. In brief the whole construction is considered as double inverted pendulum. There are random variables in the systems which required implementation of parametric probabilistic technique. For external force amplitude and frequency was random, maximum entropy rule was used [39]. Movement of double inverted pendulum in construction of such spray tower can be seen in graphics I received in result of simulation. Results of simulation can be used for construction of pulverizer of new improved models.

6.4 Triple pendulum

The last model taken is model of triple pendulum. It is just combination of three simple ones, If we talk about constructions. Movement of such system is calculated via Lyapunov method in considered model. In brief human makes a lot of action in order to move. So integral factor of such moves of human is a balance. From past experiences, Newtonian mechanics is tool applied for modeling of movement of human. So in our case we take human skeletal system, or we can say exact part of this skeletal system, since it repeats the construction of triple pendulum [40]. As it can bee seen from results, I received in previous passage, simulation was made in Python. Professor Agranta [40] took human arm as an object of his research. Moreover, he was working on analyzing the movement of human arm during dancing, so he could make conclusion about its balance. Not obvious fact but extremely important is that human arm makes the most significant and hard moves, since it is normal state for human to keep the balance in brief, so arms are crucial in this process. If we act in any way by our body, balanced hands lead to longer balance state for us. Muscles, bones and nerval system are main participants in organization of body balance. Balance allows us to burn little amount of energy to act in. a certain way and helps not to face any troubles in face of various injuries [41]. Lest imagine vertical line, in our case it is a body itself. So in short gravity line is kept at maintenance pedestal with little oscillations. So in my model 3 part are taken as parts of triple pendulum. Kinetic and potential forces sum is taken to describe external forces [42, 43]. In current thesis I used Lagrange [44] algorithm to model the equation. So in results the movement of human arm can be seen and analyzed.

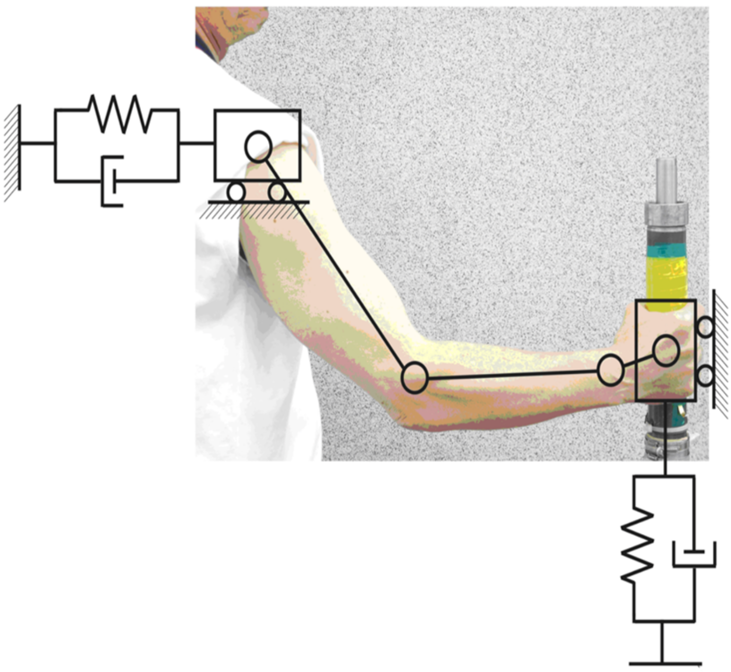

Figure 6.7: Hand-arm model for the operator - three-segment bar system in the shape of a triple pendulum [45]

Final triple pendulum is a models taken from arm movement and its construction. This simulation illustrated possible moves of it and can be applied in construction of things devoted for human use and requires keeping of human body via human arm.

CONCLUSION

As a result, I have received illustrated figures, after making simulations of single inverted, double inverted and simple triple pendulums. To solve the equations of motion i used the Runge Kutta method for solving sets of ordinary differential equa- tions. I have achieved the main purpose of this paper, which is creation of simulation of swinging pendulums. Inverted and Simple pendulums' equations of motions were applied in order to predict their swinging activity. Through this simulation we can see system's response and to get understanding about complexity of a rigid body in rotation. I reported examples with the aid of simple numerical simulations. Simulations in this work can be successfully applied in such area like robotics, balance controlling systems and human body movement imitation. The work was done via gathering of huge amount of theoretical data and analyzing articles which were done for similar topics before. Practical part included implementation of mathematical formulas, Python and Spyder. Via simulation it is possible to practice numerical tests which emulate live systems. While simulation we must into account extent of accordance of outputs among live system and simulated model. It is usually done in order to define changeability of estimators. Need of choosing of initial conditions and time makes it little more complicated to analyze results. Definitely, displacement of initial conditions can lead to incorrect results of simulations and conclusion obviously is going to be misinterpreted, whether it is for numerous runs or long-time ones. The range of simulation technique is quite wide so it can be applied in new areas where it has never been used before. The idea is quite easy to use since it is simple to understand. It bases on equations, and it considers initial conditions and time dependence. Usually there is a live system which we try to simulate basing on theoretical model, after which comparison experienced. Simulations applies mathematics and computer science. Nevertheless, models are illustrations of real system, and they can be of various types. For starting simulation process, it is required to formulate the issue after which we apply determined model in certain appropriate programming language. However, there are certain severeness such as: gives special results but not general, it is laborious work which requires time, it was required to be in touch with expert professor because of difficulty of interpretation and fixing errors while simulation. But there are some pluses as well, such as: constructing of model and simulating of it can result in improvements of systems, and better comprehension can be applied for verification of solutions.

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