ÓBUDAI EGYETEM
ÓBUDA UNIVERSITY

NEUMANN JÁNOS
FACULTY OF INFORMATICS

Annex 5

DEGREE PROJECT

THESIS

The Gravitational Tunnel Problem

OE-NIK	Student's name:	**Jetigen Bakyt uulu**
2022	Student's registration number:	**T/008789/FI12904/N**

Óbuda University
John von Neumann Faculty of Informatics
Institute of Applied Mathematics

DIPLOMA THESIS
TOPIC DESCRIPTION

Student name:	**Jetigen Bakyt uulu**
Registration number:	T/008789/FI12904/N
Neptun's code:	[redacted]

Title of diploma thesis:

The gravitational tunnel problem
A gravitációs alagút probléma

Internal supervisor:	Prof. Dr. Alexandru Kristály
External supervisor:	
Deadline:	15th of May, 2022.
Subjects of the final exam:	Engineering computational methods Optimization System and control theory Stochastics

Topic description

The topic belongs to the mainstream direction of Geometric Analysis and Calculus of Variations, combined with the theory of Partial Differential Equations. The thesis will provide an insight into the gravitation tunnel problem, whenever the geometric setting is provided by Riemannian-Finsler geometry.

The diploma thesis should cover:

- elements from Riemannian-Finsler geometry,
- geodesics,
- basic elements from Calculus of Variations, Euler-Lagrange equation,
- tunnel problem, classical approach,
- tunnel problem, Riemannian-Finsler approach,
- numerical tests.

ÓBUDAI EGYETEM Neumann János Informatikai Kar Ph. 1 Budapest

....
Dr. Péter Kárász
head of institute

Term of limitation: **15th of May, 2024.**

The diploma thesis is adequate and can be submitted:

....
external supervisor

....
internal supervisor

STUDENT'S DECLARATION

I the undersigned student hereby declare that this degree project / thesis is the result of my own work; I have disclosed the references and tools used in an identifiable manner. The results indicated in my degree project / thesis completed may be used by the university and the institution announcing the task for their own purposes free of charge.

Dated: Budapest, 4 May 2022

student's signature

Annex 8

Information on non-disclosure agreements

In a highly justified case, when required by the topic of the degree project / thesis, the external partner may initiate the classification of the degree project / thesis as confidential **at the time of announcing the topic (week 7)** in the form of a request addressed to the Deputy Dean of Studies but submitted to the Registrar's Office, which includes a detailed presentation of the reasons why the Degree project / thesis is requested to be classified as confidential.

In such case, the following shall be submitted together to the Registrar's Office:

- the student's request to have the degree project / thesis classified as confidential, having regard to the external partner's opinion, and
- the external partner's request, duly signed and sealed, which includes the reasons and information in view of which the degree project / thesis is requested to be classified as confidential.

Before such request is assessed by the Deputy Dean of Studies, the degree project / thesis may not be considered confidential.

ÓBUDAI EGYETEM
ÓBUDA UNIVERSITY

Neumann János Faculty of Informatics

Annex 9

CONSULTATION LOG

Student's name: Jetigen Bakyt uulu

Neptun code: [redacted]

Specialty: MSc Applied Mathematics

Phone number:

Mailing address (e.g. domicile):

[redacted]

Degree project / thesis[1] title in Hungarian:

A gravitációs alagút probléma

Degree project / thesis[2] title in English:

The gravitational tunnel problem

Institutional supervisor: Prof. Dr. Alexandru Kristály

External supervisor:

Please write the data in printed capital letters!

Occ.	Date	Content	Signature
1.	Mar 1, 2022	Plan the progress of the work	Kris
2.	Mar 29, 2022	Check the data in the thesis	Kris
3.	Apr 22, 2022	Consult about solutions	Kris
4.	May 6, 2022	Check the final version	Kris

The Consultation log is required to be countersigned by either supervisor on a total of 4 occasions of consultation.

The student has complied with the requirements of the subject titled Degree project I / Degree project II (BSc) or Thesis 1 / Thesis 2 / Thesis 3 / Thesis 4[3], and is allowed to proceed for reporting / defense[4].

Dated: Budapest, 9 MAY 2022

Kris

institutional supervisor

[1] Underline as appropriate
[2] Underline as appropriate
[3] Underline as appropriate
[4] Underline as appropriate

Abstract

The gravitational tunnel problem and its variation of a brachistochrone problem were introduced by P. Cooper in his paper "Through the Earth in Forty Minutes", American Journal of Physics. The solutions of this problem revealed the parametric equations of a hypocycloid. This finding was similar to the solution(the parametric equations of a cycloid) of the original brachistochrone problem. Following the Calculus of Variations, we look for the solution of the brachistochrone problem of the gravity train in different geometric settings, specifically, looking for the behavior of the minimal-time paths, expecting to find a hypocycloid-type of curve. We introduce the Finsler-Poincaré disk model, and substitute the Euclidean distance with the Finslerian distance. Numerical calculations are made to provide a better understanding of the curve.

Contents

Chapter 1

Introduction

In this paper, we study the gravity train by P. Cooper, 1966[1]. Cooper, inspired by Jules Verne's Journey to the Center of the Earth, presented calculations for a train going through the Earth in a frictionless tunnel. He suggested that it would take about 42 minutes and only require gravity as fuel. There were several variations of his solution, which gave an interesting insight into this kind of travel. At first, he calculated the straight line from one point through the center to the other side, and then by rotating the second half of the path to any point, he showed that the calculations hardly change. If we can travel to any point, then it would be rational to seek a straight path between the two points. Cooper showed that under gravity, the time it would take the train to go through this path would be approximately the same as the previous path through the center. And lastly, in pursuit of efficiency, he introduced a "minimal-time path" problem, one similar to a brachistochrone problem. That is, given two points A and B on a vertical plane, to find a curve starting from A and reaching B in the shortest time under gravity[2]. In Cooper's case, the problem was trickier because he was working with a sphere, where the "gravitational field was radial and not uniform"[1].

Cooper did not express an explicit solution to the latter, instead, he presented computer solutions in a view of a graph with several examples of the curve, it would take the train to travel between two points in the shortest time under gravity[1].

Several publications followed this one right away, suggesting modifications to the article. Some of them suggested explicit solutions to Cooper's brachistochrone problem[3]–[5]. They referred to the Calculus of Variations and based their solutions on the Euler-Lagrange equation.

The simplicity of this model gives an elegant solution to the brachistochrone problem in light of a hypocycloid[3]–[5]. Regardless, it raises questions about the model. Our attention focused on how the calculations to the brachistochrone problem would behave in other settings of this model. In hopes of breaking out of the euclidean geometry, we look for a solution in a Finsler-Poincaré disk model[6].

By changing the geometry, we replace the Euclidean notion of distance with the Finslerian distance[6]. In these settings, we will receive more complicated expressions. Following the result of Venezian, we will look for a solution to this brachistochrone problem[5]. The main question would be whether the new parametric equation is still behaving as a hypocycloid.

1.1 Literature review

The idea behind the gravity tunnel has been around before P. Cooper presented his calculations in 1966 [7][8]. Kirmser points out the fact that the time it takes to travel through the gravitational tunnel between any two points on the Earth was known back to 1898 [7], referencing a couple of old papers and books that mention these calculations. Cooper at the same time, confirms that a paper came to his knowledge, which describes the 42 minutes travel. The paper was entitled "De Paris A Rio-Janeiro en 42 minutes 11 secondes", from the journal La Nature [1]. Further commentaries to the paper included explicit solutions to the brachistochrone problem of Cooper[3][4][5]. All solutions were based on the Calculus of Variations.

Venezian's, and others, comments that the solution to the problem could be expressed using elementary functions[5]. Starting from conservation of energy following to the relation of the brachistochrone problem with the Euler-Lagrange equation, and showing the parametric equations of the solution. Venezian concludes by identifying the curve as a hypocycloid. He and Mallett noted that the exact solution is a special case brachistochrone cycloid[4]. Although Mallett also refers to the Euler-Lagrange equation, he seeks a different approach to the solution. In the end, it leads to a description of the parameters in more depth, unlike the previous solution. Interestingly, Mallett provides another fact that might intrigue the reader, in both variations of the brachistochrone problem "the time required to reach the lowest point" - of the minimal-time paths - "is the same as the time required to start from rest at any intermediate point of the path to descend to the lowest point"[4]. Laslett repeats almost the exact solution of Venezian but offers an interesting suggestion[3]. As it would be evident from the next chapter, the speed gained at the lowest point of the minimal-time curve (a hypocycloid) would highly affect the passengers of the gravity train. Laslett questions the reader what would be the g-force experienced by these passengers[3].

Another, more recent, comment on Cooper's gravity train was provided by M. Selmke [9]. He notes that, even though the papers commented on the lack of references to the 42 minutes path through the Earth, the idea of a gravity tunnel goes even further away in history. The tunnel through the Earth has been around since about the 2nd century AD [9]. Selmke claims, that the tunnel was mentioned by many since then, including Galileo. He especially describes that based on the experimentation with constant gravity, Galileo should have achieved the transit time of 54 minutes.

E. Parker in his paper [10], considers relativistic cases of gravity trains. He points out, that a particle going through the uniform Earth with a radial force of gravity takes an unrealistic harmonic oscillation, in other words, the properties of a hypocycloid are not relativistic as the initial model was highly simplified for this problem. He presents two ways to introduce the problem to relativistic settings. For the first case, he incorporates the particle into the Special Relativity, and for the second case, he incorporates the particle into the General Relativity[10]. In both cases, the property of a hypocycloid turns into something a lot more complicated, he explains. The relativistic harmonic oscillation is explained in the following way, a "relativistic particle spends almost all of its proper time near the turning points, but almost all of its coordinate time moving through" the Earth[10]. In other words, the time observed within the place of the particle motion is perceived mostly at the turning point of oscillation, whereas, the time observed away from the particle motion is perceived mostly throughout the Earth tunnels.

Chapter 2

Formulation of the problem

In this chapter we will look closer to the Brachistochrone problem. We will see its connection to the gravity train by Cooper [1]. And, in the end, we will see the model of our problem and the minimal-time path, that we are looking for.

2.1 Brachistochrone Problem

The brachistochrone problem was initially suggested in 1696 by Johann Bernoulli[2]. Erlichson explains that later in 1697, Bernoulli published his solution, which initially used "Fermat's optical principle of least time"[2]. The brachistochrone problem is one of the foundations of the calculus of variations, nowadays, the solution is based on the Euler-Lagrange equation(or Beltrami equation)[11], [12]. Let us see the construction of the problem and its calculations.

We follow the construction of the problem by Grasmair and then calculations by Fischer[11], [12]. Grasmair suggests the following mathematical model, we have two points $A = (0, 0)$ and $B = (a, b)$ on a plane, and we want to find a path, on which an object would slide from A to B in the shortest time, regardless of any friction(Figure 2.1). Assume $a > 0$, so that B is to the right of A, y points downwards and then $b > 0$ [11].

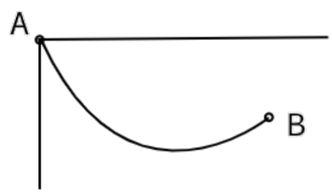

Figure 2.1: Sketch of the brachistochrone problem.

Now we are looking for the path in the form of a curve as following

$$x \mapsto \begin{pmatrix} x \\ y(x) \end{pmatrix}$$

with $y : (0, x) \to \mathbb{R}$ satisfying $y(0) = 0$ and $y(a) = b$. Grasmair claims, that this eliminates a major part of possible paths, not including our optimal solution. Now we are not looking for a path from A to B, but for a function y on $(0, a)$ satisfying boundary conditions. He notes that we will always assume that the optimal solution satisfies $y(x) \geqslant 0$ for all x[11].

Now we are deriving a formula for the travel time of an object from A to B given the function y, and according to Grasmair [11], due to conservation of energy, the following holds

$$v(x) = \sqrt{2gy(x)}$$

for all x (where g is the gravitational acceleration).
We denote by $s(x)$ the length of the path from 0 to $(x, y(x))$, s.t.

$$s(x) = \int_0^x \sqrt{1 + y'(\hat{x})^2} d\hat{x}.$$

Next, the length l of the whole path is

$$l = \int_0^a \sqrt{1 + y'(x)^2} dx.$$

We switch from variable x to t. By definition of velocity

$$v(t) = \frac{ds}{dt}$$

or

$$\frac{dt}{ds} = \frac{1}{v(s)}.$$

Consequently, by denoting the total travel time by T, Grasmair obtained

$$T = \int_0^T dt = \int_0^L \frac{1}{v(s)} ds = \int_0^a \frac{1}{\sqrt{2gy(x)}} \sqrt{1 + y'(x)^2} dx$$

Thus, we formulate our problem as the minimization of the functional

$$F(y) := \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} dx \tag{2.1}$$

with constraints $y(0) = 0$ and $y(a) = b$ [11].

If we consider F to have the form

$$F(y) = \int_0^a L(x, y(x), y'(x)) dx$$

with some function L defined on $C^2_{[0,a]}$. In this case, L is

$$L(x, y, y') = \sqrt{\frac{1 + y'^2}{2gy}}.$$

This leads us to an Euler-Lagrange equation[11].

$$\frac{\partial L}{\partial x} - \frac{d}{dx}\left(\frac{\partial L}{\partial x'}\right) = 0.$$

Notice that the integral of (2.1) does not depend directly on variable x, therefore, we can apply Beltrami Identity [12]

$$L - y'\frac{\partial L}{\partial y'} = C \tag{2.2}$$

where C is a constant.
Substitute and the resulting equation is

$$\sqrt{\frac{1 + y'^2}{2gy}} - \frac{1}{2}y'\left(\sqrt{\frac{2gy}{1 + y'^2}}\right)\frac{y'}{gy} = C.$$

Simplification gives

$$(1 + y'^2)y = \frac{1}{2gC^2} = k^2.$$

According to Fischer, this equation is well-known and the solution is a cycloid[12]. The parametric equations are

$$x(\theta) = \frac{1}{2}k^2(\theta - \sin\theta)$$
$$y(\theta) = \frac{1}{2}k^2(1 - \cos\theta)$$

These parametric equations are shown in the Figure 2.2. It means that our solution would follow a cycloid to reach point B in the shortest time possible under the gravity.

Cooper, in his findings [1], offers another variation of the Brachistochrone problem, the next chapter covers everything about it.

2.2 Gravity Train

The solution P. Cooper provided for the gravity train gathered a lot of attention due to the relation to the brachistochrone problem. In his approach to a brachistochrone problem, he was not successful in representing the solution using elementary functions. This is why, the papers that followed pointed out the exact solutions

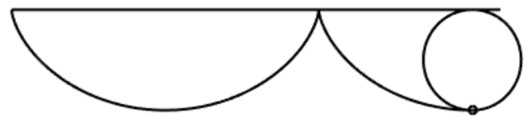

Figure 2.2: Sketch of a cycloid.

to his differential equation[3]–[5]. We break down Cooper's model and the modifications, which followed the original paper.

The model is described in the following way. By making passages through the Earth, Cooper suggests that it would be possible for men to rapidly travel in-between continents[1]. The tunnels have to be friction-less, so that an object would travel only under gravity, in no expense of any fuel. The Earth is considered to be a sphere with uniform density, radius $R = 3955$ miles (≈ 6365 kilometers), and surface gravitational constant $g = 32.16$ ft/sec^2 ($= 9.81$ m/sec^2). The radial gravitational force field is $f = -rmg/R$, where r is radial distance from the center, and m is mass. And then the idea is simple, Cooper explaining, fall into the hole on one end, exchange the gravitational potential energy to motion, reverse the exchange, and then stop on the other side of the passage[1].

Now, we go straight to Cooper's calculations[1]. We consider a straight line AC through the center O of the Earth. Starting from the surface, the velocity is

$$v = \sqrt{\frac{g}{R}(R^2 - r^2)},$$

with its maximum, at the center O, $\sqrt{gR} = 17700$ mph (≈ 28485 kph). According to Cooper, the free-fall time for this path AOC is

$$T = \pi\sqrt{\frac{R}{g}} \approx 42.2min.$$

Next, Cooper considers a different path AOB, where B is any other point on the Earth. The traveling time remains the same, but the tunnel would need a sufficiently smooth curve in the center O, so that the train does not crash[1]. In case of a straight path from A to B, Cooper claims that the time T is again ≈ 42.2 minutes, regardless of points involved (Figure 2.3).

Now, Cooper considers that the line OB is perpendicular to path AOC [1]. He suggests, that if we put a point D on the surface in-between two points B and C

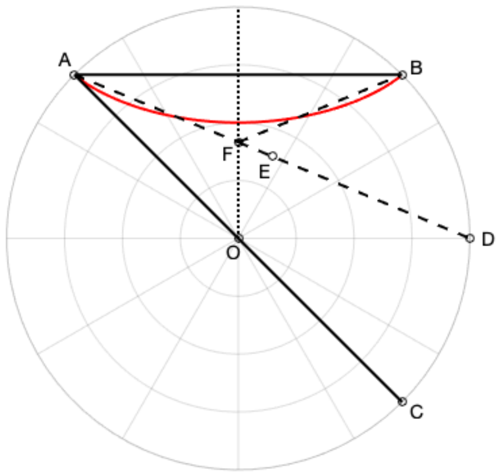

Figure 2.3: Sketch of the tunnels for the Gravity Train with a hypocycloid(red). Adapted from "Through the Earth in Forty Minutes," by P. W. Cooper, 1966, American Journal of Physics, 34, 68-70.

(Figure 2.3), then the transit time stays the same for paths AOC and AD. He puts a point E in the middle of the line AD, and a point F on the path AD where it intersects with a bisector of the angle AOB. Then the line AF should be shorter than then line AE of the line AF. If it takes $T/2$ for the path AE, it takes even less time for the path AF. Cooper notes that even though, the transit time for paths AOB and AB does not change, the path AFB requires less time.

So, once Cooper finds a shorter path from A to B, he leads us to a brachistochrone problem. He looks for the minimal-time path from A to B. Here, Cooper depending on the Calculus of Variations, presents a differential equation for such path, more precisely a curve. Unfortunately, he did not present an explicit solution[1].

Now, we consider G. Venezian paper regarding Cooper's differential equation of a curve[5]. Venezian pointed out that the solution "can be given in terms of elementary functions". The solution starts again with the equation for the conservation of energy, which gives us

$$v^2 = gR - gr^2/R. \tag{2.3}$$

Venezian considers a path $r(\theta)$. Then the time minimization problem is

$$t = \int_A^B \frac{ds}{v} = \int_0^{\theta_{AB}} \sqrt{\frac{R(r^2 + r'^2)}{g(R^2 - r^2)}} d\theta. \tag{2.4}$$

where r' stands for $dr/d\theta$. It leads us to an Euler-Langrange equation

$$L = \sqrt{\frac{r^2 + r'^2}{R^2 - r^2}},$$

and since L does not depend directly on θ, we can use a Beltrami identity (2.2)

$$L - r' \frac{\partial L}{\partial r'} = C, \tag{2.5}$$

where C is a constant, that can be expressed at radius $r = \rho$, where $r' = 0$ (the lowest point on curve $r(\theta)$)

$$C = L(\rho, 0) = \frac{\rho}{\sqrt{R^2 - \rho^2}}. \tag{2.6}$$

Now rewriting (2.5) and substituting (2.6) gives a separable ordinary differential equation, which leads to

$$d\theta = \frac{\rho}{R} \sqrt{\frac{R^2 - r^2}{r^2 - \rho^2}} \frac{dr}{r}.$$

Venezian presents the solution to this equation as

$$\theta = \arctan\left(\frac{R}{\rho}\sqrt{\frac{r^2 - \rho^2}{R^2 - r^2}}\right) - \frac{\rho}{R}\arctan\left(\sqrt{\frac{r^2 - \rho^2}{R^2 - r^2}}\right), \tag{2.7}$$

where $\theta = 0$ if $r = \rho$, and $\theta = \frac{\pi}{2}(1 - \frac{\rho}{R})$ if $r = R$[5]. Then, the total angle between two points is

$$\theta_{AB} = \pi(1 - \frac{\rho}{R}). \tag{2.8}$$

Now, if we rewrite dt using r and dr, we get

$$dt = \sqrt{\frac{R^2 - \rho^2}{Rg}} \frac{r}{\sqrt{(R^2 - r^2)(r^2 - \rho^2)}} dr.$$

The solution to this equation is

$$t = \frac{1}{2}\sqrt{\frac{R^2 - \rho^2}{Rg}} \arccos\left(\frac{R^2 + \rho^2 - 2r^2}{R^2 - \rho^2}\right), \tag{2.9}$$

where t was also evaluated at the lowest point of $r(\theta)$, which gives us the total travel time between two points

$$T = \pi\sqrt{\frac{R^2 - \rho^2}{Rg}}.$$

According to Venezian [5], from (2.7) and (2.9) we get the parametric equations of the trajectory

$$\begin{aligned} \theta &= \arctan\left(\frac{R}{\rho}\tan\omega t\right) - \frac{\rho}{R}\omega t, \\ r^2 &= \frac{1}{2}(R^2 + \rho^2) - \frac{1}{2}(R^2 - \rho^2)\cos 2\omega t, \end{aligned} \tag{2.10}$$

where $\omega = \pi/T$. Venezian states, "these parametric equations represent a hypocycloid generated by a circle of radius $\frac{1}{2}(R-\rho)$ rolling at a constant speed inside a circle of radius R"[5].

In the next section, we will introduce the model and the problem in question itself.

2.3 Finsler-Poincaré disk model

We focus completely on a Finsler-Poincaré disk model represented by D. Bao and others [6]. The example of such a model is called a Poincaré disk that is only forward complete, we follow the construction of the model. We start on a Euclidean open disk with radius 2 and center $(0,0)$ in $\mathbb{R}^2$. We introduce polar coordinates

$$X = r\cos\theta$$
$$Y = r\sin\theta.$$

Thus, getting the manifold

$$M := \{(X,Y) \in \mathbb{R}^2 : r^2 = X^2 + Y^2 < 4\}.$$

They proceed by establishing two more pieces of data, a Riemannian metric $\tilde{a}$ and a 1-form $\tilde{b}$ defined globally on M, we get a Randers space [6]. They choose $\tilde{a}$ as a Poincaré disk (with constant Gaussian curvature -1)

$$\tilde{a} := \frac{1}{(1+\frac{r^2}{4})^2}[dr \otimes dr + r^2 d\theta \otimes d\theta].$$

This Riemannian metric was taken from B. O'Neill, where it is described in more details [13]. And the 1-form $\tilde{b}$ as

$$\tilde{b} := d\left(\log\left[\frac{4+r^2}{4-r^2}\right]\right) = \frac{r}{(1+\frac{r^2}{4})(1-\frac{r^2}{4})}.$$

According to Yasuda-Shimada, "a Randers space has constant negative flag curvature $-\frac{\lambda^2}{4}$ if and only if"[14]
1) The Riemannian metric $\tilde{a}$ has constant negative sectional curvature $-\lambda^2$.
2) The 1-form $\tilde{b}$ is exact and satisfies the system of PDEs

$$\tilde{b}_{i,x^j} - \tilde{b}_k\tilde{\gamma}^k_{ij} = \lambda(\tilde{a}_{ij} - \tilde{b}_i\tilde{b}_j).$$

Bao and others show proof for both statements[6], in the example $\lambda = 1$. They reduce the choice of f to only those with r dependece, and get the following two

ODEs

$$f_r = \frac{16r}{(4+r^2)(4-r^2)},$$
$$f_{rr} - \frac{2r}{4-r^2} f_r = \frac{16}{(4-r^2)^2} - f_r f_r.$$

The solution for the first eqaution

$$f(r) = \log\left[\frac{4+r^2}{4-r^2}\right] + C,$$

where C is an arbitrary constant and "f satisfies the second ODE identically"[6].

Now they construct the Finsler function, if V is any tangent vector to M, then the Finsler function is

$$F(V) := \sqrt{\tilde{a}(V,V)} + \tilde{b}(V).$$

According to Bao and others, for this model the Riemannian norm of $\tilde{b}$ has to be less than 1 on M. They show that since $0 \leq r < 2$, then

$$||\tilde{b}|| = \frac{4r}{4+r^2} < 1.$$

According to [6], it proves that the Randers F is positive and strongly convex on $TM\backslash 0$. And therefore, F is a y-global Finsler metric.

For an arbitrary point on M, they expressed the tangent vector V with polar coordinates (r, θ) as

$$V := p\frac{\partial}{\partial r} + q\frac{\partial}{\partial \theta}.$$

Then the Finsler function for V is

$$F(V) := \frac{1}{1-\frac{r^2}{4}}\sqrt{p^2+r^2q^2} + \frac{pr}{(1-\frac{r^2}{4})(1+\frac{r^2}{4})}.$$

According to computer calculations, this F has constant negative Gaussian curvature $-\frac{1}{4}$[6].

Now we consider the Geodesics of our Randers space. According to Bao and others, our geodesics follow the trajectories of a Riemannian metric $\tilde{a}$[6]. These trajectories are well-studied by O'Neill, according to [13], they are the Euclidean straight lines that go through the origin and the circular arcs that intersect the boundary of the Poincare disk under the Euclidean right angles.

Based on what we studied, the question we are looking for an answer to is whether the solution to the time-minimization problem for the gravity train would behave similarly in the newly introduced model of the Finsler-Poincaré disk. We start with the time minimization problem (2.4) and the conservation of evergy (2.3). Substituting the Euclidean measure of distance with Finslerian arc length of some curve c that goes from A to B:

$$T = \int_A^B \frac{ds}{v} = \int_A^B \frac{F(\bar{c})dt}{v} \tag{2.11}$$

Chapter 3

Calculations

In this chapter, we show the calculations of a brachistochrone problem on a Finsler-Poincaré disk. By changing geometry, we use Finsler distance, instead of Euclidean distance, and evaluate the path between two points on our model. The expressions should be more complicated, we might not get explicit form. In that case, we will either use computer calculations to analyze the solution or examine the problem in Riemannian geometry(weaker setting).

3.1 Construction of the ODE

We continue from the last equation we obtained in the previous chapter (2.11). Substituting the Euclidean measure of distance with Finslerian arc length:

$$T = \int_A^B \frac{F(\bar{c})dt}{v} = \int_0^{\theta_{AB}} \sqrt{\frac{R}{g(R^2-r^2)}} \left[\frac{\sqrt{r'^2+r^2}}{1-\frac{r^2}{R^2}} + \frac{r'r}{(1-\frac{r^2}{R^2})(1+\frac{r^2}{R^2})} \right] d\theta$$

$$= \int_0^{\theta_{AB}} \sqrt{\frac{R}{g}} R^2 \left[\frac{\sqrt{r'^2+r^2}(R^2+r^2)+R^2r'r}{\sqrt{R^2-r^2}(R^4-r^4)} \right] d\theta \tag{3.1}$$

where r' stands for $dr/d\theta$, for some curve c

$$c = (r(t)cos(t), r(t)sin(t))$$

and its velocity according to [6] as

$$\bar{c} = \frac{dc}{dt} = \frac{dr}{dt}\frac{\partial}{\partial r} + \frac{d\theta}{dt}\frac{\partial}{\partial \theta}.$$

We can apply Euler-Lagrange equation to the integral (3.1), where

$$L = \frac{\sqrt{r'^2+r^2}(R^2+r^2)+R^2r'r}{\sqrt{R^2-r^2}(R^4-r^4)}$$

And since L does not have θ explicitly, we can reduce the problem to a Beltrami equation (2.5). Finding

$$\frac{\partial L}{\partial r'} = \frac{r'(R^2+r^2) + R^2 r\sqrt{r'^2+r^2}}{\sqrt{r'^2+r^2}\sqrt{R^2-r^2}(R^4-r^4)}$$

gives us

$$L - r'\frac{\partial L}{\partial r'} = \frac{r^2}{\sqrt{r'^2+r^2}\sqrt{R^2-r^2}(R^2-r^2)} = C.$$

where C is once again a constant, that we express using $r = \rho$ and $r' = 0$. We obtain

$$\frac{\rho}{\sqrt{R^2-\rho^2}(R^2-\rho^2)} = C.$$

Now we can combine the last two equations

$$r^2\sqrt{R^2-\rho^2}(R^2-\rho^2) = \rho\sqrt{r'^2+r^2}\sqrt{R^2-r^2}(R^2-r^2)$$

And then we formulate a first order nonlinear ordinary differential equation

$$r' = r\sqrt{\frac{r^2(R^2-\rho^2)^3}{\rho^2(R^2-r^2)^3} - 1} \tag{3.2}$$

Following the calculations of Venezian and others[3][4][5], we tried to solve this ODE by substitution and work out an explicit solution. Unfortunately, we were not able to do so as the ODE is more complicated. This is why, we turned to MatLab calculations, as the original problem by Cooper [1] also provides computer representation of its differential equation. We have also made graphics in comparison with other types of curves, hypocycloids and geodesics, for a better understanding. The details are described in the rest of the chapter.

3.2 Parameters

Now, we will discuss the parameter specifications for computer simulations. We based all solutions on (we used as the initial value) the parameter ρ, a minimum radius our curve takes where $r' = 0$, which is dependent on the parameters θ_{AB}, the total angle between two points A and B. This relation for all three types of curves is different, but since we know only the dependence for hypocycloids and geodesics, let us calculate θ_{AB} in such a way that ρ is the same for all three types of curves. For convenience in this section, θ_{hyp} will stand as the total angle between two points for the hypocycloid, and θ_{geo} will stand as the total angle between two points for the geodesics.

Variables	Values		
R	2		
ϕ	$\pi/6$	$\pi/4$	$\pi/3$
R_c	$2\sqrt{3}$	2	$2/\sqrt{3}$
θ_{geo}	$2\pi/3$	$\pi/2$	$\pi/3$
ρ	$4-2\sqrt{3}$	$2\sqrt{2}-2$	$2/\sqrt{3}$
θ_{hyp}	$\pi(\sqrt{3}-1)$	$\pi(2-\sqrt{2})$	$\pi(1-1/\sqrt{3})$

Table 3.1: Parameter values for computer simulations.

The dependence for the hypocycloid was shown in the gravity train problem 2.8 and it is easily computable. The formula is

$$\theta_{AB} = \pi(1 - \frac{\rho}{R})$$

Following Cooper's idea behind the shortest path between two points[1], we looked into some geodesics of our model. As it was mentioned previously, the length-minimizing trajectories of the Finsler-Poincaré disk model are the same as the geodesic trajectories of the Poincaré disk inside the model[6]. O'Neill discusses these geodesics more closely [13], he explains that they are the circular arcs that intersect the rim of the Poincaré disk at the right angles, which in our case directly influence θ_{AB}. Consequently, by studying the parameters of these geodesics, we were able to visualize them in the next section.

The dependence for the geodesic is a bit trickier, so, let us focus on it. This relation is developed in the following way. O'Neill claims the following, ρ is dependent on R and R_c,

$$\rho = \sqrt{R^2 + R_c^2} - R_c$$
$$R_c = R \cot\phi$$

where R is the radius of our model and R_c is the radius of the circle that intersects the rim of the Poincaré disk at the right angles. ϕ is a half of the total angle between two points on the above-mentioned circle[13], and here comes our dependence on θ_{geo}.

$$\phi = \frac{\pi}{2} - \frac{\theta_{geo}}{2}$$

We choose ϕ s.t. $\cot\phi$ are known values. Then we calculate R_2, θ_{geo} and finally ρ. All these values are represented in Table 3.1.

Once we found our desired ρ, we substituted its values to our solution of the ODE (3.2) and to the parametric equations (2.10) of hypocycloids using (2.8).

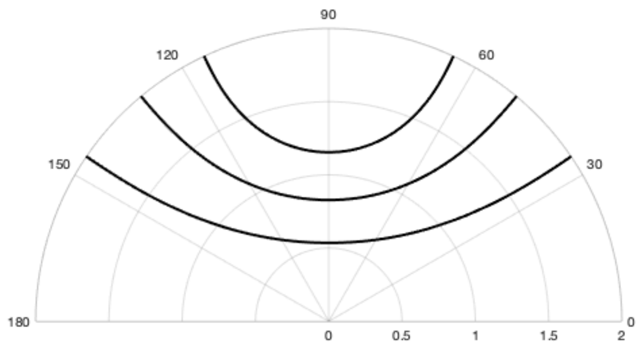

Figure 3.1: Solution of minimal-time paths with several starting points.

3.3 Simulations

In this section, we examine the behaviour of the path in our Finsler-Poincaré disk. The original problem in Cooper's paper has some elegant properties and we would like to see what happens with them under different circumstances [1]. Even though, we were not able to identify the minimal time curve, the simulations of our ODE help us to study this curve more closely.

A program has been developed using MatLab built-in functions for calculations of Ordinary Differential Equations. We were able to simulate the behavior of our ODE (Fig. 3.1). Additional graphics were produced for a visual analysis of the solution.

Figure 3.2 is a comparison of our solution with the curves of hypocycloids, the solution to the Cooper's brachistochrone problem. We used the parametric equations (2.10) provided by Venezian to reproduce the hypocycloids. Initially, we were examining if the minimal time curve in different geometric setting would be a type of a hypocycloid. The visual representation shows that the ODE does not behave like a hypocycloid.

Figure 3.3 is a comparison of our solution with the curves of geodesics of the Poincaré disk inside the Riemannian metric $\tilde{a}$. In a certain sense, it was expected that the solution would not obtain the curves of geodesics since we are working on a time-minimizing problem under the force of gravity. And as seen in the graphic, the solution does not behave like a geodesic.

All graphics in this section were produced using MatLab, see the program code in Appendix A.

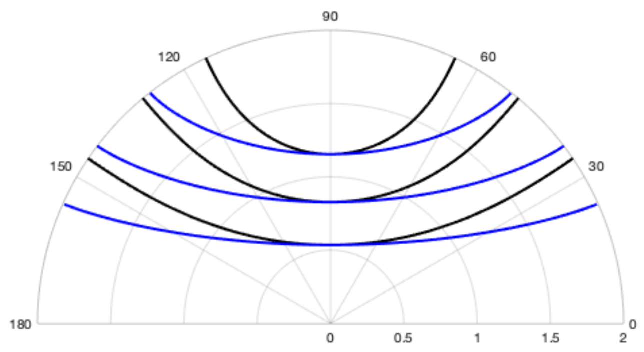

Figure 3.2: Minimal-time paths(black) vs Hypocycloids(blue).

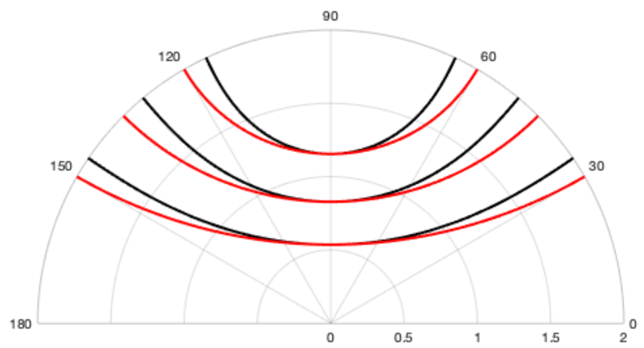

Figure 3.3: Minimal-time paths(black) vs Geodesisc(red).

Chapter 4

Conclusion

To obtain a solution to the time-minimizing problem of the gravity train in the Finsler-Poincaré disk model, a measure of Euclidean distance was replaced with Finslerian arc length based on the Finsler function of the model. After the construction of our problem, a reduced Euler-Lagrange equation was established. Following the Calculus of Variations, we obtained a first-order nonlinear ordinary differential equation. The behavior of the obtained ODE was studied under computer calculations. After reproducing solutions for several curves of our ODE, we studied the parameters of the curves. We compared them with the parameters of a known solution to the gravity train, equations of hypocycloids, and with the parameters of a known geodesic trajectory of our model. We found that our solution does not coincide with a hypocycloid, which was the initial assumption of this study, and does not coincide with a geodesic, which was expected since we are working on a time-minimization and not on a length-minimization problem. The computer-generated graphs show support for this statement. Despite, the lack of explicit solutions, simulation of our ODE provides useful information regarding the analysis of the curve.

Since further studies of our ordinary differential equations are required, we would like to describe the details for the future development of this problem. The explicit solutions could be retrieved using the Calculus of Variation, a simplification of the model to the Riemannian geometry might reveal some helpful aspects. After that, substitution to the minimal-time integral would reveal the parametric equations of the curve, at this point one might identify the curve if possible. Another way, to achieve this, could be to analyze the solution of the ODE by numerical methods, e.g. using polynomial interpolation to study the properties of the curve.

Bibliography

[1] P. W. Cooper, "Through the earth in forty minutes," *American Journal of Physics*, vol. 34, no. 1, pp. 68–70, 1966.

[2] H. Erlichson, "Johann bernoulli's brachistochrone solution using fermat's principle of least time," *European journal of physics*, vol. 20, no. 5, p. 299, 1999.

[3] L. J. Laslett, "Trajectory for minimum transit time through the earth," *American Journal of Physics*, vol. 34, no. 8, pp. 702–703, 1966.

[4] R. L. Mallett, "Comments on "through the earth in forty minutes"," *American Journal of Physics*, vol. 34, no. 8, pp. 702–702, 1966.

[5] G. Venezian, "Terrestrial brachistochrone," *American Journal of Physics*, vol. 34, no. 8, pp. 701–701, 1966.

[6] D. Bao, S.-S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*. Springer Science & Business Media, 2000, vol. 200.

[7] P. G. Kirmser, "An example of the need for adequate references," *American Journal of Physics*, vol. 34, no. 8, pp. 701–701, 1966.

[8] P. W. Cooper, "Further commentary on "through the earth in forty minutes"," *American Journal of Physics*, vol. 34, no. 8, pp. 703–704, 1966.

[9] M. Selmke, "A note on the history of gravity tunnels," *American Journal of Physics*, vol. 86, no. 2, pp. 153–153, 2018.

[10] E. Parker, "A relativistic gravity train," *General Relativity and Gravitation*, vol. 49, no. 8, pp. 1–22, 2017.

[11] M. Grasmair, *Basics of calculus of variations*, 2015.

[12] J. Fischer, *Introduction to the calculus of variations*, 1999.

[13] B. O'neill, *Elementary differential geometry*. Elsevier, 2006.

[14] H. Yasuda and H. Shimada, "On randers spaces of scalar curvature," *Reports on Mathematical Physics*, vol. 11, no. 3, pp. 347–360, 1977.

Appendix A

Program

The following program produced Figures 3.13.23.3. The parameters were estimated according to Table 3.1, see the description in the Section 3.2. By choosing the desired curve, one should comment and/or uncomment three parts of the code. The functiones called in the program are also included in this Appendix A, see them below.

```
R1 = 2.0;
g = 9.81;
phi = [pi/6 pi/4 pi/3];
R2 = R1 .* cot(phi);
theta = pi/2 - phi;
C = sqrt(R1^2 + R2.^2);
rho = C - R2;

for i = 1:3
    % MY ODE SOLUTION
    setRho(rho(i));
    y0 = getRho + 0.000001;
    [T, Y] = ode45('ODE', [pi/2 pi/2+theta(i)], y0);

    polarplot(T, Y, 'k-','LineWidth',2);
    hold on
    polarplot(-T, -Y, 'k-','LineWidth',2);

%     % GEODESICS
%     t = linspace(3*pi/2, 3*pi/2+phi(i));
%     x = R2(i) .* cos(t);
```

```
%       y = C(i) + R2(i) .* sin(t);
%       [T2, Y2] = cart2pol(x, y);
%
%       polarplot(T2, Y2, 'r-','LineWidth',2);
%       polarplot(-T2, -Y2, 'r-','LineWidth',2);
%
%       % HYPOCYCLOID
%       if i == 3
%           t = linspace(0, 0.55);
%       else
%           t = linspace(0, 0.64);
%       end
%       w = sqrt(R1 * g) / sqrt(R1^2-getRho^2);
%       T3 = atan((R1/getRho)*tan(w*t)) - (getRho/R1)*w*t;
%       Y3 = sqrt(0.5*(R1^2+getRho^2) - ...
%           0.5*(R1^2-getRho^2)*cos(2*w*t));
%
%       polarplot(T3 + pi/2, Y3, 'b-','LineWidth',2);
%       polarplot(-T3-pi/2, -Y3, 'b-','LineWidth',2);
end
rlim([0 2])
thetalim([0 180])
hold off
```

The following function sets the global variable rho:

```
function setRho(r0)

global rho
rho = r0;
```

The following function returns the global variable rho:

```
function r = getRho()

global rho
r = rho;
```

The following function returns the first derivative 3.2:

```
function yp = ODE(t, y)

R = 2.0;
```

```
rho = getRho;

yp = y * sqrt((y^2 * (R^2 - rho^2)^3) / ...
     (rho^2 * (R^2 - y^2)^3) - 1);
```