ÓBUDAI EGYETEM
ÓBUDA UNIVERSITY

NEUMANN JÁNOS
FACULTY OF INFORMATICS

Annex 5

INTRODUCTION TO LIE ALGEBRAS

THESIS

OE-NIK	Student’s name:	**Daniel Mjomba Nderitu**
2022	Student’s registration number:	**T/008804/FI12904/N**

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DIPLOMA THESIS
TOPIC DESCRIPTION

Student name: **Daniel Mjomba Nderitu**
Registration number: T/008804/FI12904/N
Neptun's code: [redacted]

Title of diploma thesis:

Introduction to Lie algebras
Bevezetés a Lie-algebrákba

Internal supervisor: Dr. Magdolna Szőke
External supervisor:

Deadline: 15th of May, 2022.

Subjects of the final exam: Engineering computational methods
Optimization
System and control theory
Stochastics

Topic description:

The concept of Lie algebra was introduced in the 1870s. Since then, a beautiful theory of Lie algebras has been worked out. As an example, classification of simple Lie algebras can be mentioned. Furthermore, Lie algebras have several applications. In this thesis, the graduand shall give an introduction to this nice topic from an algebraic point of view.

The diploma thesis should cover:

- basic definitions concerning Lie algebras (notion of Lie algebra, subalgabras, ideals, nilpotency, solubility, simplicity, etc.),
- a number of examples of Lie algebras,
- a number of examples explaining the concepts concerning Lie algebras,
- solution of some exercises chosen and worked out by the graduand,
- presentation of some applications of Lie algebras.

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Ph.

....

Dr. Péter Kárász
head of institute

Term of limitation: **15th of May, 2024.**

The diploma thesis is adequate and can be submitted:

....
external supervisor

....
internal supervisor

STUDENT'S DECLARATION

I Daniel Mjomba Nderitu the undersigned student hereby declare that this thesis is the result of my own work; I have disclosed the references and tools used in an identifiable manner. The results indicated in my thesis completed may be used by the university and the institution announcing the task for their own purposes free of charge.

Dated: Budapest, 13 MAY 2022

student's signature

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Log of consultations

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INTRODUCTION TO LIE ALGEBRAS

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Ocassion	Date	Topics discussed	Signature
1.	14. 09	INTRODUCTION, LITERATURE	
2.	19. 10	BASIC DEFINITIONS	
3.	12. 11	NILPOTENT LIE ALGEBRAS	
4.	10. 12	REPRESENTATIONS	

This log should be signed on each consultation (by a supervisor), 4 times altogether.

The student fulfilled the requirements of the Thesis work subject, I allow his/her participation on the presentation/defence[1].

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Budapest, 2021 December, 15.

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Ocassion	Date	Topics discussed	Signature
1.	15· 02	SIMPLICITY	
2.	18· 03	SOME EXAMPLES	
3.	12· 04	DYNKIN DIAGRAMS	
4.	16 · 05	FINAL REVIEW / CORRECTIONS	

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Budapest, 20.22. MAY 13.

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DEDICATION

This thesis is dedicated to my devoted mother and beloved wife, without whom this thesis paper would not have been possible. They never fail to inspire me. At the same time, I'd like to thank my siblings, their advice was invaluable for this thesis paper.

ACKNOWLEDGMENT

This project would not have been possible without the support of my supervisor Dr. Magdolna Szőke who read my numerous revisions and helped make some sense of the confusion. Thank you for your patience, guidance, and support. I have greatly benefited from your wealth of knowledge.

ABSTRACT

Since its introduction in the 19^{th} century, the concept of Lie algebra has gradually evolved into an important tool in the study of local group actions on smooth manifolds. When we have a local group action on a smooth manifold, we obtain a Lie algebra of vector fields from which we can reconstruct the local group action. As a result, the Lie algebra is more easily accessible than the group.

In this study the concept of Lie algebras is introduced from an algebraic point of view. The focus here is on introducing the concept of a Lie algebra and its basic properties, as well as identifying common concrete examples of Lie algebras.

Contents

List of Figures

Chapter 1

Introduction

Sophus Lie introduced Lie algebras in the 19^{th} century as algebraic structures used to study the concept of infinitesimal transformations, also known as Lie groups and their associated Lie algebras. The tangent space of a Lie group at the identity element has the natural structure of a Lie algebra which is known as the infinitesimal group.

Sophus Lie began researching on all possible local group actions on manifolds in the 19^{th} century. His plan was to investigate the action infinitesimally. If we have a local action by $\mathbb{R}$ on a smooth manifold, it results in a vector field on the manifold which integrates to capture the action of the local group. In the general case, we obtain a Lie algebra of vector fields, which allows us to reconstruct the local group action. Because it is a linear object, the Lie algebra is more easily accessible than the group. As a result, before classifying all group actions, one should first classify all finite dimensional real Lie algebras. The gradual evolution of Lie's ideas demonstrated the importance of determining all simple Lie algebras.

This study aimed at introducing Lie algebras, with topics ranging from basic definitions and examples of Lie algebras,to solvable and nilpotent Lie algebras and their representations, to the theory of root systems and classification of simple Lie algebras of finite dimension. This study helps us to appreciate the fact that working with Lie algebra is much easier than working directly with the corresponding Lie group.

The classical algebras play an important role in the study of Lie algebras. In this study we will use them to identify simple Lie algebras, derive isomorphism and determine the solvability of simple Lie algebras. The study of Lie Algebra needs an in-depth understanding of linear algebra, ring theory and group theory. Unless otherwise stated, our basic definitions for the introduction Chapter was derived from the book Lie algebras of finite and affine type by Roger William Carter. [Car05]

Definition 1.0.1 (**Associative algebra**). An algebra A is said to be an associative algebra over a field $\mathbb{F}$, if A is a vector space over $\mathbb{F}$ together with a $\mathbb{F}$-bilinear multiplication $A \times A \to A$ such that the associative law holds:

$$(xy)z = x(yz) \quad \forall\, x, y, z \in A.$$

Bilinearity is defined as follows:

$$\begin{aligned} (x+y)z &= xz + yz \qquad & x(y+z) &= xy + xz \\ (cx)z &= c(xz) & x(cz) &= c(xz) \end{aligned}$$

For all $x, y, z \in A$ and $c \in \mathbb{F}$.

If we have an identity element 1 in A such that $x1 = 1x = x \quad \forall x \in A$, then A is called a unitary associative algebra. This kind of algebras contains all elements a of the field $\mathbb{F}$ by identification with $a1$.

Example 1.0.2. Examples of associative algebras.

- Complex numbers over the real numbers form a 2-dimensional unitary associative algebra.
- $n \times n$ square matrices with entries from the field $\mathbb{F}$ form a unitary associative algebra over $\mathbb{F}$
- Polynomials with coefficients from any field $\mathbb{F}$ form a unitary associative algebra over $\mathbb{F}$.

Next we define the binary operation $[*, *]$ called the *Commutator*.

Definition 1.0.3 (**Commutator**). The commutator of two elements x and y of an algebra is defined as

$$[x, y] = xy - yx.$$

Claim 1.0.4. The commutator has the following properties:

1. Alternativity: $\quad [x, x] = 0.$
2. Anticommutativity: $\quad [x, y] = -[y, x].$
3. Jacobi Identity: $\quad [x, [y, z]] + [y[z, x]] + [z, [x, y]] = 0.$

Proof: ***Alternativity:*** $[x, x] = 0.$

$$[x, y] = xy - yx,$$

hence

$$[x, x] = xx - xx = 0.$$

Anticommutativity:

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y]$$

since $[x, x] = 0$ and $[y, y] = 0$

$$0 + [x, y] + [y, x] + 0 = 0$$

hence

$$[x, y] = -[y, x]$$

Jacobi Identity: $[x,[y,z]]+[y[z,x]]+[z,[x,y]]=0$

$$\begin{aligned}[[xy]z]+[[yz]x]+[[zx]y] &= (xy-yx)z-z(xy-yx)\\ &\quad+(yz-zy)x-x(yz-zy)\\ &\quad+(zx-xz)y-y(zx-xz)=0\end{aligned}$$

There are some important algebraic structures which do not satisfy the associative law. These algebraic structures are called *non-associative algebras*.

Definition 1.0.5 (**Non-associative algebras**)**.** An algebra A is said to be a non-associative algebra over a field $\mathbb{F}$ if A is a vector space over a field $\mathbb{F}$ together with an $\mathbb{F}$-bilinear multiplication operation $A \times A \to A$ which is not assumed to be associative. That is, associativity does not necessarily hold.

So far we have defined associative algebras, the commutator, non-associative algebras and their properties. With this information we can now define Lie algebras as follows.

Definition 1.0.6 (**Lie algebra**)**.** A *Lie algebra* is defined as a vector space V over a field $\mathbb{F}$ together with an operation $[,]: V \times V \to V$ known as the *Lie bracket* satisfying the following conditions:

1. $[ax+by,z]=a[x,z]+b[y,z]$
2. $[x,x]=0$
3. $[x,[y,z]]+[y[z,x]]+[z,[x,y]]=0$
4. $[x,y]=-[y,x]$

$\forall a,b \in \mathbb{F}$ and all elements $x,y,z \in V$.

Theorem 1.0.7. Alternativity and bilinearity imply anticommutativity and If char $\mathbb{F} \neq 2$ then anticommutativity implies alternativity.

Proof: We first prove that condition 2. implies condition 4. provided condition 1. holds.

$$\begin{aligned}0 &= [x+y,x+y]\\ &= [x,x+y]+[y,x+y]\\ &= [x,x]+[x,y]+[y,x]+[y,y]\\ &= [x,y]+[y,x]\end{aligned}$$

since $[x,x]=0$ and $[y,y]=0$. Therefore, $[x,y]=-[y,x]$. This shows that alternativity and bilinearity together imply anticommutativity.

If char $\mathbb{F} \neq 2$ then

$$[x,x]=-[x,x]$$

implies

$$2[x,x]=0,$$

hence

$$[x,x]=0.$$

Remark 1.0.8. Any associative algebra becomes a Lie algebra with the commutator as Lie bracket.

We now look at examples of Lie algebras.

Example 1.0.9 (**Abelian Lie algebras**)**.** A Lie algebra L is said to be Abelian if

$$[x, y] = 0 \ \ \forall\, x, y \in L.$$

For example let x and y be 3×3 upper triangular matrices given by

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \ y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the vector space spanned by x and y is an Abelian Lie algebra with respect to the usual commutator since

$$[x, y] = xy - yx = 0$$

as it can be easily checked.

Let's consider the 2×2 matrices

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & -4 \\ -4 & -2 \end{bmatrix}.$$

We also check that it satisfies the Abelian Lie algebra condition under the commutator, that is,

$$[A, B] = AB - BA = 0.$$

Example 1.0.10 (**General linear Lie algebras**)**.** Let $\mathbb{F}$ be a field, n a positive integer. The general linear algebra denoted by $gl_n(\mathbb{F})$ comprises of all $n \times n$ real matrices with the commutator serving as the Lie bracket.

Let V be a vector space of finite dimension over a field $\mathbb{F}$ and let $gl(V)$ be the Lie algebra of the linear transformation $V \mapsto V$ with the commutator. Then we identify the Lie algebra $gl(V)$ with set $n \times n$ matrices $gl_n(\mathbb{F})$, where n is the dimension of V.

Definition 1.0.11 (**Trace of a square matrix**)**.** This is the sum of elements on the main diagonal of an $n \times n$ matrix A.

$$tr(A) = \sum_{i=1}^{n} a_{ij} = a_{11} + a_{22} + \cdots + a_{nn}$$

where a_{ij} denotes the entry on the ith row and jth column of A.

Example 1.0.12 (**Special linear Lie algebras**). Let $\mathbb{F}$ be a field, n a positive integer. The special linear Lie algebra denoted $sl_n(\mathbb{F})$ consists of $n \times n$ matrices with trace 0 with the commutator as Lie bracket. This is indeed a Lie algebra since when both x and y are $n \times n$ matrices, the trace of the commutator of x and y vanishes because:

$$tr(xy) = tr(yx)$$

and tr is linear.

Example 1.0.13. Heisenberg Lie algebra. A Lie algebra $\mathcal{H}$ is called Heisenberg Lie algebra if it can be represented by the space of 3×3 matrices of the form:

$$\begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$. The following 3×3 matrices form a basis for $\mathcal{H}$:

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

These matrices satisfy the following commutator relations:

$$[e_1, e_2] = e_3; \quad [e_1, e_3] = 0; \quad [e_2, e_3] = 0.$$

Definition 1.0.14 (**Derivation**). Let A be a (non-associative) algebra. A map $D \in gl(A)$ that satisfies the condition

$$D(ab) - D(a)b - aD(b) = 0$$

for all $a, b \in A$ is a derivation of A.

Derivations of the Heinsenberg Lie algebras. Consider the Heisenberg Lie algebra H. Let $H = \langle e_1, e_2, e_3 \rangle$,

where

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are a basis.

Let $D : H \mapsto H$ be a derivation. Then it has to satisfying

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all x, y. Then since this condition is linear in x and y, we only have to check it for a basis H

Let

$$e_1 \to \begin{bmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix} = A, \quad e_2 \to \begin{bmatrix} 0 & b_1 & b_3 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{bmatrix} = B, \quad e_3 \to \begin{bmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} = C.$$

$$x = e_1, \quad y = e_2$$

$$[e_1, e_2] = e_3 \mapsto C$$

$$[A, e_2] + [e_1, B] = (Ae_2 - e_2A) + (e_1B - Be_1)$$

$$= \begin{bmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix}_, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b_1 & b_3 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & b_1 & b_3 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & b_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_1 + b_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = e_1, y = e_3[e_1, e_3] = 0$

$$[A, e_3] + [e_1, C] = (Ae_3 - e_3A) + (e_1C - Ce_1)$$

$$= \begin{bmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix}_, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_1 & a_3 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \implies c_2 = 0$$

$x = e_2, y = e_3[e_2, e_3] = 0$

$$[B, e_3] + [e_2, C] = (Be_3 - e_3B) + (e_2C - Ce_2)$$

$$= \begin{bmatrix} 0 & b_1 & b_3 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & c_1 & c_3 \\ 0 & 0 & c_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \implies c_1 = 0.$$

1.1 Other methods of forming Lie algebras

We have seen how to create a Lie algebra L from an associative algebra A by using the commutator as a Lie bracket. Following that, we will look at other methods for forming Lie algebras.

Lie Algebras of Derivations

Lie algebras are often formed as the algebra of derivations of a given algebra. This correlates with the use of vector fields in geometry.

Theorem 1.1.1. The set of derivations $Der(A)$ of an algebra A is a Lie subalgebra of $gl(A)$.

Proof: Let us fix a and b. We find that the left hand side of the above equation is linear in D, hence the set of endomorphism satisfying the above equation is a subspace. We also find that the set of derivations is the intersection over all a and b in A of these subspaces and hence it is a subspace. Next we confirm that the bracket of two derivations is a derivation for all $a, b \in A$ and $D_1, D_2 \in Der(A)$.

$$\begin{aligned}
[D_1, D_2](ab) &= D_1(D_2(a)b + aD_2(b)) - D_2(D_1(a)b + aD_1(b)) \\
&= D_1D_2(a)b + D_2(a)D_1(b) + D_1(a)D_2(b) + aD_1D_2(b) \\
&\quad - D_2D_1(a)b - D_1(a)D_2(b) - D_2(a)D_1(b) - aD_2D_1(b) \\
&= D_1D_2(a)b - D_2D_1(a)b + aD_1D_2(b) - aD_2D_1(b) \\
&= [D_1, D_2](a)b + a[D_1, D_2](b)
\end{aligned}$$

Therefore the set of derivations is closed under the commutator and this creates a Lie subalgebra of $gl(A)$.

Lie Algebras via Structure Constants

Given a basis $e_1, \ldots, e_n$ of a Lie algebra L over $\mathbb{F}$, the bracket is determined by the structure constants C_{ijk}, defined by:

$$[e_i, e_j] = \sum_k C_{ijk} e_k.$$

The structure constants must meet the obvious skew-symmetry requirement.

$$C_{iik} = 0, \;\; C_{ijk} = 0 \;\text{ and }\; C_{ijk} = -C_{ijk}$$

and a quadratic condition corresponding to the Jacobi identity.

Example 1.1.2. Abelian Lie algebra: $C_{ijk} = 0$ for all possible i, j, k.

Example 1.1.3. Heisenberg Lie algebra: $C_{123} = 1$, $C_{213} = -1$, all the others are zero.

Example 1.1.4. $sl_2(\mathbb{F})$:

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

For the indices (1,1),(1,2) and (3,1) for i, j we have

$$[e_1, e_1] = \underbrace{C_{111}}_{0} e_1 + \underbrace{C_{112}}_{0} e_2 + \underbrace{C_{113}}_{0} e_3 = 0$$
$$[e_1, e_2] = \underbrace{C_{121}}_{0} e_1 + \underbrace{C_{122}}_{0} e_2 + \underbrace{C_{123}}_{1} e_3 = e_3$$
$$[e_3, e_1] = \underbrace{C_{211}}_{0} e_1 + \underbrace{C_{212}}_{0} e_2 + \underbrace{C_{213}}_{\text{-1}} e_3 = -e_3$$

Chapter 2

Ideals and Homomorphisms

Unless otherwise stated, information in this Chapter was derived from [Jac62] apart from the self-made examples, definitions or exercises.

Remark 2.0.1. If X, Y are subspaces of a Lie algebra L, then $[X, Y]$ denotes the span of all vectors $[x, y]$ where $x \in X, y \in Y$.

Definition 2.0.2 (Subalgebra). Let L be a Lie algebra. A subspace $S \subset L$ is called a subalgebra if and only if $[S, S] \subset S$.

Definition 2.0.3 (Ideal). A subset I of L is called an ideal of L if I is a subspace of L and $[I, L] \subset I$.

Example 2.0.4. $L \triangleleft L$. L itself is an ideal of L for all Lie algebras L.

Example 2.0.5. $0 \triangleleft L$. The zero subspace is an ideal for all Lie algebras L.

Example 2.0.6. $sl_n(\mathbb{F}) \triangleleft gl_n(\mathbb{F})$ where $n, \mathbb{F}$ are arbitrary. The simple linear Lie algebra is an ideal of the general linear lie algebra.

Proof: Let $x \in sl_n(\mathbb{F})$ and $y \in gl_n(\mathbb{F})$ then

$$[x, y] \in sl_n(\mathbb{F})$$

.

Example 2.0.7. In the Heisenberg Lie algebras $\mathcal{H}$ we have the following ideals:

1. $\{0\}$.
2. $\langle e_3 \rangle \triangleleft \mathcal{H}$.
3. $\langle e_2, e_3 \rangle \triangleleft \mathcal{H}$.
4. $\langle e_1, e_3 \rangle \triangleleft \mathcal{H}$. Moreover,
5. $\langle ae_1 + be_2, e_3 \rangle \triangleleft \mathcal{H}$ for all $a, b \in \mathbb{R}$.
6. $\mathcal{H}$ itself.

where $e_1, e_2$and e_3 are 3×3 matrices satisfying 1.0.13.

Since e_3 is a multiple of any two non-trivial commutator, it must be contained in any non-trivial ideal. Since the commutator of e_3 by every element of L is 0, the above are indeed ideals.

$$[e_3, e_3] = [e_2, e_2] = [e_2, e_3] = [e_1, e_3] = 0 \text{ and } [e_1, e_2] = e_3.$$

Definition 2.0.8 (**Proper ideal**). $I \triangleleft L$ is called proper if and only if $L \neq I$.

Theorem 2.0.9. Let L be a Lie algebra. Then $[L, L]$ is an Ideal of L.

Proof: Let $a \in L, b \in [L, L] \subset L$. Then $[a, b] \in [L, L]$.

Example 2.0.10. We now classify the Lie algebras in dimensions 1 and 2.

$\dim L = 1$: $L = \mathbb{F}a, [a, a] = 0$. So the Abelian Lie algebra is the only one (up to isomorphism).

$\dim L = 2$: We prove $[L, L] \subsetneq L$. Let $L = \mathbb{F}x + \mathbb{F}y$. By the linearity of the Lie bracket $[L, L] = \mathbb{F}[x, y]$. Hence

$$\dim[L, L] \leq 1,$$

so $[L, L] \subsetneq L$.

Case 1: $\dim[L, L] = 0$. Then L is an Abelian Lie algebra.

Case 2: $\dim[L, L] = 1$. Then $[L, L] = \mathbb{F}b$, with some $b \neq 0$. Take $a \in L \backslash \mathbb{F}b$. Then $[a, b] \in [L, L]$, hence $[a, b] = \lambda b$ and $\lambda \neq 0$, otherwise $[L, L] = 0$. So replacing a by $\lambda^{-1}a$, we get $[a, b] = b$. Hence we have found a basis of L with bracket $[a, b] = b$. This algebra is isomorphic to the subalgebra

$$\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \subset gl_2(\mathbb{F}),$$

since for

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

we have

$$[a, b] = b.$$

□

To understand and visualize Lie algebra homomorphisms we need the following definitions.

Definition 2.0.11 (**homomorphism**, **isomorphism**, **monomorphism**, **epimorphism**). Let L_1 and L_2 be the Lie algebras over the field $\mathbb{F}$. A map $\theta\colon L_1 \to L_2$ is called a **homomorphism** of Lie algebras if θ is linear and

$$\theta[x, y] = [\theta x, \theta y] \text{ for all } x, y \in L_1.$$

A map $\theta\colon L_1 \to L_2$ is called an **isomorphism** of Lie algebras if θ is a bijective homomorphism of Lie algebras.

If θ is a homomorphism of a Lie algebra L_1 into L_2 then the set K of all those elements of L_1 which is mapped by θ onto the 0 of L_2 is called the **kernel** of the homomorphism θ denoted by Ker θ. $Im\ \theta$ is the range of θ.

A map $\theta : L_1 \to L_2$ is called a **monomorphism** of Lie algebras if θ is an injective homomorphism of Lie algebras, that is, its kernel is zero.

A map $\theta : L_1 \to L_2$ is called an **epimorphism** of Lie algebras if θ is a surjective homomorphism of Lie algebras, that is $Im\ \theta = L_2$.

Example 2.0.12. Since an associative ring can be thought of as a Lie algebra using the bracket $[x, y] = xy - yx$, any ring homomorphism will induce a Lie algebra homomorphism.

Theorem 2.0.13. Let $\theta\colon L_1 \to L_2$ be a homomorphism of Lie algebras. Then the kernel of θ is an ideal of L_1, $Im\ \theta$ is a subalgebra of L_2 and $L_1/Ker\ \theta$ is isomorphic to $Im\ \theta$.

Proof: $Im\ \theta$ is a subalgebra of L_2 because it is a subspace and for $x, y \in L_1$

$$[\theta(x), \theta(y)] = \theta[xy]\ \in Im\ \theta.$$

Clearly $Ker\ \theta$ is a subspace of L_1. Let $x\ \in\ Ker\ \theta$ and $y\ \in L_1$.

$$\theta[x, y] = [\theta(x), \theta(y)] = [0, \theta(y)] = 0$$

Therefore, $[xy]\ \in Ker\ \theta$ and so $Ker\ \theta$ is an ideal of L_1.

We let $x, y\ \in\ L_1$. Then

$$\begin{aligned}\theta(x) = \theta(y) &\Leftrightarrow\ \theta(x-y) = 0\\ &\Leftrightarrow\ x - y\ \in\ Ker\ \theta\\ &\Leftrightarrow\ Ker\ \theta + x = Ker\ \theta + y\end{aligned}$$

This implies there is a bijective map $\theta(x) \mapsto Ker\ \theta\ + x$ between $Im\ \theta$ and $L_1/Ker\ \theta$. This bijection is an isomorphism of Lie algebras since it is linear and given $x, y, z \in L_1$, we have

$$\begin{aligned}[\theta(x), \theta(y)] = \theta(z) &\Leftrightarrow \theta[xy] = \theta(z)\\ &\Leftrightarrow Ker\ \theta + [xy] = Ker\ \theta + z\\ &\Leftrightarrow [Ker\ \theta + x, Ker\ \theta + y] = Ker\ \theta + z\end{aligned}$$

Chapter 3

Solvable and Nilpotent Lie algebras

Unless otherwise stated, information in this Chapter was derived from [Hal03] apart from the self-made examples, definitions or exercises.

Definition 3.0.1 (Derived series). Given a Lie algebra L over a field $\mathbb{F}$, the derived series of L is a sequence of ideals of L given by

$$L^{(0)} = L,\ L^{(1)} = [L, L],\ L^{(2)} = [L^{(1)}, L^{(1)}],\ \dots\ , L^{(k)} = [L^{(k-1)}, L^{(k-1)}],$$

where

$$L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \cdots \supseteq L^{(k)}$$

and

$$L^{(k+1)} = L^{(k)}.$$

Lemma 3.0.2. Suppose I is an ideal of L. Then L/I is Abelian if and only if I contains the derived algebra $L^{(1)}$.

Proof:

$$L^{(1)} = [L, L] = \langle\ [x, y] | x, y \in L\ \rangle$$

$$L/I \ \text{ Abelian } \iff [x + I, y + I] = [x, y] + I = I$$

hence

$$[x, y] \in I \iff L^{(1)}$$

3.1 Solvable Lie algebras

Definition 3.1.1 (Solvable Lie algebra). A Lie algebra is called solvable if its derived series terminates in the zero subalgebra:

$$L^{(k)} = 0.$$

Example 3.1.2. For all Abelian Lie algebras.

Lemma 3.1.3. If L is a Lie algebra with ideals $I_0, I_1, \cdots, I_m$ with

$$L = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_{m-1} \supset I_m = 0$$

and I_{k-1}/I_k is Abelian for $1 \leq k \leq m$ then L is solvable.

Proof: We need to show that

$$L^{(k)} \subset I_k$$

for $k = 1, 2, \cdots, m$. Since $L/I_1 = I_0/I_1$ is assumed to be Abelian $\implies L^{(1)} \subseteq I_1$ by Lemma 3.0.2.

By induction suppose

$$L^{(k-1)} \subset I_{k-1}$$

where $k \geq 2$.

The Lie algebra I_{k-1}/I_k is Abelian. Apply Lemma 3.0.2 to I_{k-1} then

$$(I_{k-1})^{(1)} \subseteq I_k$$

therefore

$$[I_{k-1}, I_{k-1}] \subseteq I_k$$

but

$$L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \subseteq [I_{k-1}, I_{k-1}] \subseteq I_k$$

hence $L^{(k)} \subseteq I_k$.

Let $k = m$ then

$$L^{(m)} \subseteq I_m = 0$$

and therefore L is solvable.

Example 3.1.4. Heinsenberg Lie algebra. $H = \langle e_1, e_2, e_3 \rangle = L$

$$[e_1, e_2] = z \implies L^{(1)} = \langle z \rangle \implies L^{(2)} = [L^{(1)}, L^{(1)}] = 0$$

Proposition 3.1.5. Let $\Phi : L_1 \mapsto L_2$ be a surjective homomorphism of Lie algebras. Show that $\phi(L_1^{(k)}) = (L_2)^{(k)}$.

Proof:

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $y \in L - 2 \; \exists \; x \in L_1$ such that $\phi(x) = y$. Now consider $k = 1$ then

$$\begin{aligned}
L_1^{(1)} &= \langle \; [x, y] \mid x, y \in L_1 \; \rangle \\
\phi(L_1^{(1)}) &= \langle \; \phi([x, y]) \mid x, y \in L_1 \; \rangle \\
&= \langle \; [\phi(x), \phi(y)] \mid x, y \in L_1 \; \rangle \\
&= \langle \; [\phi(x), \phi(y)] \mid \phi(x), \phi(y) \in L_1 \; \rangle \\
&= L_2^{(1)}
\end{aligned}$$

hence

$$\phi(L_1^{(1)}) = \phi([L_1, L_1]) = [\phi(L_1), \phi(L_1)] = [L_2, L_2] = L_2^{(1)}.$$

Suppose inductively that $\phi(L_1^{(k)}) = L_2^{(k)}$ for $k = n$ then

$$\begin{aligned}\phi(L_1^{(n+1)}) &= \phi([L_1^{(n)}, L_1^{(n)}]) \\ &= [\phi(L_1^{(n)}, \phi(L_1^{(n)})] \\ &= L_2^{(n+1)}.\end{aligned}$$

Thus by proof of mathematical induction the Proposition is proved.

Theorem 3.1.6. Let L be a Lie algebra and $I \subset L$ be an ideal. If I is solvable and L/I is solvable then L is solvable.

Proof: Suppose $I \subseteq L$ is a solvable ideal, such that L/I is also solvable. Then $L^{(n)} = 0$ and $(L/I)^{(m)} = 0$ for some n, m. We consider the natural homomorphism

$$\pi : L \to L/I.$$

By the previous paragraph,

$$\pi(L^{(m)}) = (L/I)^{(m)} = 0 \ \text{ therefore } \ L^{(m)} \subseteq I.$$

Hence

$$(L^{(m)})^{(n)} = L^{(m+n)} \subseteq I^{(n)} = 0.$$

This means L is solvable.

3.2 Nilpotent Lie algebras

Definition 3.2.1 (**Descending central series**)**.** Let L be a Lie algebra. The descending central series of L is the sequence of subalgebras recursively defined by:

$$L^{k+1} = [L, L^k]$$

with

$$L^0 = L, \ \ L^1 = [L, L], \ \ L^2 = [L, L^1], \ \dots \ , L^k = [L, L^{k-1}].$$

Definition 3.2.2 (**Nilpotent Lie algebras**)**.** A Lie algebra L is called nilpotent if for some $k > 0$, $L^k = 0$. This clearly shows that all subalgebras and all factor algebras of a nilpotent Lie algebra is also nilpotent.

Example 3.2.3. Abelian Lie algebras $L^{(k)} = 0$.

Example 3.2.4. An example of a nilpotent Lie algebra is the Lie algebra of strictly upper triangular matrices over a field $\mathbb{F}$. It is true in general, but we show it in the 3×3 case, which is again

Heisenberg.

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \mid a, b, c \in \mathbb{F} \right\}.$$

We calculate

$$\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & az \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & cx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & az - cx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & az - cx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & az - cx \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore $L^{(2)} = 0 \implies$ solvable. Then we check

$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $L^2 = 0 \implies$ nilpotent.

Proposition 3.2.5. $L^k \subseteq L^{(k)}$ for all k.

Proof: We use induction on k to show that $L^{(k)} \subseteq L^k$. For $k = 1$

$$L^{(1)} = L^1 \ \therefore\ L^{(1)} \subseteq L^1.$$

Then we assume $L^{(k)} \subseteq L^k$ for $k \geq 1$. Consider $L^{(k+1)}$

$$L^{(k+1)} = [L^{(k)}, L^{(k)}] \subseteq [L, L^k] = L^{k+1}.$$

Corollary 3.2.6. This implies that nilpotent Lie algebras are solvable.

However, solvable Lie algebras are not necessarily nilpotent, as the following example shows.

Example 3.2.7. Let L be the Lie algebra of the 2 *by* 2 upper triangular matrices. Then

$$L^1 = [L, L] = \{\text{strictly upper triangular matrices}\}.$$

$L^2 = [L, L^1]$ is the same. However

$$L^{(2)} = [L^1, L^1] = 0.$$

So this algebra is solvable but not nilpotent.

Theorem 3.2.8. Let L be a Lie algebra and I an Ideal of L. All elements of the descending central series of I is also an ideal of L.

Proof: Since I is an ideal of L, $I^0 = I$ is also an ideal of L. Assume $I^{(k-1)} \triangleleft L$. We prove that $I^k = [I, I^{k-1}]$ is an ideal of L. Let $x \in L, y \in I, z \in I^{k-1}$.

$$\begin{aligned}
[x[yz]] &= -[y[zx]] - [z[xy]] \\
&\in [y, I^{k-1}] + [z, I] \text{ since } [z, x] \in I^{k-1}, [x, y] \in I \\
&[I, I^{k-1}] + [I^{k-1}, I] \text{ since } y \in I, z \in I^{k-1} \\
&= I^{k-1+1} + I^{k-1+1} \\
&= I^K + I^k \\
&= I^k.
\end{aligned}$$

Thus I^K is an Ideal and all elements of the descending central series of I is an Ideal of L.

3.3 Representations of solvable and nilpotent Lie algebras

Definition 3.3.1 (**Representation of a Lie algebra**.)**.** Let L be a Lie algebra and V be a vector space. Let $gl(V)$ denote the space of all linear maps of V to itself. $gl(V)$ is transformed into a Lie algebra by the commutator

$$[\rho, \sigma] = \rho \circ \sigma - \sigma \circ \rho \ \ \forall \rho, \sigma \in gl(V).$$
$$\rho \colon L \mapsto gl(V),$$

where ρ is a linear map and satisfies the condition

$$\rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x) \ \ \forall\, x, y \in L.$$

Definition 3.3.2 (**L-module**)**.** An L-module is defined as the vector space V together with the representation ρ.

Definition 3.3.3 (**Adjoint representaion of Lie algebras**)**.** Let L be a Lie algebra over a filed $\mathbb{F}$. Let $x \in L$. The adjoint action of x on L denoted ad_x or $ad(x)$ is the map

$$ad_x : L \mapsto L$$

with

$$ad_x(y) = [x, y]$$

for all $y \in L$.

Example 3.3.4. $sl_2(\mathbb{C})$ where we have a basis

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The matrix of ad with respect to the given basis are:

$$\mathrm{ad}_{e_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathrm{ad}_{e_2} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{ad}_{e_3} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} ad_{e_1}(e_1) &: e_1 \mapsto [e_1, e_1] = 0 \\ ad_{e_1}(e_2) &: e_2 \mapsto [e_1, e_2] = 2e_2 \\ ad_{e_1}(e_3) &: e_3 \mapsto [e_1, e_3] = -2e_3 \end{aligned}$$

Example 3.3.5. By the definition of the Heisenberg algebra we have:

$$\mathrm{ad}_{e_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathrm{ad}_{e_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

ad_{e_3} is the zero matrix.

Definition 3.3.6 (**Direct sum**)**.** A direct sum of Lie algebras L_1 and L_2 denoted $L_1 \oplus L_2$ is the vector space of all pairs (x, y) with $x \in L_1$ and $y \in L_2$. For $(x_1, y_1), (x_2, y_2) \in L_1 \oplus L_2$, the Lie bracket is given by

$$[(x_1, x_2), (y_1, y_2))] = ([x_1, y_1], [x_2, y_2]).$$

3.3.1 Representations of solvable Lie algebras

Let the base field $\mathbb{F}$ be the field $\mathbb{C}$ of complex numbers. Also let L be a finite dimensional Lie algebra over $\mathbb{C}$. We begin by considering a 1-dimensional representation of a Lie algebra L given by the linear map

$$\theta : L \to \mathbb{C} \text{ such that } \theta[xy] = [\theta(x), \theta(y)] \text{ for all } x, y \in L$$

Lemma 3.3.7. A linear map $\theta : L \to \mathbb{C}$ is a 1-dimensional representation of L if and only if θ vanishes on L^2.

Proof: Let θ be a representation. Then for $x, y, \in L$ we have

$$\theta[xy] = [\theta(x), \theta(y)] = \theta(x)\theta(y) - \theta(y)\theta(x) = 0,$$

hence θ vanishes on L^2. In contrast, assume θ disappears on L^2. Then

$$\theta[xy] = 0 = [\theta(x), \theta(y)]$$

and hence θ is a representation of L.

Lemma 3.3.7 enables us to prove a theorem of Lie which shows that any irreducible representation of a soluble Lie algebra is 1-dimensional.

3.3.2 Representations of nilpotent Lie algebras

The purpose of studying the results on representations of nilpotent Lie algebras is because it's important in helping us understand semi-simple Lie algebras.

Definition 3.3.8 (Jordan canonical form). This is a particular type of upper triangular matrix's representation of a linear transformation over a finite-dimensional complex vector space. Up to the order of blocks, each such linear transformation has a Jordan canonical form.

Therefore any $n \times n$ matrix over $\mathbb{C}$ is similar to a diagonal sum of Jordan block matrixes of form

$$\begin{bmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & 0 & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & 1 \\ & 0 & & & & \lambda & 1 \\ & & & & & & \lambda \end{bmatrix}.$$

We also note that any linear transformation $\theta : V \to V$ on a finite dimensional vector space V over $\mathbb{C}$ gives rise to a Jordan decomposition of V.

Proposition 3.3.9. Let $\theta : V \to V$ and the distinctive polynomial

$$\chi(t) = (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \dots (t - \lambda_q)^{m_q}$$

where $\lambda_1, \dots, \lambda_q$ are the distinct eingenvalues of θ and $m_1, \dots, m_q$ are their Multiplicities in algebra. Let V_i be the set of all $v \in V$ annihilated by some power of $\theta - \lambda_i 1$. Then we have

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_q.$$

Moreover dim $V_i = m_i$, $\theta(V_i) \subset V_i$ and the characteristic polynomial of θ on V_i is $(t - \lambda_i)^{m_i}$.

[Car05]

The subspace V_i is called the generalised eigenspace of V with eigenvalue λ_i. Thus, the generalised eigenspace contains the ordinary eigenspace of λ_i. It is not in general true that V is the direct sum of its eigenspaces with respect to its different eigenvalues, but Proposition 3.3.9 shows that this result is true if the eigenspaces are replaced by the generalized eigenspaces.

The importance of the decomposition into generalised eigenspaces for the representations of nilpotent Lie algebras is shown by the following theorem.

Theorem 3.3.10. Let V be an L-module and L be a nilpotent Lie algebra. Let $x \in L$ and let $p(x) : V \to V$ be the map $v \mapsto xv$. All the submodules of V are the generalised eigenspaces V_i of V associated with $p(x)$.

[Car05]

Chapter 4

Simple and semisimple Lie algebras

Unless otherwise stated, information in this Chapter was derived from [Hum12] apart from the self-made examples, definitions or exercises.

Definition 4.0.1. A Lie algebra L is said to be *simple* if it satisfies the following conditions:

1. dim $L > 1$, that is $[L, L] \neq 0$. Hence L is not abelian. Then we have
$$[L, L] = L.$$
2. The only ideals of L are 0 and L itself.

Definition 4.0.2 (semisimple Lie algebras). A Lie algebra L is called semisimple if it is isomorphic to a direct sum of simple Lie algebras.

Proposition 4.0.3. If L is semisimple then $[L, L] = L$.

Proof: We write L as a direct sum of simple Lie algebras, that is
$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$$
with all L_i simple. Then we compute the derived algebra L as follows.
$$\begin{aligned}[L, L] &= \bigoplus_{i,j=1}^{k} [L_i, L_j] \\ &= \bigoplus_{i,j=1}^{k} [L_i, L_i] \\ &= \bigoplus_{i,j=1}^{k} L_i \\ &= L.\end{aligned}$$
Hence the proof.

Lemma 4.0.4. $gl_n(\mathbb{F})$ is never semisimple.

Proof: This is immediate since
$$[gl_n, gl_n] = sl_n \neq gl_n.$$

Lemma 4.0.5. Solvable algebras are not semisimple.

Proof: This is immediate since
$$[L, L] \subsetneq L.$$

Proposition 4.0.6. Let I and J be solvable ideals of a Lie algebra L. Then $I+J$ is also a solvable ideal.

Proof: Clearly $I + J$ is an ideal. $I + J \supset I \supset 0$ since I is solvable and

$$\frac{(I+J)}{I} \cong \frac{J}{I \cap J}$$

is also solvable. We have $I + J$ is solvable.

Corollary 4.0.7. If L is finite dimensional then it has a unique solvable ideal which consists of all other solvable ideals of L.

Definition 4.0.8 (**Radical of L**)**.** The unique largest solvable ideal in Corollary 4.0.7 is called the radical of L and denoted by $Rad(L)$

Proposition 4.0.9. If L is semisimple and finite dimensional then $Rad(L) = 0$.

Proof: If L is semisimple it implies that the adjoint module is semisimple. The adjoint module is written as
$$L \cong \oplus Rad(L).$$
Then
$$[a, Rad(L)] = 0.$$
Therefore

$$\begin{aligned} L &= [L, L] \\ &= [a, a] \oplus [Rad(L), Rad(L)] \subset a \oplus Rad(L) \\ &= L. \end{aligned}$$

Hence $Rad(L)$ is solvable $\implies$ $Rad(L) = 0$.

4.1 Examples of Simple Lie algebras

Simple Lie algebras over an algebraic closed field of $\text{char}(\mathbb{F}) = 0$ can be classified as follows:

1. Special linear Lie algebra $A_n = sl_{n+1}$. This is true for only $n \geq 1$ since a_0 is the *trivial Lie algebra.*
2. Special orthogonal Lie algebra $B_n = so_{2n+1}$. This true for only $n \geq 2$ since $b_n = a_n$ for $n < 2$
3. Sympletic Lie algebra $C_n = sp_n$. This is true for only $n \geq 3$ since $c_n = b_n$ for $n < 3$.
4. Even-dimensional special orthogonal Lie algebra $D_n = so_{2n}$. This is true for only $n \geq 4$ since $d_n \cong a_n$ for $n < 2$ and $n = 3$ while $d_2 \cong a_2 \oplus a_2$(which is not simple).
5. Exceptional Lie algebras. We have five of them. E_6, E_7, E_8, F_4 and G_2.

Proposition 4.1.1. sl_2 is simple. Let

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \neq 0 \ \in I \triangleleft sl_2 \ (\text{char}(\mathbb{F}) \neq 2).$$

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Let I be a non-zero ideal of $sl_2(\mathbb{F})$. Let $A \in I$. Then

$$[e_1, A] = (e_1 A) - (A e_1) = \begin{bmatrix} a & b \\ -c & a \end{bmatrix} - \begin{bmatrix} a & -b \\ c & a \end{bmatrix} = \begin{bmatrix} 0 & 2b \\ -2c & 0 \end{bmatrix} = B \in I.$$

$$[e_2, B] = (e_2 B) - (B e_2) = \begin{bmatrix} -2c & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & -2c \end{bmatrix} = \begin{bmatrix} -2c & 0 \\ 0 & 2c \end{bmatrix} \in I.$$

Hence if $c \neq 0$, then $e_1 \in I$. Similarly, if $b \neq 0$, then $e_1 \in I$. If $b = c = 0$ then $a \neq 0$. So again $e_1 \in I$.

But

$$[e_1, e_2] = e_1 e_2 - e_2 e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \in I = 2e_2 \in I.$$

$$[e_1, e_3] = e_1 e_3 - e_3 e_1 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \in I = -2e_3 \in I.$$

$$[e_2, e_3] = e_2 e_3 - e_3 e_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = e_1.$$

Therefore

$$\langle e_1, e_2, e_3 \rangle \subseteq I,$$

where

$$\langle e_1, e_2, e_3 \rangle = sl_2(\mathbb{F}).$$

Hence $I = sl_2(\mathbb{F})$ and $sl_2(\mathbb{F})$ is indeed simple.

Chapter 5

Cartan subalgebras

Unless otherwise stated, information in this Chapter was derived from [Bou89] apart from the self-made examples, definitions or exercises.

Definition 5.0.1 (Normalizer). Let L be a Lie algebra and let H be a subalgebra of L. The normalizer of H denoted by $N(H)$ is

$$N(H) = \{x \in L; [hx] \in H\} \text{ for all } h \in H.$$

Lemma 5.0.2. For each subalgebra H of L, $N(H)$ is a subalgebra of L.

Proof: Let $x, y \in N(H)$ then by the linearity of the Lie bracket, $N(H)$ is a subspace

$$[h[xy]] = [[yh]x] + [[hx]y] \in H.$$

Hence $[xy] \in N(H)$ and $N(H)$ is a subalgebra.

Definition 5.0.3 (Cartan subalgebra). Let L be a finite dimensional Lie algebra over some field $\mathbb{F}$ and let H be a subalgebra of L. H is called a cartan subalgebra if it is nilpotent and equal to its normalizer that is if $H = N(H)$.

Definition 5.0.4 (Null component of L). Let $x \in L$ and $\mathrm{ad}_x : L \mapsto L$ be a linear map. The null component of L with respect to x denoted $L_{0,x}$ is the eingenspace of ad_x with eingen value 0.

$$L_{0,x} = \{y \in L; \exists\, n \text{ such that } (\mathrm{ad}_x)^n y = 0\}.$$

Definition 5.0.5 (Regular element). Let $x \in L$. x is called a regular element of L if $\dim L_{0,x}$ is as minimal as possible.

Lemma 5.0.6. Let L be a nilpotent Lie algebra and $H \subset L$ a subalgebra such that $H \neq L$. Then $H \subsetneq N_L(H)$.

Proof: Consider the central series

$$L = L^1 \supset L^2 = [L, L] \supset L^3 \cdots \supset L^n = 0.$$

The last equality is true for some $n \in \mathbb{N}$ since L is a nilpotent Lie algebra. Take i to be the maximal possible positive integer such that $L^i \nsubseteq H$. Then we have $1 < i < n$. However $[L^i, H] \subset L^{i+1} \subset H$ by the choice made of i. Hence $L^i \subset N_L(H)$ which is not a subspace of H. We can therefore conclude that $H \subsetneq N_L(H)$.

Proposition 5.0.7. Let $L \subset gl_n(\mathbb{F})$ be a subalgebra containing a diagonal matrix $A = \text{diag}(a_1, \ldots, a_n)$ with distinct a_i, and let H be the subspace of all diagonal matrices in L. Then H is a Cartan subalgebra.

Proof: We check that H is a nilpotent Lie algebra and that $N_L(H) = H$. H is clearly Abelian hence it is nilpotent. So we only have to check that H is equal to its normalizer $N_L(H) = H$. Let

$$B = \sum_{i,j=1}^{n} b_{ij} E_{ij} \in N_L(H).$$

Then $[A, B] \in H$ for all $A \in H$. but

$$[A, B] = \left[\sum_k a_k E_{kk}, \sum_{i,j} b_{ij} E_{ij}\right] = \sum_{i,j} (a_i - a_j) b_{ij} E_{ij}$$

which will be non-diagonal if $b_{ij} \neq 0$ for some $i \neq j$. Therefore we can conclude that $B \in H$ and thus $N_L(H) = H$.

By Lemma 5.0.6 a Cartan subalgebra is a maximal nilpotent subalgebra. However the converse is false.

To give an example, we first prove the following lemma.

Lemma 5.0.8. Fix $l \geq 0$. Let n_1 be strictly upper triangular matrix. Let n_2 be a matrix where all $(i, i+t)$ entries are 0 if $t = 0, 1, \ldots, l$. Then the $(i, i+t)$ entries of $n_1 \cdot n_2$ and $n_2 \cdot n_1$ are 0 for $t = 0, 1, \ldots, l+1$.

Proof: Observe that the first i entry in the i^{th} row of n_1 is 0 and the only non-zero entries in the j^{th} column of n_2 are the first $j - l$ ones. Hence if $i \geq j - l - 1$ then the (i, j) entry of $n_1 \cdot n_2$ is 0

Then

$$i \geq j - l - 1 \iff j \leq i + l + 1$$

hence the claim follows for $n_1 n_2$. See below example of such matrices.

$$\begin{bmatrix} 0 & & \\ & \ddots & a_{ij} \\ 0 & & \\ & & \ddots \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 & & \\ & \ddots & b_{ij} \\ 0 & & \\ & & \ddots \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & & \\ \ddots & \ddots & * & \\ 0 & & \ddots & \\ & & \ddots & 0 \\ & & & 0 \end{bmatrix}$$

Multiplying the two upper-triangular matrices "pushes" the first non-zero "diagonal" further up until we get the zero matrix. See the example below.

Exercise 5.0.9. Let $L = gl_n(\mathbb{F})$ with char $\mathbb{F} \neq 2$. Let $H = n + \mathbb{F}I$, where $n \in N$ and N is the subalgebra of strictly upper triangular matrices. Then this is a maximal nilpotent subalgebra but not a Cartan subalgebra.

Solution: We first show that H is nilpotent.

$$h_1 = a_1 I + n_1 \ \in H$$

$$h_2 = a_2 I + n_2 \ \in H$$

where n_1 and n_2 are strictly upper triangular matrices. Then

$$\begin{aligned}
[h_1, h_2] &= [n_1, n_2] \\
[h_1, h_2] &= [a_1 I + n_1, a_2 I + n_2] \\
&= [a_1 I, a_2 I] + [n_1, a_2 I] + [a_1 I, n_2] + [n_1, n_2] \\
&= [n_1, n_2]
\end{aligned}$$

By lemma 5.0.8 the $(i, i+1)$ entry of this matrix is 0.

Then we proceed by induction. Let $h \in H$ and n be a matrix whose $(i, i+t)$ entries are 0 for $t \leq l$.

Then by lemma 5.0.8 and the first part of this proof $[h, n]$ has one more zero diagonal. After $n-1$ steps we get the 0 matrix hence H is nilpotent.

Assume H is not maximal, that is, $H \subsetneq L$ where L is nilpotent. Then $N_L(H) \supsetneq H$ where $N_L(H)$ is nilpotent.

We calculate $N_L(H)$. Let $A \in N_L(H)$. We calculate the commutators (Lie brackets) of A by E_{ij}, where $i < j$. We perform the precise calculations for E_{12}. Then

$$[A, E_{12}] = AE_{12} - E_{12}A$$

$$A \cdot E_{12} - E_{12}A = \begin{bmatrix} 0 & a_{11} & \\ \vdots & a_{21} & \\ & \vdots & 0 \\ \vdots & & \\ 0 & a_{n1} & \end{bmatrix} - \begin{bmatrix} a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ & & 0 & & \\ & & & & \end{bmatrix}$$

$$
= \begin{bmatrix} a_{21} & a_{11}-a_{22} & -a_{23} & \dots & -a_{2n} \\ 0 & -a_{21} & & & \\ \vdots & \vdots & & & \\ & & & 0 & \\ \vdots & & & & \\ 0 & -a_{n1} & & & \end{bmatrix} \in H
$$

Hence we have

$$
a_{31}, \dots, a_{n1} = 0
$$
$$
-a_{21} = a_{21} \;\to\; a_{21} = 0
$$

Performing the same calculations with $E_{23}, E_{34}, \dots, E_{(n-1)n}$. We obtain that A is an upper triangular matrix.

Conclusion: $N_L(H) =$ is the set of upper triangular matrices.

We claim $N_H(L)$ is not nilpotent. Moreover if $K = \langle A, H \rangle$ is nilpotent, then $A = aI$ is a scalar matrix.
Fix i, j such that $a_i \neq a_j$. E_{ij} $i < j$. Then

$$
[A, E_{ij}] = (a_i - a_j)E_{ij}.
$$

$$
K^1 = [K, K] \;\ni\; E_{ij}.
$$
$$
K^2 = [K, K^1] \;\ni\; E_{ij}.
$$

Hence $E_{ij} \in K^t$ for each t and so $K^t \neq 0$ for all t. We can therefore conclude that H is not Cartan.

5.1 Derivations and automorphisms

We recall that. A ***derivation*** on a Lie algebra L is a linear map $\delta\colon L \to L$ such that the following condition is satisfied, that is

$$
\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]
$$

for all $x, y \in L$.

The ***inner derivation*** on any $x \in L$ is the adjoint mapping defined by:

$$
ad_x(y) := [x, y],
$$

which is the derivation as a result of the Jacobi identity.

Outer derivations on L are derivations which do not come from the adjoint representation of the Lie algebra L.

Lemma 5.1.1. Let $I \triangleleft L$ be a Lie algebra ideal. Then the adjoint representation ad_L of L acts as derivations on I since $[x, i] \subset I$ for any $x \in L$ and $i \in I$.

Example 5.1.2. Let B_n be a Lie algebra of upper triangular matrices in $gl_n(\mathbb{F})$. B_n has an ideal N_n of strictly upper triangular matrices. The commutator of elements in B_3 and N_3 gives:

$$\left[\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}, \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} a & ax & ay+bz \\ 0 & 0 & dz \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & dx & ex+yf \\ 0 & 0 & fz \\ 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & (a-d)x & (a-f)y - ex + bz \\ 0 & 0 & (d-f)z \\ 0 & 0 & 0 \end{bmatrix},$$

this shows there exists outer derivations of B_3 in $\mathrm{Der}(N_3)$.

5.2 Conjugacy and Decomposition of Cartan subalgebras

Definition 5.2.1 (Conjugacy of Cartan subalgebras). Let L be a Lie algebra. Given an element $x \in L$ it can be shown that any Cartan subalgebra is the null component $(L_{0,x})$ of an element $x \in L$. Then given two regular elements in L their null components are conjugate in L.

Definition 5.2.2 (Rank of a Lie algebra). The dimension of the Cartan subalgebra H of L is called the rank of L.

Proposition 5.2.3. For a semisimple Lie algebra L over an algebraically closed field of characteristic 0, the Cartan subalgebra H has the properties.

1. H is abelian.
2. The image ad_H of the adjoint representation $ad : L \to gl(L)$ consists of diagonizable matrices.

Proof of property 1 of proposition 5.2.3: A semisimple Lie subalgebra is automatically abelian over an algebraically closed field. A Cartan subalgebra can thus be defined as a semisimple subalgebra over an algebraically closed field of characteristic zero.

Example 5.2.4. Examples of Cartan subalgebras.

1. If a Lie algebra is nilpotent, then it is its own Cartan subalgebra.
2. For $sl_n(\mathbb{F})$ and $gl_n(\mathbb{F})$ the diagonal matrices form a Cartan subalgebra. This follows from proposition 5.0.7.

Let L be a Lie algebra and H a Cartan subalgebra of L. We consider L as an H-module. By Theorem 3.3.10 we have a decomposition of L into indecomposable summands given by

$$L = H \oplus (\oplus .... L_\lambda).$$

Here, H is the 0 component and L_λ belongs to a 1-dimensional representation $\lambda : H \to C$.

More precisely, L_λ consists of elements of L which are annihilated by some power of $ad_x - \lambda I$ for all $x \in H$.

Definition 5.2.5 (**Roots, Cartan decomposition**)**.** The 1-dimensional representations $\lambda H \to C$ occuring in the above decomposition of L are called roots of L.

The subspaces L_λ are the weight subspaces or root subspaces of H in L.

The decomposition

$$L = H \oplus (\oplus .... L_\lambda).$$

is called the Cartan decomposition of L with respect to H.

Proposition 5.2.6. For a simple Lie algebra each L_λ is indeed 1-dimensional.

Chapter 6

The Killing form

Unless otherwise stated, information in this Chapter was derived from [Car05] apart from the self-made examples, definitions or exercises.

Definition 6.0.1 (**Adjoint map**)**.** Let $(L, [\ ,\ ])$ be a complex Lie algebra and $x \in L$, then define.

$$\begin{gathered} \mathrm{ad}_x \colon L \mapsto L \\ L \mapsto \mathrm{ad}_x(L) \colon = [x, L], \end{gathered}$$

or simply $\mathrm{ad}_x \colon = [x, \cdot]$ is called the adjoint map with respect to $x \in L$.

Let L be a Lie algebra and $x, y \in L$. The Killing form of L is the bilinear map $L \times L \mapsto \mathbb{C}$ given by

$$\langle x, y \rangle = tr(\mathrm{ad}_x \quad \mathrm{ad}_y).$$

Theorem 6.0.2. The Killing form $\langle x, y \rangle = tr(\mathrm{ad}_x \quad \mathrm{ad}_y)$ satisfies the following properties:

1. Bilinearity: $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z,)$.
2. Symmetry: $\langle x, y \rangle = \langle y, x \rangle$.
3. Antisymmetric: $(\mathrm{ad}_z(x), y) = -(x, \mathrm{ad}_z(y))$.
4. $\langle x, y \rangle$ is invariant, that is $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$

where $\alpha, \beta \in \mathbb{R}$, $x, y, z \in L$.

Proof:

Bilinearity of the Killing form is obvious from the linearity of the operator and the linearity of the trace.

Symmetric since L is a finite dimensional vector space and tr is cyclic hence $tr(AB) = tr(BA)$ for every matrix A,B.

Antisymmetric property is a consequence of the Jacobi identity.

Invariant property of the Killing form can be proven as follows.

$$\begin{aligned}
\langle [x,y],z\rangle &= tr(\mathrm{ad}[x,y] \ \ \mathrm{ad}_z) \\
&= tr((\mathrm{ad}_x \ \ \mathrm{ad}_y - \mathrm{ad}_y \ \ \mathrm{ad}_x)\,\mathrm{ad}_z) \\
&= tr(\mathrm{ad}_x \ \ \mathrm{ad}_y \ \ \mathrm{ad}_z) - tr(\mathrm{ad}_y \ \ \mathrm{ad}_x \ \ \mathrm{ad}\,z) \\
&= tr(\mathrm{ad}_x \ \ \mathrm{ad}_y \ \ \mathrm{ad}_z) - tr(\mathrm{ad}_x \ \ \mathrm{ad}_z \ \ \mathrm{ad}\,y) \\
&= tr(\mathrm{ad}_x \ \ tr(\mathrm{ad}_y \ \ \mathrm{ad}_z - \mathrm{ad}_z \ \ \mathrm{ad}_y)) \\
&= tr(\mathrm{ad}_y \ \ \mathrm{ad}[y,z]) \\
&= \langle x,[y,z]\rangle
\end{aligned}$$

Definition 6.0.3 (**Non-degenerate bilinear form**)**.** A bilinear form is non-degenerate on a vector space V if whenever

$$\langle x,y\rangle = 0, \ \text{ then } \ x = 0 \ \ \forall\, y \in V$$

Example of $sl_2(\mathbb{R})$ non-degenerate Killing form. $sl_2(\mathbb{F})$ has the basis e_1, e_2, e_3 from example 3.3.4. The adjoint representation with respect to this basis is given by

$$\mathrm{ad}_{e_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \ \mathrm{ad}_{e_2} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathrm{ad}_{e_3} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Therefore, the matrix of the Killing form of $sl_2(\mathbb{F})$ is

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

Definition 6.0.4 (**Identically zero bilinear form**)**.** A bilinear form of L is identically zero if

$$\langle x,y\rangle = 0 \ \ \forall\, x,y \in L.$$

See Example 3.3.5

Example 6.0.5. Let L be a Lie algebra with a $\mathbb{C}$ basis given by $\{e_1, e_2, e_3\}$ and a Lie bracket given by

$$[e_1,e_2] = e_2 \ \ [e_1,e_3] = \alpha e_3 \ \text{ and } \ [e_2,e_3] = 0$$

where $\alpha \in \mathbb{C}$ is fixed. Its clear that every L_k contains e_2 and therefore L is not nilpotent. The adjoint matrices are given by

$$\mathrm{ad}_{e_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}.$$

$$\mathrm{ad}_{e_2} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\mathrm{ad}_{e_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha & 0 & 0 \end{bmatrix}.$$

Therefore in the same basis the Killing form matrix

$$K = \begin{bmatrix} 1+\alpha^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

when $1 + \alpha^2 = 0$ then $K = 0$

Proposition 6.0.6. Given I as an Ideal of L and $x, y, z \in L$. Then the Killing form of L restricted to I is the Killing form of I. That is

$$\langle x, y \rangle_I = \langle x, y \rangle_L$$

.

Proof: We begin by selecting an I basis and extending it to a L basis. Hence with respect to this basis $\mathrm{ad}_x \colon L \to L$ is represented by a matrix of the form.

$$\begin{bmatrix} D_1 & D_2 \\ 0 & 0 \end{bmatrix}$$

since $x \in I$. Likewise, $\mathrm{ad}_y \colon L \to L$ is represented by a matrix of form

$$\begin{bmatrix} E_1 & E_2 \\ 0 & 0 \end{bmatrix}$$

hence $\mathrm{ad}_x \ \ \mathrm{ad}_y \colon L \to L$ is represented by a matrix of the form

$$\begin{bmatrix} D_1E_1 & D_1E_2 \\ 0 & 0 \end{bmatrix}$$

thus we have

$$\begin{aligned} tr_L(\mathrm{ad}_x \ \ \mathrm{ad}_y) &= tr D_1 E_1 \\ &= tr_I(\mathrm{ad}_x \, \mathrm{ad}_y) \end{aligned}$$

and therefore

$$\langle x, y \rangle_L = \langle x, y \rangle_I$$

Chapter 7

Root Systems

Unless otherwise stated, information in this Chapter was derived from [Hum12] apart from the self-made examples, definitions or exercises.

7.1 Root Systems

Before we define a root system we need to understand a few terms that will be vital in our definition.

Definition 7.1.1 (Euclidean space). A Euclidean n-space denoted $\mathbb{E}^n$ is a finite-dimensional vector space over $\mathbb{R}^n$ endowed with an inner product $\langle \cdot, \cdot \rangle$.

Definition 7.1.2 (Reflction in Euclidean space). A reflection in a Euclidean space($\mathbb{E}^n$) is an invertible linear transformation which fixes pointwise some hyperplane $P_\alpha = \{\beta \in \mathbb{E} | (\beta, \alpha) = 0\}$ and maps all orthogonal vectors to P_α to their negatives.

Let $\mathbb{E}$ be a Euclidean space, $\langle \beta, \alpha \rangle$ be the dot product, and denote the reflection in the hyperplane $P_\alpha = \{\beta \in \mathbb{E} | (\beta, \alpha) = 0\}$ by

$$\begin{aligned} \sigma_\alpha(\beta) &= \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha \\ &= \beta - \langle \beta, \alpha \rangle \alpha, \end{aligned}$$

where

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

Definition 7.1.3 (Root System). A subset Φ of the Euclidean space $\mathbb{E}$ is called a *root system* in $\mathbb{E}$ if the following conditions are satisfied:

1. Φ is finite, spans $\mathbb{E}$ and does not contain 0.
2. If $\alpha \in \Phi$, the reflection σ_α, leaves Φ invariant.
3. If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.
4. If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

Definition 7.1.4 (**A total ordering**)**.** Let $\mathbb{E}$ be a Euclidean space containing all the roots $\alpha \in \Phi$ and let $\lambda, \mu \in \mathbb{E}$. A total ordering on $\mathbb{E}$ is defined as the relation $<$ on $\mathbb{E}$ satisfying the following conditions.

1. $\lambda < \mu$ and $\mu < v$ implies $\lambda < v$ where $v \in \mathbb{E}$.
2. For every set of elements $\lambda, \mu \in \mathbb{E}$ only one of the conditions $\lambda < \mu, \lambda = \mu, \mu < \lambda$ holds.
3. If $\lambda < \mu$ then $\lambda + v < \mu + v$.
4. If $\lambda < \mu$ and $\xi \in \mathbb{R}$ with $\xi > 0$ then $\xi\lambda < \xi\mu$ and if $\xi < 0$ then $\xi\mu <$
$xi\lambda$.

Definition 7.1.5 (**Positive system** $\Phi^+ \subset \Phi$)**.** is defined as the set of all roots $\alpha \in \Phi$ satisfying $0 < \alpha$ for some total ordering on V.

Definition 7.1.6 (**A fundamental system** $\Pi \subset \Phi^+$)**.** Let $\alpha \in \Phi^+$. The fundamental system $\Pi \subset \Phi^+$ is defined as $\alpha \in \Pi$ if and only if $\alpha \in \Pi^+$ and α can not be expressed as the sum of two elements of Φ^+.

Definition 7.1.7 (**Fundamental roots**)**.** A set of fundamental roots is a subset of all the roots $(\Pi \subset \Phi)$ such that the following conditions are satisfied.

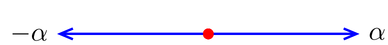
1. Π is linearly independent, $\Pi = \{\Pi_1, \cdots, \Pi_f\}$.
2. Every $\lambda \in \Pi$: $\exists n_1, \cdots, n_f \in \mathbb{N}$ such that $\exists\ \epsilon \in \{+1, -1\}$: $\lambda = \epsilon \sum_{i=1}^{f} n_i \Pi_i$.

Example 7.1.8. Let $n = 1$, $\alpha \in \mathbb{E}^1$. Then

$$\Phi = \{\alpha, -\alpha\}$$

is a root system, $\Phi^+ = \{\alpha\}$, $\Phi^- = \{-\alpha\}$ and $\Pi = \{\alpha\}$ is a fundamental system. This root system is denoted by A_1.

Remark: Root system diagram A_1



Suppose that we have two root systems $(\mathbb{E}_1, \Phi_1), (\mathbb{E}_2, \Phi_2)$ where $\mathbb{E}_1, \mathbb{E}_2$ are real vector spaces and Φ_1, Φ_2 are root systems. Then

$$(\mathbb{E}_1 \oplus \mathbb{E}_2, \Phi_1 \cup \Phi_2)$$

is a root system. It is called the direct sum of Φ_1 and Φ_2.

The inner producton $\mathbb{E}_1 \oplus \mathbb{E}_2$ is the natural one induced from $\langle , \rangle_{\mathbb{E}_1}$ and $\langle , \rangle_{\mathbb{E}_2}$.

Thus

$$\langle (e_1, e_2), (e_1', e_2') \rangle = \langle e_1, e_1' \rangle_{\mathbb{E}_1} + \langle e_2, e_2' \rangle_{\mathbb{E}_2}.$$

$$\Phi_1 \subset \mathbb{E}_1 \subset \mathbb{E}_1 \oplus \mathbb{E}_2 \text{ by the identification } e_1 = (e_1, 0)$$

$$\Phi_2 \subset \mathbb{E}_2 \subset \mathbb{E}_1 \oplus \mathbb{E}_2 \text{ by the identification } e_2 = (0, e_2)$$

Also note that $\mathbb{E}_1 \perp \mathbb{E}_2$ since

$$\langle (e_1, 0), (0, e_2) \rangle = \langle e_1, 0 \rangle_{\mathbb{E}_1} + \langle 0, e_2 \rangle_{\mathbb{E}_2} = 0.$$

Definition 7.1.9 (**Reducible root system**). A root system $(\mathbb{E}, \Phi)$ is reducible if there exists an orthogonal decomposition

$$\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2,$$

such that $\dim \mathbb{E}_1, \dim \mathbb{E}_2 > 0$ and for all $\alpha \in \Phi$, either $\alpha \in \mathbb{E}_1$ or $\alpha \in \mathbb{E}_2$. Otherwise $(\mathbb{E}, \Phi)$ is called *irreducible*. $(\mathbb{E}, \Phi)$ is reducible with the decomposition $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$.

Let

$$\Phi_1 = \Phi \cap \mathbb{E}_1, \Phi_2 = \Phi \cap \mathbb{E}_2,$$

then

$$\Phi = \Phi_1 \cup \Phi_2.$$

In this case $(\mathbb{E}_1, \Phi_1), (\mathbb{E}_2, \Phi_2)$ are root systems and $(\mathbb{E}, \Phi)$ is the direct sum of (E_1, Φ_1) and (E_2, Φ_2)

$$(\mathbb{E}, \Phi) = \Phi_1 \oplus \Phi_2$$

Therefore a root system $(\mathbb{E}, \Phi)$ is reducible if $(\mathbb{E}, \Phi)$ is a direct sum of two non-trivial root systems.

Definition 7.1.10 (**Isomorphic root systems**). Two root systems are $(\mathbb{E}, \Phi), (F, S)$ are isomorphic if there exists an invertible linear transformation: $A : \mathbb{E} \mapsto F$ such that $A(\Phi) = S$ and for all $\alpha \in \Phi, \beta \in \mathbb{E}$,

$$A(S_\alpha \cdot \beta) = S_{A\alpha} \cdot (A\beta).$$

Such a map A is called an isomorphism.

Proposition 7.1.11. Let α, β be roots. Suppose $\alpha \neq c\beta$. Assume that $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then one of the following condition holds.

1. $\langle \alpha, \beta \rangle = 0$

2. $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$, $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

3. $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$, $\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

4. $\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle$, $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Proof: Let

$$m_1 = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}, \quad m_2 = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}$$
$$m_1 m_2 = 4\frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = 4cos^2\theta \leq 4$$

If $m_1 m_2 = 0 \implies cos\theta = 0 \implies \langle \alpha, \beta \rangle = 0$. proves conditon 1.

Suppose $\langle \alpha, \beta \rangle \neq 0$. Then

$$\begin{aligned} \frac{m_2}{m_1} &= \frac{\langle \alpha, \alpha \rangle}{\langle \beta, \beta \rangle} \\ &\geq 1, \end{aligned}$$

and hence $0 \leq m_1 m_2 \leq 4$.

If $m_1 m_2 = 4 \implies cos\theta = 1 \implies \alpha = \pm\beta$ which is a contradiction.

If $m_1 m_2 = 1 \implies cos\theta = \frac{1}{4} \implies \theta = \frac{\pi}{3}$ or $\frac{2\pi}{3} \implies (m_1, m_2) = (1, 1)$ or $(-1, -1)$.

$(m_1, m_2) = (1, 1) \implies \theta = \frac{\pi}{3}$.

$(m_1, m_2) = (-1, -1) \implies \theta = \frac{2\pi}{3}$ this proves condition 2.

If $m_1 m_2 = 2$ then $cos\theta = \pm 1/sqrt(2)$, so the angle is $\pi/4$. On the other hand, the product can only be 2 if $m_1 = 1$ and $m_2 = 2$ or their negatives. So the ratio of the lengths is $\sqrt{(2)}$.

If $m_1 m_2 = 3 \implies \theta = \frac{\pi}{6}$.

Corollary 7.1.12. Let α, β be roots and θ be an angle between α, β.

If $\frac{\pi}{2} < \theta < \pi$ then $\alpha + \beta$ is a root.

If $0 < \theta < \frac{\pi}{2}$ then $\alpha - \beta$, $\beta - \alpha$ are roots.

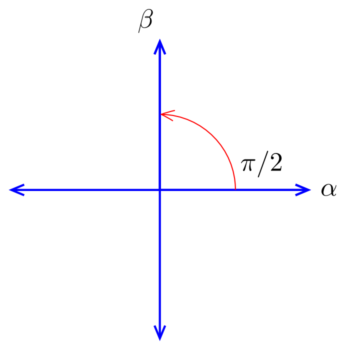
Proof: Consider condition 2 from Proposition 7.1.11 with $0 < \theta < \frac{\pi}{2} \implies \theta = \frac{\pi}{3}$

$$\begin{aligned} s_\alpha \cdot \beta &= \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \\ &= \beta - \alpha \in R \end{aligned}$$

Let us now determine all root systems of rank 2 where

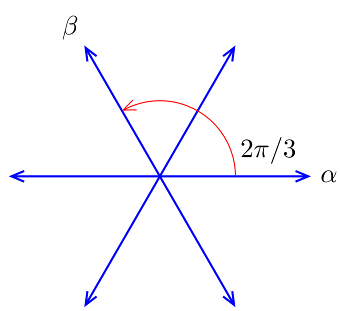
$$\{\alpha, \beta\} \in \Pi, \qquad \mathbb{E}^2 = \langle \alpha, \beta \rangle.$$

Case 1: Root system of type $A_1 \times A_1$, $\theta = \frac{\pi}{2}$. The length can be arbitrary.



Case 2: Root system with $\theta = \frac{2\pi}{3}$. This root system is denoted by A_2 and corresponds to the simple Lie algebra $sl_3(\mathbb{C})$ as it is noted later.

A_2



The root system Φ given by

$$\Phi = \{\alpha, \beta, \alpha + \beta, -\alpha, -\beta, -\alpha - \beta\}$$

We check the properties of Φ.

1. It is clear that the vectors span $\mathbb{E}^2$

2. With the reflections corresponding to the remaining roots, we have a similar result. If we perform the reflection σ_α, we see that $\sigma(\Phi) = \Phi$.
3. $(\alpha, \beta) = -\frac{1}{2}|\alpha||\beta| = -\frac{1}{2}(\alpha, \alpha)$. Hence
$$\langle \alpha, \beta \rangle = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)} = -1.$$

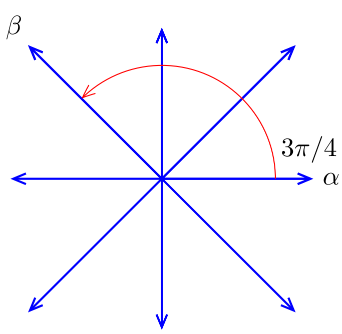
4. The positive root system and the fundamental root is given by
$$\Phi^+ = \{\alpha, \beta, \alpha + \beta\} \text{ and } \Pi = \{\alpha, \beta\}$$

Case 3: $\theta = 2\pi/4$. This root system is called B_2. The roots have two different lengths such that

$$\| \textit{ long root } \| = \| \textit{ short root } \| \cdot \sqrt{2}.$$

By repeatedly reflecting with respect to the hyperplanes orthogonal to the roots, we obtain the following:

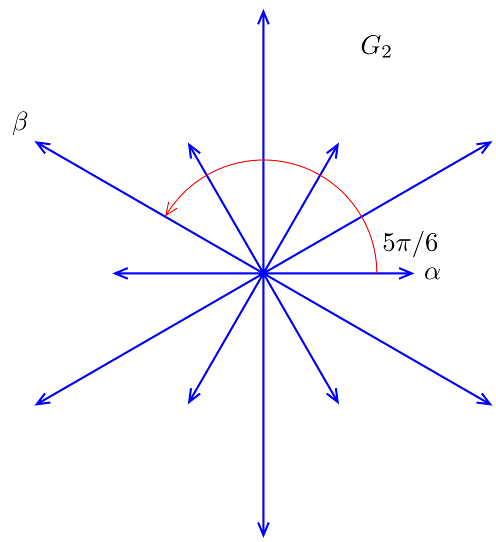
B_2



Case 4: $\theta = \frac{5\pi}{6}$. This root system is called G_2 The roots have two different lengths such that

$$\| \textit{ long root } \| = \| \textit{ short root } \| \cdot \sqrt{3}.$$

By repeatedly reflecting with respect to the hyperplanes orthogonal to the roots, we obtain the following:



Since we have no other possibility for θ, we have determined all root systems of rank 2 up to isomorphism.

7.2 Weyl group

Definition 7.2.1 (**Weyl group**). Let L be a simple Lie algebra, let H be a Cartan subalgebra of L, and let Φ be the set of roots of L with respect to H. For every $\alpha \in \Phi$, define $s_\alpha \colon H \to H$ by

$$s_\alpha(h) = h - \frac{2\langle \alpha, H\rangle}{\langle \alpha, \alpha\rangle}\,\alpha,$$

where $s_\alpha \in GL(H)$ and $\langle \cdot, \cdot \rangle$ is the inner product on H. The ***Weyl group*** of Φ is the subgroup W of $GL(h)$ generated by s_α, $\alpha \in \Phi$. From the definition of a root system s_α preserves Φ and W is a finite group.

Theorem 7.2.2. For all $w \in W$, $\alpha \in \Phi$. then $w.\alpha \in \Phi$.

Note: W acts on Φ, so $W.\Phi = \Phi$ since $s_\alpha \cdot s_\alpha \cdot w = w$. This means that if you apply the reflection twice you get the identity map.

Corollary 7.2.3. W is finite.

Proof: We know that $H = span(\Phi)$ and $w \in W$, $w \colon H \to H$ is determined by $w \cdot \alpha$, $\alpha \in \Phi$ but we also know that $w \cdot \alpha \in \Phi$. Then w is determined by the permutation $w\Phi \colon \Phi \to \Phi$ which is a bijection hence Φ is finite which implies w is finite.

$$\mid \phi \mid < \infty \implies |w| < \infty$$

Chapter 8

Cartan Matrix

Unless otherwise stated, information in this Chapter was derived from [Car05] apart from the self-made examples, definitions or exercises.

Let $\mathbb{E}$ be a Euclidean space in terms of the scalar product $\langle , \rangle$. Let Φ be a root system. Then any fundamental system $\Pi \subset \Phi$ forms a basis in $\mathbb{E}$ and the roots of Φ span $\mathbb{E}$.

We begin by considering the angles that can exist between pairs of roots α, β as well as the relative lenghts of the roots α, β. The angle will be chosen to satisfy $0 \leq \theta \leq \pi$

Recall Proposition 7.1.11. Then let $\Pi = \{\alpha_1, \cdots, \alpha_l\}$ be a fundamental system of roots. We represent the angles between α_i and their relative lenghts in form of a matrix given by

$$C_{ij} = 2\,\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} \text{ where } C_{ij} \in \mathbb{Z} \quad i, j = 1, \cdots, l.$$

The $l \times l$ matrix $C = C_{ij}$ is called the ***Cartan matrix***.

Proposition 8.0.1. Properties of the Cartan matrix.

1. $C_{ii} = 2$ for all i.
2. $C_{ij} \in \{0, -1, -2, -3\}$ if $i \neq j$
3. $C_{ij} = -2$ or -3 then $C_{ji} = -1$.
4. $Cij = 0$ if and only if $C_{ji} = 0$

Definition 8.0.2. [Generator] Q is called a generator of L if each element of L is a linear combination of elements of Q.

A semisimple Lie algebra L can be associated with an $n \times n$ Cartan matrix, where n denotes the rank of L. The simple roots of the Lie algebra are the basis vectors and we can determine the elements C_{ij} by their inner product when using the Killing form

$$C_{ij} = 2\,\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

Using the generators $\{q_i, r_i, s_i\}$ we can reconstruct the Lie algebra L up to isomorphism. That is

$$L = q \oplus r \oplus s$$

where q, r, s are Lie subalgebras that are generated by the generators of the same letter.

Examples of Cartan matrices.

Case 1: Type A_1. The only 1×1 such matrix is (2).

Case 2: If we have any two distinct simple roots α_i and α_j then their corresponding 2×2 Cartan matrix is of the form

$$\begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix},$$

and has the following possibilities

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

For $A_1 \times A_1$ it is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

because the two simple roots are orthogonal, the numerator's inner product is 0 in

$$2\,\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

Hence the off-diagonal entries are 0.

For A_2 it is

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

For B_2 it is

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

For G_2 it is

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Chapter 9

Dynkin diagrams

The Dynkin diagram classifies every semisimple Lie algebra L. A Dynkin diagram is a graph that has several different types of edges. The graph's connected components correspond to L's irreducible subalgebras. As a result, the Dynkin diagram of a basic Lie algebra has only one component. The rules are stringent. In fact, each component has just a few possibilities, which corresponds to the classification of semi-simple Lie algebras.

Definition 9.0.1 (Bond number). The bond number is given by

$$\begin{aligned} n_{ij} &= C_{ij}C_{ji} \text{ for } i \neq j \\ &= 4\,\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha} \frac{\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} \\ &= 4cos^2\theta, \end{aligned}$$

where θ is the angle between α, β. Therefore $0 \leq n_{ij} \leq 4$, if $i \neq j$ and since $0 \leq cos^2\theta \leq 1$.

Therefore, the bond number is $n_{ij} = 0, 1, 2, 3$.

The table below shows the values of $n_{ij} = C_{ij} \cdot C_{ji}$

C_{ij}	C_{ij}	n_{ij}
0	0	0
-1	-1	1
-1	-2	2
-2	-1	2
-1	-3	3
-3	-1	3

Rules for drawing dynkin diagrams.

1. For every fundamental root draw a circle.
2. If two circles represent $\alpha, \beta \in \Pi$, draw n_{ij} lines between them.
3. If there are 2 or 3 lines between two roots, write an arrow directed to the longer or shorter root.

Theorem 9.0.2. Any finite dimension simple $\mathbb{C}$ Lie algebra can be reconstructed from it's set of fundamental roots and only comes in the following forms. (**see figures below**).

Figure 9.1: A_n where $n \geq 1$. Below are examples of Dynkin diagrams of the form A_l.

$A_1 \times A_1$

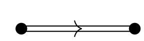
A_2

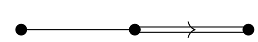
A_3

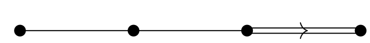
A_4

A_5

Figure 9.2: B_n where $l \geq 2$. Below are examples of Dynkin diagrams of the form B_n.

B_2 

B_3 

B_4 

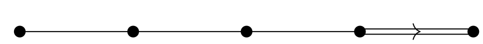
B_5 

Figure 9.3: C_n where $l \geq 3$. Below is an example of a Dynkin diagram of the form C_n.

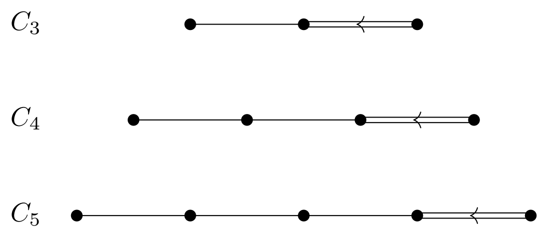


Figure 9.4: D_n where $l \geq 4$. Below are examples of Dynkin diagrams of the form D_n.

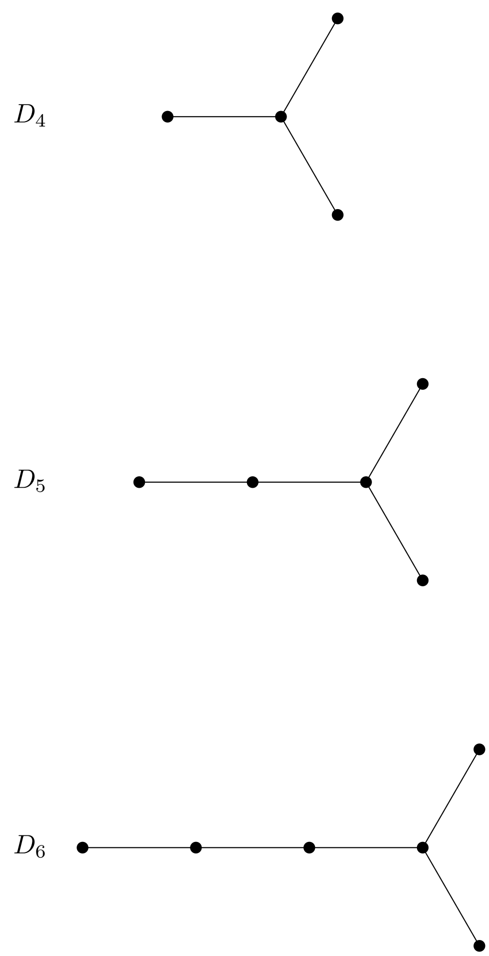
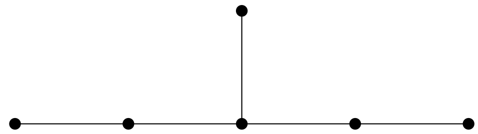
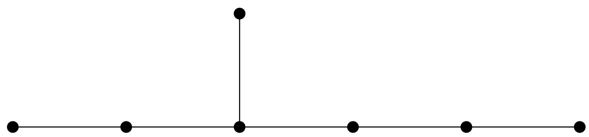


Figure 9.5: We also have the exceptional Lie algebras which have Dynkin diagrams of the form E_n where $6 \leq n \leq 8$. Below are examples of Dynkin diagrams of the form E_n.

E_6



E_7



E_8

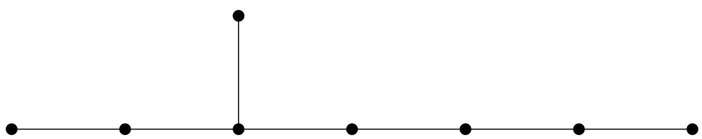


Figure 9.6: Below is a Dynkin diagram of the form F_4.

F_4

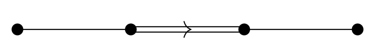


Figure 9.7: Below is a Dynkin diagram of the form G_2.

G_2

Chapter 10

Applications of Lie algebras

A very first application of Lie algebras is already in the area of algebra, more precisely, of group theory. A large amount of simple groups are closely related to simple Lie algebras and in fact can be constructed with their help. These simple groups are called simple groups of Lie type. For example, the simple groups $SL_n(q)$ correspond to the Lie algebra $sl_n(C)$. However, the construction of these simple groups is rather involved and lies beyond the framework of this thesis.

10.1 Computer Vision

Recently there has been extensive studies in Lie group theory with the aim of solving problems in computer vision. In the dynamical analysis of computer vision, transformation matrices play an important role. These matrices consists of elements known as Lie groups.The Lie group tangent space at the identity is known as the Lie algebra. The exponential map exp is a transformation from Lie algebra to Lie group elements.The product of Lie group element becomes the sum of Lie algebra elements using exponential mappings, making dynamical analysis simple.

The main problems in computer vision include **object tracking** and **pattern recognition**.

Object tracking.

Object tracking is employed in a range of applications, including vision-based robot control, medical applications, and so forth. This issue has been solved in a variety of ways, the most common of which are feature monitoring and template tracking. In the template tracking method, an iterative least squares problem was used to minimize the sum of squared differences between the template and image intensities.The method is slow since it takes each iteration to compute the image gradient, Jacobian, and Hessian. These methods use linearization to estimate the additive updates to the motion parameters. Lie group methods use Lie algebra to estimate additive updates to motion parameters.

As opposed to the conventional approach. When Lie algebra representation is used each intermediate transform is a rigid transformation as well. This means the sequence elements remain within the subgroup. The Lie algebra-defined process is optimal in that it corresponds to the shortest geodesics linking the two transforms on the affine manifold. This makes the study of Lie groups theory very useful in motion estimation.

Pattern recognition.

One challenge in constructing statistical models for parameterized geometric transformations is that their inherent group structure makes it difficult to use statistical methods that depend on underlying vector spaces. Many of these problems are alleviated by Lie algebra theory. Its main purpose is to build equivalent representations in vector space while remaining connected to the geometric transform group. As a result, we obtain a vector representation of each transform, to which many commonly used statistical models can be easily applied.[XM12]

10.2 Mathematical Modelling in Natural Sciences.

A wide range of mathematical models expressed in terms of nonlinear differential equations can successfully be solved by Lie group methods. Lie group analysis is particularly useful in the study of nonlinear differential equations because its algorithms behave as consistently in this case as they do in linear cases. Lie group analysis proposes a precise mathematical formulation of intuitive ideas of symmetry and provides analytical methods for solving non-linear differential equations. Lie group analysis enables the discovery of symmetries in differential equations that would otherwise be hidden.

Knowledge of group analysis is essential for developing and testing nonlinear mathematical models of natural and engineering problems. Lie symmetries can be used to investigate a wide range of physical phenomena in order to find various group invariant solutions.The group invariance principle can be used to derive differential equations, conservation laws, solutions to boundary value problems, and so on.

Lie group analysis has been used in the following fields of mathematical modeling and natural science. Applications of Lie theory to hydrodynamics and finite-difference equations, nonlinear optics and acoustics, diffusion and wave phenomena, industrial mathematics, field theory, astrophysics and Earth sciences. [II12]

10.3 Mathematical Finance and Economics.

One of the most important applications of Lie Theory to Economics is the application of Lie groups of partial differential equations to multidimensional screening problems. Many industries have a non-proportional relationship between the price paid by customers and the quantity purchased. Furthermore, in the case of nonlinear tariffs, the payment is frequently determined as a function of several characteristics, each of which is valued differently by each customer. As a result, a one-dimensional characteristic is insufficient to represent the customer typology in general. This creates a multi-dimensional screening problem.

Profit maximization is always one of the monopolist's primary objectives.An intriguing way to accomplish this is through the appropriate selection of a tariff. Keeping in mind that a tariff is nothing more than a real function on a set of goods bundles. The tariff enables us to calculate the price a consumer will pay for a specific bundle of goods. Solving this problem usually require solving a boundary problem involving a system of nonlinear partial differential equations. When symmetries exist, the problem can be significantly simplified and even explicitly solved. This is equivalent to explicitly solving a PDE system using Lie Theory. [HMNT09]

Chapter 11

Bibliography

[Bou89] Nicolas Bourbaki. *Lie Groups and Lie Algebras: Chapters 1-3*, volume 1. Springer Science & Business Media, 1989.

[Car05] Roger W. Carter. *Lie Algebras of Finite and Affine Type.* Number 96 in Cambridge studies in advanced Mathematics. Cambridge University Press, Cambridge, 2005.

[Hal03] Brian C. Hall. *Lie groups, Lie lgebras, and representationc*, volume 222 of *Graduate Texts in Mathematics.* Springer Verlag, New York, 2003.

[HMNT09] Isabel Hernández, Consuelo Mateos, Juan Núñez, and Ángel F Tenorio. Lie theory: applications to problems in mathematical finance and economics. *Applied Mathematics and Computation*, 208(2):446–452, 2009.

[Hum12] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[II12] NH Ibragimov and RN Ibragimov. Applications of lie group analysis to mathematical modelling in natural sciences. *Mathematical Modelling of Natural Phenomena*, 7(2):52–65, 2012.

[Jac62] N. Jacobson. *Lie Algebras.* Dover Publications Inc. New York, 1962.

[XM12] Qiang Xu and Dengwu Ma. Applications of lie groups and lie algebra to computer vision: A brief survey. In *2012 International Conference on Systems and Informatics (ICSAI2012)*, pages 2024–2029. IEEE, 2012.