Image Processing on Tensor-Product Basis

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***Abstract**: The paper introduces a tensor-product based representation of digital images and shows how their processing can be performed. The image function in this case is expressed by one-variable smooth functions forming an orthonormal basis, which are specific to the expressed function and therefore less number of components is needed to achieve the same approximation accuracy than in case of trigonometric functions or orthonormal polynomials. It will also be shown how these one-variable functions can be determined using the higher order singular value decomposition (HOSVD). The proposed techniques work well even in cases, when beside the color further attributes are assigned to the pixels, e.g. temperature, various type of labels, etc. Finally the results are compared to the techniques working in the frequency domain. It can be observed that in many cases the usage of the proposed domain is more advantageous.*

***Keywords**: scaling; smoothing; compression; tensor-product; approximation; HOSVD*

1 Introduction

There are numerous digital image processing tasks, e.g. image smoothing, edge detection, etc. which efficiency strongly depends on the domain they are working on, e.g. image resolution enhancement, filtering [1], image compression [2], etc. The two-dimensional array of pixels is the most natural way to represent discrete images, applicable for example for histogram modification, pixel and neighbor operations, etc. [10]. On the other hand, in some applications, the data are actually collected in the frequency domain, specifically in the form of Fourier coefficients, e.g. MR or CT imaging and spectral methods for PDEs [11]. Another type of representation is based on Wavelet transforms which are multi-resolution representations of signals and images decomposing them into multi-scale details [12]. Neural network based representation techniques stand for a further group of image representations, e.g non-linear image representations based on pyramidal decomposition with neural network [3], etc. The main aim of this paper is to propose a tensor-product based representation domain, in which the image can be expressed by less number of components than for example the frequency based representation requires by supporting the efficient multidimensional filtering, compression and rescaling. There are numerous image processing tasks which can be performed more efficiently when switching to another domain, e.g. in the well known frequency domain the image filtering or compression. On the other hand, to represent the image in frequency domain without meaningful quality decline, relatively large number of components is needed in contrast to the proposed tensor-product based representation, where the number of components to achieve the same approximation accuracy than in case of trigonometric ones is much more less. As shown in the upcoming sections, any *n*-variable smooth function can be expressed with the help of a system of orthonormal one-variable smooth functions on higher order singular value decomposition (HOSVD) basis. The main aim of the paper is to numerically reconstruct these specially determined one-variable functions using the HOSVD and to show how this approach can give support for certain image processing tasks and problems. The paper is organized as follows: Section 2 deals with the reconstruction of the one-variable functions in detail, Section 3 shows how this representation can be applied in image processing for resolution enhancement and filtering, while in Section 4 the properties of the proposed method are compared to the well known Fourier transformation. Finally in Section 5 experimental results and conclusions are reported.

2 The HOSVD-based Representation of Functions

The approximation methods of mathematics are widely used in theory and practice for several problems. If we consider an *n*-variable smooth function

$f(\mathbf{x}), \mathbf{x} = (x_1, ..., x_N)^T, x_n \in [a_n, b_n], 1 \leq n \leq N,$

then we can approximate the function $f(\mathbf{x})$ with a series

$$f(\mathbf{x}) = \sum_{k_1=1}^{I_1} ... \sum_{k_N=1}^{I_N} \alpha_{k_1,...,k_n} p_{1,k_1}(x_1) \cdot ... \cdot p_{N,k_N}(x_N). \tag{1}$$

where the system of orthonormal functions $p_{n,k_n}(x_n)$ can be chosen in classical way by orthonormal polynomials or trigonometric functions in separate variables and the numbers of functions I_n playing role in (1) are large enough. With the help of Higher Order Singular Value Decomposition (HOSVD) a new approximation method was developed in [7], [5] in which a specially determined system of orthonormal functions can be used depending on function $f(\mathbf{x})$, instead of some other systems of orthonormal polynomials or trigonometric functions.

Assume that the function $f(\mathbf{x})$ can be given with some functions $\tilde{w}_{n,i}(x_n), x_n \in [a_n, b_n]$ in the form

$$f(\mathbf{x}) = \sum_{k_1=1}^{I_1} \dots \sum_{k_N=1}^{I_N} \alpha_{k_1,\dots,k_n} \tilde{w}_{1,k_1}(x_1) \cdot \dots \cdot \tilde{w}_{N,k_N}(x_N). \tag{2}$$

Denote by $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_N}$ the N-dimensional tensor determined by the elements $\alpha_{i_1,\dots,i_N}, 1 \le i_n \le I_n, 1 \le n \le N$ and let us use the following notations (see : [4]).

- $\mathcal{A} \boxtimes_n \mathbf{U}$: the n-mode tensor-matrix product,
- $\mathcal{A} \boxtimes_{n=1}^{N} \mathbf{U}_n$: the multiple product as $\mathcal{A} \boxtimes_1 \mathbf{U}_1 \boxtimes_2 \mathbf{U}_2 \dots \boxtimes_N \mathbf{U}_N$.

The n-mode tensor-matrix product is defined by the following way. Let $\mathbf{U}$ be an $K_n \times M_n$-matrix, then $\mathcal{A} \boxtimes_n \mathbf{U}$ is a $M_1 \times \dots \times M_{n-1} \times K_n \times M_{n+1} \times \dots \times M_N$ -tensor for which the relation

$$(\mathcal{A} \boxtimes_n \mathbf{U})_{m_1,\dots,m_{n-1},k_n,m_{n+1},\dots,m_N} \stackrel{def}{=} \sum_{1 \le m_n \le M_n} a_{m_1,\dots,m_n,\dots,m_N} U_{k_n,m_n}$$

holds. Detailed discussion of tensor notations and operations is given in [4]. We also note that we use the sign $\boxtimes_n$ instead the sign $\times_n$ given in [4]. Using this definition the function (2) can be rewritten as a tensor product form

$$f(\mathbf{x}) = \mathcal{A} \boxtimes_{n=1}^{N} \tilde{w}_n(x_n), \tag{3}$$

where $\tilde{w}_n(x_n) = (\tilde{w}_{n,1}(x_n), \dots, \tilde{w}_{n,I_n}(x_n))^T, 1 \le n \le N$. Based on HOSVD it was proved in [6] that under mild conditions the (3) can be represented in the form

$$f(\mathbf{x}) = \mathcal{D} \boxtimes_{n=1}^{N} w_n(x_n), \tag{4}$$

where

- $\mathcal{D} \in \mathbb{R}^{r_1 \times \dots \times r_N}$ is a special (so called core) tensor with the properties:

  (a) $r_n = rank_n(\mathcal{A})$ is the n-mode rank of the tensor $\mathcal{A}$, i.e. rank of the linear space spanned by the n-mode vectors of $\mathcal{A}$:

  $$\{(a_{i_1,\dots,i_{n-1},1,i_{n+1},\dots,i_N}, \dots, a_{i_1,\dots,i_{n-1},I_n,i_{n+1},\dots,i_N})^T : 1 \le i_j \le I_n, 1 \le j \le N\},$$

  (b) all-orthogonality of tensor $\mathcal{D}$: two subtensors $\mathcal{D}_{i_n=\alpha}$ and $\mathcal{D}_{i_n=\beta}$ (the nth indices $i_n = \alpha$ and $i_n = \beta$ of the elements of the tensor $\mathcal{D}$ keeping fix) orthogonal for all possible values of n, α and $\beta : \langle \mathcal{D}_{i_n=\alpha}, \mathcal{D}_{i_n=\beta} \rangle = 0$

when $\alpha \neq \beta$. Here the scalar product $\left\langle \mathcal{D}_{i_n=\alpha}, \mathcal{D}_{i_n=\beta} \right\rangle$ denotes the sum of products of the appropriate elements of subtensors $\mathcal{D}_{i_n=\alpha}$ and $\mathcal{D}_{i_n=\beta}$,

(c) ordering: $\left\| \mathcal{D}_{i_n=1} \right\| \geq \left\| \mathcal{D}_{i_n=2} \right\| \geq \cdots \geq \left\| \mathcal{D}_{i_n=r_n} \right\| > 0$ for all possible values of n ($\left\| \mathcal{D}_{i_n=\alpha} \right\| = \left\langle \mathcal{D}_{i_n=\alpha}, \mathcal{D}_{i_n=\alpha} \right\rangle$ denotes the Kronecker-norm of the tensor $\mathcal{D}_{i_n=\alpha}$).

- Components $w_{n,i}(x_n)$ of the vector valued functions

$$w_n(x_n) = (w_{n,1}(x_n), ..., w_{n,r_n}(x_n))^T, 1 \leq n \leq N,$$

are orthonormal in L_2-sense on the interval $[a_n, b_n]$, i.e.

$$\forall n: \int_{a_n}^{b_n} w_{n,i_n}(x_n) w_{n,j_n}(x_n) dx = \delta_{i_n, j_n}, \quad 1 \leq i_n, j_n \leq r_n,$$

where $\delta_{i,j}$ is a Kronecker-function ($\delta_{i,j} = 1$, if $i = j$ and $\delta_{i,j} = 0$, if $i \neq j$) The form (4) was called in [6] HOSVD canonical form of the function (2).

Let us decompose the intervals $[a_n, b_n]$, $n = 1..N$ into M_n number of disjunct subintervals Δ_{n,m_n}, $1 \leq m_n \leq M_n$ as follows:

$$\xi_{n,0} = a_n < \xi_{n,1} < \ldots < \xi_{n,M_n} = b_n, \Delta_{n,m_n} = [\xi_{n,m_n}, \xi_{n,m_n-1}).$$

Assume that the functions $w_{n,k_n}(x_n)$, $x_n \in [a_n, b_n]$, $1 \leq n \leq N$ in the equation (2) are piece-wise continuously differentiable and assume also that we can observe the values of the function $f(\mathbf{x})$ in the points

$$\mathbf{y}_{i_1,...,i_N} = (x_{1,i_1}, ..., x_{N,i_N}), 1 \leq i_n \leq M_n \tag{5}$$

where

$$x_{n,m_n} \in \Delta_{n,m_n}, \quad 1 \leq m_n \leq M_n, 1 \leq n \leq N$$

Based on the HOSVD a new method was developed in [6] for numerical reconstruction of the canonical form of the function $f(\mathbf{x})$ using the values $f(\mathbf{y}_{i_1,...,i_N})$, $1 \leq i_n \leq M_n$, $1 \leq i_n \leq N$. We discretize function $f(\mathbf{x})$ for all grid points as:

$$b_{m_1,..,m_N} = f(\mathbf{y}_{m_1,..,m_N}).$$

Then we construct N dimensional tensor $\mathcal{B} = (b_{m_1,\dots,m_N})$ from the values $b_{m_1,\dots,m_N}$. Obviously, the size of this tensor is $M_1 \times \dots \times M_N$. Further, we discretize vector valued functions $\mathbf{w}_n(x_n)$ over the discretization points x_{n,m_n} and construct matrices $\mathbf{W}_n$ from the discretized values as:

$$\mathbf{W}_n = \begin{pmatrix} w_{n,1}(x_{n,1}) & w_{n,2}(x_{n,1}) & \cdots & w_{n,r_n}(x_{n,1}) \\ w_{n,1}(x_{n,2}) & w_{n,2}(x_{n,2}) & \cdots & w_{n,r_n}(x_{n,2}) \\ \vdots & & \ddots & \vdots \\ w_{n,1}(x_{n,M_n}) & w_{n,2}(x_{n,M_n}) & \cdots & w_{n,r_n}(x_{n,M_n}) \end{pmatrix} \tag{6}$$

Then tensor $\mathcal{B}$ can simply be given by (4) and (6) as

$$\mathcal{B} = \mathcal{D} \boxtimes_{n=1}^{N} \mathbf{W}_n. \tag{7}$$

Matrices $\mathbf{W}_n$ and tensor $\mathcal{D}$ can be obtained by HOSVD of $\mathcal{B}$. For further details see [5].

3 Image Scaling in the HOSVD-Based Domain

Let $f(\mathbf{x}), \mathbf{x} = (x_1, x_2, x_3)^T$ represent the image function, where x_1 and x_2 correspond to the vertical and horizontal coordinates of the pixel, respectively. x_3 is related to the color components of the pixel, i.e. the red, green and blue color components in case of *RGB* image. Function $f(\mathbf{x})$ can be approximated (based on notes discussed in the previous section) in the following way:

$$f(\mathbf{x}) = \sum_{k_1=1}^{I_1}\sum_{k_2=1}^{I_2}\sum_{k_3=1}^{I_3} \alpha_{k_1,k_2,k_3} w_{1,k_1}(x_1) \cdot w_{2,k_2}(x_2) \cdot w_{3,k_3}(x_3). \tag{8}$$

The red, green and blue color components of pixels can be stored in a $m \times n \times 3$ tensor, where n and m correspond to the width and height of the image, respectively. Let $\mathcal{B}$ denote this tensor. The first step is to reconstruct the functions $w_{n,k_n}, 1 \le n \le 3, 1 \le k_n \le I_n$ based on the HOSVD of tensor $\mathcal{B}$ as follows:

$$\mathcal{B} = \mathcal{D} \boxtimes_{n=1}^{3} \mathbf{W}^{(n)} \tag{9}$$

where $\mathcal{D}$ is the so called core tensor. Vectors corresponding to the columns of matrices $\mathbf{W}^{(n)}, 1 \le n \le 3$ as described in the previous section are representing the discretized form of functions $w_{n,k_n}(x_n)$ corresponding to the appropriate dimension n, $1 \le n \le 3$.

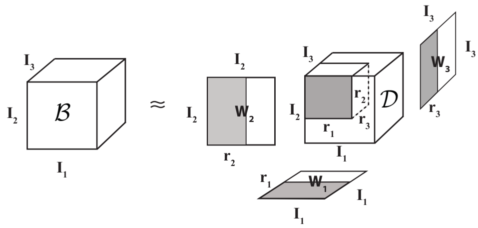

Figure 1
Illustration of the higher order singular value decomposition for a 3-dimensional array. Here ***D*** is the core tensor, the W_n -s are the *n*-mode singular matrices.

Our goal is to demonstrate the effectiveness of image scaling in the proposed domain. Let $s \in \{1, 2, ...\}$ denote the scaling factor of the image. First, let us consider the first column $W_1^{(1)}$ of matrix $\mathbf{W}^{(1)}$. Based on the previous sections, it can be seen, that the value $w_{1,1}(1)$ corresponds to the 1st element of $W_1^{(1)}$, $w_{1,1}(2)$ to the 2nd element,..., $w_{1,1}(M_n)$ to the M_n th element of $W_1^{(1)}$. To enlarge the image by a factor *s*, the $\mathbf{W}^{(i)}, i = 1..2$ matrices should be updated, based on the scaling factor *s*, as follows: The number of columns remains the same, the number of lines will be extended according to the factor *s*. For example let us consider the column $W_1^{(1)}$ of $\mathbf{W}^{(1)}$. $W_1^{(1)}(1)$ does not change, $W_1^{(1)}(s) := W_1^{(1)}(2), W_1^{(1)}(2s) := W_1^{(1)}(3), ..., W_1^{(1)}((M_n - 1)s) := W_1^{(1)}(M_n)$.

The missing elements $W_1^{(1)}(2), W_1^{(1)}(3), ..., W_1^{(1)}(s-1), W_1^{(1)}(s+1), ..., W_1^{(1)}(2s-1)$, $W_1^{(1)}(2s+1), ..., W_1^{(1)}((M_n - 1)s - 1)$ can be determined by interpolation. In the paper the cubic spline interpolation was applied. The remaining columns should be processed similarly. After every matrix element has been determined the enlarged image can be obtained using the equation (9).

4 HOSVD vs. Frequency Domain

Comparing the proposed representation to the frequency domain, similarities can be observed in their behaviour. As it is well known, the Fourier Transformation is connected to trigonometric functions, while in case of HOSVD approach the functions $w_{n,i}(x_n)$ are considered, which are specific to the approximated *n*-variable function. In both cases the functions are forming an orthonormal basis. Let us mention some common, widely used applications of both approaches.

In case of the Fourier based smoothing, some of the high frequency components are dismissed, resulting a smoothed image (low pass filter). In case of the HOSVD similar effect can be observed when dismissing functions corresponding to smaller singular values. In the opposite case, i.e. dismissing small frequencies yields an edge detector (high pass filter) which in HOSVD case is equivalent to dismissing components corresponding to higher singular values [13].

5 Examples

5.1 Part-1 (Approximation)

In this section some approximations can be observed, performed by the proposed and by the Fourier-based approach. As the number of the used components decreases, the observable differences in quality become more significant. In the examples below in both the HOSVD-based and Fourier-based cases the same number of components have been used in order to show how the form of determined functions influences the quality.

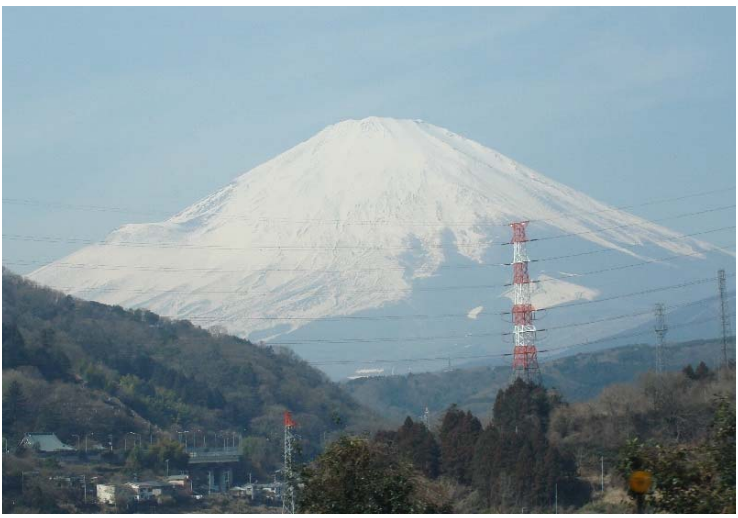

Figure 2
Original image (24bit RGB)

Figure 3
HOSVD-based approximation using 2700 components composed from polylinear functions on HOSVD basis

Figure 4
Fourier-based approximation using 2700 components composed from trigonometric functions

5.2 Part-2 (Scaling)

The pictures are illustrating the effectiveness of the image scaling by applying the proposed approach. The result is compared to the output obtained by the bilinear and bicubic image interpolation methods.

Figure 5
The original image

Figure 6
Enlarged segment using bilinear interpolation

Figure 7
Enlarged segment using bicubic interpolation

Figure 8
Enlarged segment using the proposed HOSVD-based method. Smoother edges can be observed

Conclusions

In the present paper a new image representation domain and reconstruction technique has been introduced. The results show that how the efficiency of the certain tasks depends on the applied domain. Image rescaling has been performed using the proposed technique and has been compared to other well known image interpolation methods. Using this technique the resulted image maintains the edges more accurately then the other well-known image interpolation methods. Furthermore, some properties of the proposed representation domain have been compared to the corresponding properties of the Fourier-based approximation. The results show that in the proposed domain some tasks can be performed more efficiently than in other domains.

Acknowledgement

The research was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences and in part by the Óbuda University.

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