HOSVD-based Color Image Representation in Relation to Fourier Image Processing

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***Abstract**: The paper introduces a new data representation domain and approximation approach based on Higher Order Singular Value Decomposition (HOSVD). The results have been compared to the Fourier representation domain. Based on measurements a special set of one variable orthonormal functions is numerically reconstructed, which represents the key point of the proposed method. The reconstruction is performed by the mentioned HOSVD. The such determined one variable functions can more efficiently be applied for data approximation than trigonometric functions or orthogonal polynomials. Additionally the number of one-variable functions necessary to approximate the input n dimensional data will be relatively lower than in case of other techniques. This property of the approach enables to perform much more efficient data compression by maintaining the accuracy of the approximation at a higher level.*

***Keywords**: Multidimensional data approximation; HOSVD; Fourier domain; Numerical reconstruction*

1 Introduction

In the field of image processing the representation forms of digital images play an important topic. Usually digital images are represented by a rectangular grid of pixels, where to each pixel an intensity value (grayscale image) or in case of color images – depending on the applied color model – several components may be assigned.

In many cases some tasks can be performed more efficiently in other domains, e.g. frequency domain, than in the spatial one.

An important consideration when developing an algorithm to process image data is the used representation form appropriate for the concrete task [1], [2]. Many representation forms are related to expressing the image intensity function as a linear combination of simpler functions or components having useful predefined properties.

A very frequently used representation form of this type is the well known frequency domain related to Fourier series. In this representation domain the components are trigonometric functions. In such a domain many signal processing related tasks (e.g. filtering, data compression, etc.) can more effectively be performed than in the spatial one. On the other hand to represent the image in the frequency domain without meaningful quality decline, relatively large number of trigonometric components is needed.

Another concepts of representing digital images are coming from the field of soft computing. These are using neural networks and other soft computing techniques like fuzzy to represent image data [2].

In this paper a novel approach is proposed for representing digital images and having similar properties – from the application point of view – than the Fourier based approach. The proposed representation is based on polylinear functions on Higher Order Singular Value Decomposition (HOSVD) basis [3], [4].

The main goal concerning the proposed method is to numerically reconstruct the above mentioned polylinear functions based on pixels of the original input image and to point out the effectiveness of this concept in comparison with the well known Fourier based representation. In case of HOSVD-based representation it can be observed, that the number of polylinear functions (components) expressing the image without disturbing its quality will be much more less then the number of trigonometric components needed to achieve the same image quality in case of Fourier based representation. The proposed approach can effectively be applied in many applications, e.g. point cloud processing, resolution enhancement, vision based 3D reconstruction [6], etc.

The paper is organized as follows: Section 2 describes in detail, how multivariable nonlinear functions can be approximated using a specially determined system of orthonormal functions, based on which, Section 3 deals with the HOSVD based representation of RGB images together with its possible applications and comparison to the Fourier-based domain. Section 4 shows the experimental results and finally conclusions are reported.

2 Multivariable Nonlinear Functions on HOSVD Basis

Let consider an *n*-variable smooth function

$$f(x),\ x=(x_1,...,x_N)^T,\ x_n\in[a_n,b_n],\ 1\le n\le N,$$

then it can be approximated with the series

$$f(x)=\sum_{k_1=1}^{I_1}...\sum_{k_N=1}^{I_N}\alpha_{k_1,...,k_n}\,p_{1,k_1}(x_1)\cdot...\cdot p_{N,k_N}(x_N). \tag{1}$$

where the system of orthonormal functions $p_{n,k_n}(x_n)$ can be chosen in classical way by orthonormal polynomials or trigonometric functions in separate variable and the numbers of functions I_n playing role in (1) large enough. The approximation error also strongly depends on the from of the one variable functions in (1). Later on we will see, that much more number of trigonometric functions is needed in order to achieve the same approximation accuracy then in case when these functions are specifically determined.

With the help of Higher Order Singular Value Decomposition (HOSVD) a new approximation method was developed in [7], [4] in which a specially determined system of orthonormal functions can be used depending on function *f*(*x*), instead of some system of orthonormal polynomials or trigonometric functions.

Assume that the function *f*(*x*) can be given with some functions $\widetilde{w}_{n,i}(x_n), x_n\in[a_n,b_n]$ in the form

$$f(x)=\sum_{k_1=1}^{I_1}...\sum_{k_N=1}^{I_N}\alpha_{k_1,...,k_n}\,\widetilde{w}_{1,k_1}(x_1)\cdot...\cdot\widetilde{w}_{N,k_N}(x_N). \tag{2}$$

Denote by $\mathcal{A}\in\mathbb{R}^{I_1\times...\times I_N}$ the *N*-dimensional tensor determined by the elements $\alpha_{i_1,...,i_N}, 1\le i_n\le I_n, 1\le n\le N$ and let us use the following notations (see : [5]).

- $\mathcal{A}\boxtimes_n\mathbf{U}$: the n-mode tensor-matrix product,
- $\mathcal{A}\boxtimes_{n=1}^N\mathbf{U}_n$: multiple product as $\mathcal{A}\boxtimes_1\mathbf{U}_1\boxtimes_2\mathbf{U}_2...\boxtimes_N\mathbf{U}_N$.

The n-mode tensor-matrix product is defined by the following way. Let **U** be an $K_n\times M_n$-matrix, then $\mathcal{A}\boxtimes_n\mathbf{U}$ is an $M_1\times...\times M_{n-1}\times K_n\times M_{n+1}\times...\times M_N$ -tensor for which the relation

$$(\mathcal{A}\boxtimes_n\mathbf{U})_{m_1,...,m_{n-1},k_n,m_{n+1},...,m_N}\overset{def}{=}\sum_{1\le m_n\le M_n}a_{m_1,...,m_n,...,m_N}U_{k_n,m_n}$$

holds. Detailed discussion of tensor notations and operations is given in [5]. We also note that we use the sign $\boxtimes_n$ instead the sign $\times_n$ given in [5] Using this definition the function (2) can be rewritten as a tensor product form

$$f(x) = \mathcal{A} \boxtimes_{n=1}^{N} \widetilde{w}_n(x_n), \tag{3}$$

where $\widetilde{w}_n(x_n) = (\widetilde{w}_{n,1}(x_n), ..., \widetilde{w}_{n,I_n}(x_n))^T, 1 \le n \le N$. Based on HOSVD it was proved in [7] that under milde conditions the (3) can be represented in the form

$$f(x) = \mathcal{D} \boxtimes_{n=1}^{N} w_n(x_n), \tag{4}$$

where

- $\mathcal{D} \in \mathbb{R}^{r_1 \times ... \times r_N}$ is a special (so called core) tensor with the properties:
  (a) $r_n = rank_n(\mathcal{A})$ is the n-mode rank of the tensor $\mathcal{A}$, i.e. rank of the linear space spanned by the n-mode vectors of $\mathcal{A}$:

  $\{(a_{i_1,...,i_{n-1},1,i_{n+1},...,i_N}, ..., a_{i_1,...,i_{n-1},I_n,i_{n+1},...,i_N})^T : 1 \le i_j \le I_n, 1 \le j \le N\}$,

  (b) all-orthogonality of tensor $\mathcal{D}$: two subtensors $\mathcal{D}_{i_n=\alpha}$ and $\mathcal{D}_{i_n=\beta}$ (the n-th indices $i_n = \alpha$ and $i_n = \beta$ of the elements of the tensor $\mathcal{D}$ keeping fix) orthogonal for all possible values of n, α and $\beta : \langle \mathcal{D}_{i_n=\alpha}, \mathcal{D}_{i_n=\beta} \rangle = 0$ when $\alpha \ne \beta$. Here the scalar product $\langle \mathcal{D}_{i_n=\alpha}, \mathcal{D}_{i_n=\beta} \rangle$ denotes the sum of products of the appropriate elements of subtensors $\mathcal{D}_{i_n=\alpha}$ and $\mathcal{D}_{i_n=\beta}$,

  (c) ordering: $\left\| \mathcal{D}_{i_n=1} \right\| \ge \left\| \mathcal{D}_{i_n=2} \right\| \ge \cdots \ge \left\| \mathcal{D}_{i_n=r_n} \right\| > 0$ for all possible values of n ($\left\| \mathcal{D}_{i_n=\alpha} \right\| = \langle \mathcal{D}_{i_n=\alpha}, \mathcal{D}_{i_n=\alpha} \rangle$ denotes the Kronecker-norm of the tensor $\mathcal{D}_{i_n=\alpha}$).
- Components $w_{n,i}(x_n)$ of the vector valued functions

  $w_n(x_n) = (w_{n,1}(x_n), ..., w_{n,r_n}(x_n))^T, 1 \le n \le N,$ are orthonormal in L_2-sense on the interval $[a_n, b_n]$.

The form (4) was called in [7] HOSVD canonical form of the function (2).

Let us decompose the intervals $[a_n, b_n]$, $n = 1..N$ into M_n number of disjunct subintervals Δ_{n,m_n}, $1 \le m_n \le M_n$ as follows:

$$\xi_{n,0} = a_n < \xi_{n,1} < \ldots < \xi_{n,M_n} = b_n, \ \Delta_{n,m_n} = [\xi_{n,m_n}, \xi_{n,m_n-1}).$$

Assume that the functions $w_{n,k_n}(x_n),\ x_n \in [a_n, b_n], 1 \le n \le N$ in the equation (2) are piece-wise continuously differentiable and assume also that we can observe the values of the function $f(x)$ in the points

$$y_{i_1,\dots,i_N} = (x_{1,i_1},\dots,x_{N,i_N}), 1 \le i_n \le M_n. \tag{5}$$

where

$$x_{n,m_n} \in \Delta_{n,m_n}, \quad 1 \le m_n \le M_n, 1 \le n \le N$$

Based on the HOSVD a new method was developed in [7] for numerical reconstruction of the canonical form of the function $f(x)$ using the values $f(y_{i_1,\dots,i_N}), 1 \le i_n \le M_n, 1 \le i_n \le N.$ We discretize function $f(x)$ for all grid points as:

$$b_{m_1,\dots,m_N} = f(\mathbf{y}_{m_1,\dots,m_N}).$$

Then we construct N dimensional tensor $\mathcal{B} = (b_{m_1,\dots,m_N})$ from the values $b_{m_1,\dots,m_N}$. Obviously the size of this tensor is $M_1 \times \dots \times M_N$. Further, discretize vector valued functions $\mathbf{w}_n(x_n)$ over the discretization points x_{n,m_n} and construct matrices $\mathbf{W}_n$ from the discretized values as:

$$\mathbf{W}_n = \begin{pmatrix} w_{n,1}(x_{n,1}) & w_{n,2}(x_{n,1}) & \cdots & w_{n,r_n}(x_{n,1}) \\ w_{n,1}(x_{n,2}) & w_{n,2}(x_{n,2}) & \cdots & w_{n,r_n}(x_{n,2}) \\ \vdots & & \ddots & \vdots \\ w_{n,1}(x_{n,M_n}) & w_{n,2}(x_{n,M_n}) & \cdots & w_{n,r_n}(x_{n,M_n}) \end{pmatrix} \tag{6}$$

Then tensor $\mathcal{B}$ can simply be given by (4) and (6) as

$$\mathcal{B} = \mathcal{D} \boxtimes_{n=1}^{N} \mathbf{W}_n. \tag{7}$$

The HOSVD decomposition of the discretization tensor can be written as

$$\mathcal{B} = \mathcal{D}^d \boxtimes_{n=1}^{N} \mathbf{U}^{(n)} \tag{8}$$

where $\mathcal{D}^d$ is the so-called core tensor, and $\mathbf{U}^{(n)} = \begin{pmatrix} U_1^{(n)} & U_2^{(n)} & \dots & U_{M_n}^{(n)} \end{pmatrix}$ is an $M_n \times M_n$-size orthogonal matrix ($1 \le n \le N$). Further details regarding this approximation approach can be found in [7].

3 HOSVD-based RGB Image Representation

Let $f(x), x = (x_1, x_2, x_3)^T$ stand for the image function, where x_1 and x_2 correspond to the vertical and horizontal coordinates of the pixel respectively. Variable x_3 is related to the color components of the pixel, i.e. in case of RGB image there are three possible elements for this dimension, i.e. the red, green and blue color components. Function $f(x)$ can be approximated (based on notes discussed in the previous section) in the following way:

$$f(x) = \sum_{k_1=1}^{I_1}\sum_{k_2=1}^{I_2}\sum_{k_3=1}^{I_3}\alpha_{k_1,k_2,k_3}\, \tilde{w}_{1,k_1}(x_1)\cdot \tilde{w}_{2,k_2}(x_2)\cdot \tilde{w}_{3,k_3}(x_3). \tag{9}$$

The color components of each pixel can be stored in a $m \times n \times 3$ tensor, where n and m correspond to the width and height of the image respectively. Let $\mathcal{B}$ denote this tensor. The first step is to reconstruct the functions $\tilde{w}_{n,k_n}, 1 \le n \le 3, 1 \le k_n \le I_n$ by decomposing the tensor $\mathcal{B}$ using the HOSVD as follows:

$$\mathcal{B} = \mathcal{D}^d \boxtimes_{n=1}^{3} \mathbf{U}^{(n)} \tag{10}$$

where $\mathcal{D}^d$ is the so called core tensor. Vectors corresponding to the columns of matrices $\mathbf{U}^{(n)}, 1 \le n \le 3$ as described in the previous section are representing the discretized form of functions $\tilde{w}_{n,k_n}(x_n)$ corresponding to the appropriate dimension *n*, $1 \le n \le 3$. It means, we will have as many functions for a dimension as many columns there are in the orthonormal matrix corresponding to that dimension. The number of these functions can be further decreased by dismissing some columns from the orthonormal matrices obtained by HOSVD (see. Fig.). Let C_n, $0 \le C_n \le I_n$, $n = 1..N$ stand for the number of dismissed columns in *n*th dimension. The approximation in this case can be performed as follows:

$$f(x) = \sum_{k_1=1}^{I_1-C_1}\sum_{k_2=1}^{I_2-C_2}\sum_{k_3=1}^{I_3-C_3}\alpha_{k_1,k_2,k_3}\, \tilde{w}_{1,k_1}(x_1)\cdot \tilde{w}_{2,k_2}(x_2)\cdot \tilde{w}_{3,k_3}(x_3). \tag{11}$$

3.1 HOSVD vs. Fourier-based Approach - Discussion

It is notorious that the Fourier Transform is related to trigonometric functions forming an orthonormal basis.

In case of HOSVD-based approach instead of trigonometric functions polylinear eigenfunctions are used, forming an orthonormal basis, as well. As introduced in the previous sections these functions are specific ones, they have specific properties as described in the second section. These properties of the approach ensure that the number of functions used for the approximation can be kept at lower level in order to achieve the same output then in case of classical approaches, e.g. Fourier-based one. Let mention some common widely used applications of both approaches.

In case of Fourier-based smoothing, some of higher frequencies from the frequency domain are dismissed, due to which the singularities are eliminated, i.e. as result a smoothed image can be obtained.

In case of HOSVD considering only polylinear eigenfunctions corresponding to the larger singular values for certain dimensions will have similar effect than the above mentioned low pass frequency filtering. The same concept can be used also for data compression.

In the opposite case, i.e. when maintaining only the functions corresponding to smaller singular values, an edge detector is yielded. In case of Fourier approach detecting edges in an image is equivalent to dismissing the smaller frequency components, i.e performing the so called high pass filtering.

The examples show that in case of HOSVD much smaller number of one-variable basis functions is enough to represent the image without significant information loss. In case of Fourier-based approach much larger number of trigonometric functions is needed in order to maintain the same quality. Additionally there is a frequency threshold depending on the concrete image, (when dismissing high frequency components) below which any further dismiss of frequencies results well observable waves in the image as noise.

4 Examples

In the section the results of the approximation are compared, obtained by the proposed approach and by the Fourier-based one. As the number of components decreases, the differences in quality become more significant. Fig. 1 illustrates the original image, which has been approximated using the proposed method and compared to the results achieved by the Fourier-based approach. It can be seen, that in case of smaller number of components, the Fourier approach produces well observable waves in the image. Due to the special system of orthnormal functions in case of the proposed method this behavior is eliminated.

Figure 1
Original image (24bit RGB)

Figure 2
HOSVD-based approximation using 7500 components (left);
Fourier-based approximation using 7500 components (right)

Figure 3
HOSVD-based approximation using 2700 components (left);
Fourier-based approximation using 2700 components (right)

Conclusions

The paper introduces a new concept based on HOSVD for approximation of multi variable functions. The main advantage of the proposed method is that due to specifically determined orthonormal system of functions more accurate approximation can be achieved, than in case of trigonometric functions or orthogonal polynomials. The necessary number of components to achieve similar accuracy than in case of Fourier-based approach is significantly lower.

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