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Counting graphs

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***Abstract*— In this paper we will present some problems which show the connection between algebraic formulas and combinatorial principles. Each problem is based on the idea counting the number of special graphs or subgraphs.**

I. INTRODUCTION

Using combinatorial arguments to prove algebraic identities is an effective method. In this paper we will solve some problems in which both algebraic and combinatorial methods are involved. We will see that the ideas we use are very general.

Our methods are elementary but one can have benefits from the ideas presented below. We tried to search problems on which we have worked.

The first theorem is a result of Borbély, the proof of the third theorem presented in the paper was given by Borbély and deviates from the official one. With similar approaches one could prove other formulas of this type..

II. THE THEOREMS

As warming-up we prove the following theorem, which was a problem in the OKTV in 2013, proposed by Borbély.

Theorem 1: *For ever positive integer n, the number of simple graphs on 2n labeled vertices, whose vertices have only odd degrees, is exactly* $2^{\binom{2n-1}{2}}$.

Proof:

We will find bijection between the simple graphs on (2n-1) labeled vertices and the simple graphs on 2n labeled vertices with vertices having only odd degrees.

Let be the vertices 1,2,...,2n.

Let G be a labeled graph on the vertices 1,2,...,2n-1. Then we can connect the vertices of G exactly in a unique to get a graph with only odd degrees (we connect the vertices with even degree in G with the edge 2n). If we do that, then the vertex 2n will have an odd degree, too, because in G there mut be an odd number of vertices with odd degree, thus there must be an odd number of vertices with even degree.

Conversely, if G* is a graph on the vertices 1,2,.., 2n with the desired property, then by deleting the vertex 2n with the incidental edges we get a unique graph G on the vertices 1,2,...,2n-1. QED

Secondly we will prove the following theorem, which appeared in 1979 in the Schweitzer competititon [see 2]. We will use a clever combinatorial argument in order to prove an algebraic formula about directed graphs.

The proposer of this problem was Schrijver.

Theorem 2: *If for fixed positive integers n and k we denote the number of simple strongly connected directed graphs with k edges on n labeled vertices by G(n,k), then*

$\sum_{k=n}^{n^2-n} (-1)^k G(n,k)$=*(n-1)! .*

Proof:

We will prove the statement by induction on n. For n=1 the statement of the theorem holds. Let us assume that we proved the statement for simple directed graphs with 1,2,...,n-1 vertices (n≥2). We will prove the statement for n, too.

Let 1,2,...,n be the vertices.

We will handle two distinct cases.

Case I:

If exactly two edges are incident to the vertex n, and their other endpoints coincide, then let us denote the other endpoint by w. If the graph is simple and strongly connected, then both nw and wn must be edges. Thus, in this case, if we proceed from a strongly connected graph, then by deleting the vertex n and the edges incident to it, we get a strongly connected graph with (k-2) edges on the vertices 1,2,..., (n-1). Using the induction hypothesis, our sum is in this case (n-2)!. The vertex w can be chosen in (n-1) different ways, so the sum gives (n-1)(n-2)!=(n-1)!, if two edges are incident to the vertex n, and their other endpoints are the same.

Case II:

If the vertex n does not fulfill the conditions described in Case I, then there must be different vertices u and v such that un and nv are edges of the graph (here we used that the graph is strongely connected). Let A be the set of graphs which correspond to this description and uv is an edge from them, and let B be the set of graphs which correspond to this description and uv is not an edge from them. It is easy to see that a graph G is an element of A iff (G-uv) is an element of B. Thus in this case our sum gives 0. QED

The following theorem was proposed by Dályay Pál Péter in the journal “A matematika tanítása” (see [1]), and solved in the following way by the author Borbély (this proof deviates from the official one).

Theorem 3: *For every integer greater than one holds the identity*

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}^2 k! = \sum_{k=0}^{n-2}(-1)^{k+1}\binom{n-1}{k+1}\binom{n}{k}(k+1)!.$$

Proof:

Throughout of the whole proof, the $\sum a_k$ will denote the sum of a_i's for which the sum is interpreted.

Clearly, the right hand side can be written as

$$\sum(-1)^{k+1}\binom{n-1}{k+1}\binom{n}{k}(k+1)!.$$

Let us write the right hand side in the following way:

$\sum(-1)^{k+1}\binom{n-1}{k+1}\binom{n}{k}(k+1)!=$

$=\sum(-1)^{k+1}\frac{n-1}{k+1}\binom{n-2}{k}\binom{n}{k}(k+1)!=$

$=-(n-1)\sum(-1)^k\binom{n-2}{k}\binom{n}{k}(k)!.$

Thus we have to prove the identity

$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 k! = -(n-1)\sum(-1)^k\binom{n-2}{k}\binom{n}{k}(k)!.$

Substract from both sides the term

$$\sum(-1)^k\binom{n-2}{k}\binom{n}{k}(k)!.$$

So we get on the left hand side

$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 k! - \sum(-1)^k\binom{n-2}{k}\binom{n}{k}(k)! =$

$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}k!\left(\binom{n}{k}-\binom{n-2}{k}\right)=$

$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}k!\left(\binom{n-2}{k-1}-\binom{n-1}{k-1}\right)=$

$=\sum_{k=0}^{n}(-1)^k\frac{n}{k}\binom{n-1}{k-1}k!\left(\binom{n-2}{k-1}-\binom{n-1}{k-1}\right)=$

$=(-n)\sum(-1)^k\binom{n-1}{k-1}\binom{n-2}{k-1}(k-1)!\ +$

$(-n)\sum(-1)^k\binom{n-1}{k-1}\binom{n-1}{k-1}(k-1)! =$

$=(-n)\sum(-1)^k\binom{n-1}{k}\binom{n-2}{k}(k)!\ +$

(- n) $\sum(-1)^k \binom{n-1}{k}\binom{n-1}{k}(k)!$,

and on the right hand side

- (n) $\sum(-1)^k \binom{n-2}{k}\binom{n}{k}(k)!$.

Thus we have to prove the identity

(- n) $\sum(-1)^k \binom{n-1}{k}\binom{n-2}{k}(k)!$ +

+(-n)$\sum(-1)^k \binom{n-1}{k}\binom{n-1}{k}(k)!$=

=(-n) $\sum(-1)^k \binom{n-2}{k}\binom{n}{k}(k)!$.

Dividing both sides by (-n), we have to prove that

$\sum(-1)^k \binom{n-1}{k}\binom{n-2}{k}k!$ +

+$\sum(-1)^k \binom{n-1}{k}\binom{n-1}{k}k!$ =

= $\sum(-1)^k \binom{n-2}{k}\binom{n}{k}k!$.

For fixed positive integers r and s let $K_{r,s}$ denote the complete bipartite graph with labeled vertices and two vertex-classes having respectively r and s vertices. Let p_k(r,s) be the number of k-matchings in $K_{r,s}$ (i.e. the number of matchings that cover 2k vertices). In order to construct such a k-matching, we have to choose k vertices from both of the vertex-classes and we have to decide how to establish pairs.

Thus p_k(r,s) = $\binom{r}{k}\binom{s}{k}$k!.

In order to prove our theorem we have to verify that

$\sum(-1)^k p_k(\mathrm{n}-1,\mathrm{n}-2)$ +$\sum(-1)^k p_k(\mathrm{n}-1,\mathrm{n}-1)$ =

=$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n})$.

Let A and B be the two vertex-classes of the complete bipartite graph $K_{r,s}$ with |A|=r and |B|=s. Let x be a vertex in A.

The number of k-matchings, which do not cover the vertex x, is exactly

p_k(r-1,s).

The number of k-matchings, which do cover the vertex x, is exactly

(s-k+1)p_{k-1}(r-1,s).
Thus we have the recursion

p_k(r,s) = p_k(r-1,s) + (s-k+1)p_{k-1}(r-1,s).

Thus we get

$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n})$ =

$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n}-1)$ +

+ $\sum(-1)^k (n-k-1)p_{k-1}(\mathrm{n}-2,\mathrm{n}-1)$ =

=$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n}-1)$ +

+ $\sum(-1)^{k+1}(n-k-2)\, p_k(\mathrm{n}-2,\mathrm{n}-1)$ =

=$\sum(-1)^{k+1}(n-k-3)p_{k-1}(\mathrm{n}-2,\mathrm{n}-1)$.

On the other side we have

$\sum(-1)^k p_k(\mathrm{n}-1,\mathrm{n}-2)$ +$\sum(-1)^k p_k(\mathrm{n}-1,\mathrm{n}-1)$ =

=$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n}-1)$ +

+$\sum(-1)^k p_k(\mathrm{n}-1,\mathrm{n}-1)$ =

=$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n}-1)$ +

+$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n}-1)$ +

+ $\sum(-1)^k (n-k)p_{k-1}(\mathrm{n}-2,\mathrm{n}-1)$ =

=$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n}-1)$ +

+$\sum(-1)^k p_k(\mathrm{n}-2,\mathrm{n}-1)$ +

+$\sum(-1)^{k+1}(n-k-1)p_k(\mathrm{n}-2,\mathrm{n}-1)$ =

= $\sum(-1)^{k+1}(n-k-3)p_{k-1}(\mathrm{n}-2,\mathrm{n}-1)$.

Thus the two sides are equal, and so we proved the statement of the theorem. QED

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[1] A matematika tanítása, issue 2010/3, p.27

[2] Gábor J. Székely (editor), *Contests in higher mathematics*, Springer, 1996.