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Finding pairs in mathematical games

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Abstract— **In this paper we will present some two-personal mathematical games. The problems will be taken from mathematical competitions and we will give a winning strategy for special cases in every game presented below. Some of the problems were invented by the author.**

I. INTRODUCTION

In this paper we will present some abstract strategy games, where no chance is involved. In all of the investigated games we will use some "pairing principle", i.e. we will establish disjoint pairsto describe how one can win the game.

The tools used to solve these problems are totally elementary, but one needs to have a good idea to find the winning strategy (or to find out which player has a winning strategy). The problems were posed in mathematical competitions like Nemzetközi Magyar Matematikaverseny (NMMV), Országos középiskolai tanulmányi verseny (OKTV), or the Tournament of the Towns (TT).

Let us begin with a theorem that will ensure the determinancy of our games (see [1]):

Theorem 1 (Zermelo, 1912) *In any combinatorial game, at least one of the players has a non-losing strategy.*

Theorem 1 is called the fundamental theorem of combinatorial games. Using this theorem, we always can

be sure that in the investigated game one of the players has a winning strategy.

We will call the players in all of the games "first player" and "second player". The "first player" will be tha player who makes the first step in the game.

II. FINDING PAIRS

The following is a generalization of a problem posed in the OKTV competititon in 2014 (the proposer was Borbély).

Game 1: *Two players alternately delete a number from the set {1,2,..., n} until two numbers remain. If the sum of the two remaining numbers is a square number, then the second player wins, otherwise the first player wins.*

We will prove the following result:

Theorem 2: *If n is divisible by 8, then in Game 1 the second player has a winning strategy.*

Proof:

We will prove by induction on k, that the elements of the set {1,2,..., 8k} can be ordered into pairs in such a way that the sum of the numbers in every pair equals a square number.

Note that this would imply the result of the theorem, because player two has the possibility to erase in each of his step the number, that was the pair of the number deleted by player one in the previous step.

For k=1 the statement holds, because the pairs (1,8), (2,7), (3,6), (4,5) correspond to the conditions.

Let us assume that we proved the statement for 1,2,...,k-1. We will prove the statement for k, too. Let s be the smallest odd square number greater then 8k (where k≥2). We will show that s<16k.

Let s be s=$(2m+1)^2$=$4m^2$+4m+1.

Note that if k≥2, then m≥3. If m≥ 3, then we have the estimates

$$\frac{s}{8k}=\frac{4m^2+4\mathrm{m}+1}{8k}< \frac{4m^2+4\mathrm{m}+1}{4m^2-4\mathrm{m}+1}=1+$$

$$+\frac{8\mathrm{m}}{4m^2-4\mathrm{m}+1}\leq 1+\frac{(4\mathrm{m}-4)\mathrm{m}}{4m^2-4\mathrm{m}+1}<1+1=2,$$

which implies the inequality s<16k.

It is easy to see that (s-1) is divisible by 8, thus we can make pairs in the following way:

(8k, s-8k), (8k-1, s-8k+1),..., ($\frac{s-1}{2}$, $\frac{s+1}{2}$) . In these pairs the sum of the numbers is a square (namely the number s). The number (s-8k-1) is divisible by 8, thus by the induction hypothesis we can make pairs from the elements of the set {1,2,...,s-8k-1} in the appropriate way. So our proof is complete, we proved the required result.

Remark: it is an open question which of the two players has a winning strategy if n is not divisible by 8.

The following is a generalization of a problem posed in the NMMV competititon in 2012 (the proposer was Borbély)

Game 2: *n andk are fixed positive integers. Two players alternately delete a number from the set {1,2,..., n} until two numbers remain. If the absolute value of the difference of the two remaining numbers is a prime number greater than k, then the second player wins, otherwise the first player wins.*

We will prove the following result:

Theorem 3: *If n is even,* $\frac{n}{2}$ *is a composite number, q is the greatest prime number not exceeding* $\frac{n}{2}$ *- 1 , and q+*$\frac{n}{2}$ *is also a prime number, then in Game 2 the second player has a winning strategy iff k≤q-1 .*

Proof:

1. If k≥q, then the first player can play in such a way that he always deletes the greatest number that was not

deleted. If he plays so, then the absolute value of the difference of the two remaining numbers cannot exceed $\frac{n}{2}$, thus (by the definition of q) cannot be a prime number greater than q.

2. Now assume that k≤q-1.

We will order the elements of the set {1,2,...,n} into disjoint pairs in such a way that the absolute value of the differences for every pair is a prime number. That implies that in this case the second player has a winning strategy, because player two has the possibility to erase in each of his step the number, that was the pair of the number deleted by player one in the previous step. Let us make the pairs in the following way:

First we establish the pairs

(1, q+$\frac{n}{2}$ +1), (2, q+$\frac{n}{2}$ +2),...,($\frac{n}{2}-q, n$) , then we establish the pairs

($\frac{n}{2}-q+1, \frac{n}{2}+1$) , ($\frac{n}{2}-q+2$, $\frac{n}{2}+2$), ..., ($\frac{n}{2}$, $\frac{n}{2}+$ $+q$).

In our construction in the first group the difference is always q+$\frac{n}{2}$, which is by our assumption a prime number. In the second group the difference is always q. The pairs are disjoint, thus we proved the desired result.

Remark: it is an open question which of the two players has winning strategy if n and k have other number theoretical properties.

The following is a generalization of a problem posed in the OKTV competition in 2005:

Game 3: *Let n be a fixed positive integer. Two player write alternately numbers on the board. Player one writes down the number 1. Then in each turn the following player writes down a number that is a sum or a product of two already written numbers, but he cannot write down numbers that are already on the board and he cannot write down a number greater than n. The player who writes down the number n wins the game.*

We will prove the following result:

Theorem 4: *If n is an odd prime number, then in Game 3 player one has a winning strategy.*

Proof:

Note that if n is a prime number, then the number n can be generated only by addition.

Let us order the elements of the set {1,2,...,n} into disjoint pairs with the exception of the element n:

(1, n-1), (2, n-2),..., ($\frac{n-1}{2}$, $\frac{n+1}{2}$).

Let player one play in the following way: after each step of player two he shall write down the least positive integer x such that neither x nor n-x were written down. If he always can do that, then he wins the game because n is odd. Thus we have only to prove that player one can use this strategy throughout the whole game.

We prove this by induction on the number of the steps. In the first step he can use this strategy (he writes down the number 1). Let k≥2 and let us assume that player one was able to apply his strategy. Let us assume, that each of the players made k steps, and it is player one's turn. Let x be the least positive integer such that neither x nor n-x were written down (if no such x exists, then player ine can write down the number n).

x≤2k+1, because in the set {1,2,...,2k+1} there must be a number y such that neither y nor n-y were written down (2k numbers were written down). By the strategy of player one, he chose all his k numbers from the set {1,2,...x-1}. Player two had to choose as his first step the number 2, thus at least k+1 elements of the set {1,2,...x-1} are already written on the board. That means that player one can establish the number x by addition, because x≤2k+1.

Thus player one has a winning strategy.

Remark: one can similarly show that if n is the double of a prime number, then player two has a winning strategy. But generally it is an open question which of the two players has a winning strategy.

Finally we present two games from the reputed competition Tournament of the Towns, and we will show that the two games have a common root.

The following game was presented in the TT competititon in 1987:

Game 4: *The game is played on an 8*8 chessboard. Player one places a knight on the board. Then each player in turn moves the knight, but cannot place it on a square where it has been before. The player who cannot make a move looses.*

The following game was presented at the TT competititon in 2009:

Game 5: *Player one and player two visit Archipelago with 2009 islands. Some pairs of islands are conneceted by boots which run both ways. The folllowing game is played:*

Player one chooses the first island where they arrive by plane. Then player two chooses the next island they will visit. Thereafter, the two take turns choosing an island which they not have yet visited. When they arrive at an island which is connected only to islands they had already visited, whoever's turn to choose next would be the looser.

And now we will show the connection between Game 4 and Game 5. The problems will be reformulated in the language of graphs.

Game 6: *Player one and player two alternately select distinct vertices v_1, v_2,... of a graph in such a way that v_i and v_{i+1} always must be adjacent vertices. The last player able to choose a vertex wins the game.*

We will exactly determine when player one and when player two in Game 6 has a winning strategy. Using our result we can easily determine which of the players has a winning strategy in Game 4 and game 5.

Theorem 5: *In Game 6 player two has a winning strategy iff the graph they use for the game has a perfect matching.*

Proof:

If the graph has a perfect matching, then player two clearly has a winning strategy: he can always choose the pair of the vertex chosen by player one, and thus cannot loose the game.

If in the graph there are not any perfect matchings, then let M be a maximal matching in the graph. Player one begins with a vertex that is not covered by M. Then player two has to choose a vertex covered by M (because M was maximal). In his next turn, player one chooses the pair of the vertex chosen by player two.

We will show that player one can continue the game in such a way. More exactlx, we state that player two cannot choose a vertex not covered by M.

Namely, if in any of his steps player two would be able to choose a vertex not covered by M, then let us consider the tour already made by the players. In the tour they used an odd number of edges, that were alternately not in M and in M (in this order). Thus changing the rules of the edges in this tour we could find a matching greater than M, which is impossible.

Thus player two can only choose vertices covered by M and player one has a winning strategy.

Remark: using the result of Theorem 5, it is easy to see that in Game 5 the first player has a winning strategy. In Game 4 one can find a proper pairing which shows that here the second player has winning strategy (see [2]).

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