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A new nontriangulable polyhedron

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***Abstract*—Triangulations of 3-dimensional polyhedron are partitions of the polyhedron with tetrahedra in a face-to-face fashion without introducing new vertices. We give a new family of nontriangulable polyhedra.**

1 INTRODUCTION

It is well known that every planar polygon can be partitioned into triangles without introducing new vertices. The triangles in these triangulations automatically meet in edge-to-edge fashion, moreover the number of triangles is dependent on the side number of the original polygon. Surprisingly none of the analogous statements remain true when we move up by one dimension. For more information regarding these properties the texts [1], [2], and [3] provide a detailed analysis of triangulations. In the rest of this paper we consider polyhedra in 3-space.

Definition 1. A **triangulation** of a polyhedron *P* is a collection of tetrahedra that satisfies the following three properties:

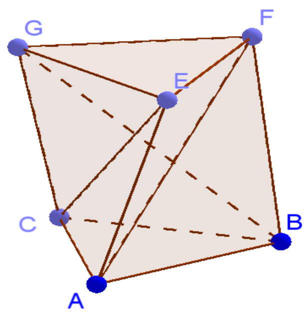

Figure1. Schönhardt's polyhedron

Vertex property: The vertices of each tetrahedron are vertices of *P*.
Union property: The union of all tetrahedra is *P*.
Face-to-face property: Any pair of tetrahedra intersect in a common face (possibly empty).

First Lennes [4] in 1911 and after that Schönhardt [5] in 1927 produce nontriangulable polyhedron. The example of Lennes was quite difficult so we write

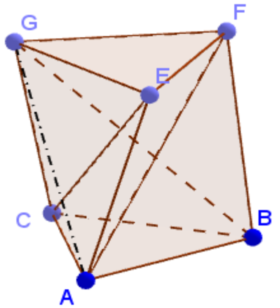

Figure2. The tetrahedron ABCG in the Schönhardt's polyhedron
solid line – visible edge
dash line – nonvisible edge
dashdot line – exterior diagonal

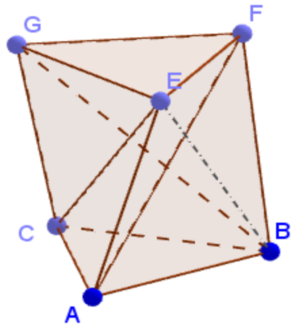

Figure3. The tetrahedron ABCE in the Schönhardt's polyhedron
solid line – visible edge
dash line – nonvisible edge
dashdot line – exterior diagonal

down the Schönhardt's example only. He observed that in the nonconvex twisted triangular prism (subsequently called "Schönhardt's polyhedron") every diagonal that is not a boundary edge lies completely in the exterior. (E.g. every tetrahedron with base triangle ABC has an exterior edge Fig. 2-4.) This implies

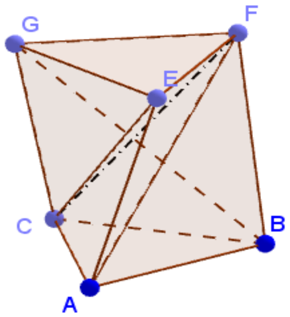

Figure4. The tetrahedron ABCF in the Schönhardt's polyhedron
solid line – visible edge
dash line – nonvisible edge
dashdot line – exterior diagonal

that there can be no triangulation of it without new vertices because there is simply no interior tetrahedron: all possible tetrahedra spanned by four of its six vertices would introduce new edges.

Bagemihl [6] modified Schönhardt's nonconvex prism to get a new polyhedron. Bagemihl's polyhedron has

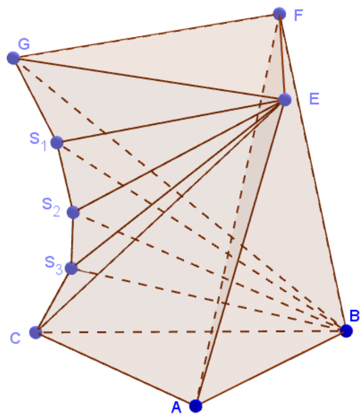

Figure 5. Bagemihl's construction

vertices more than six. He replaced a

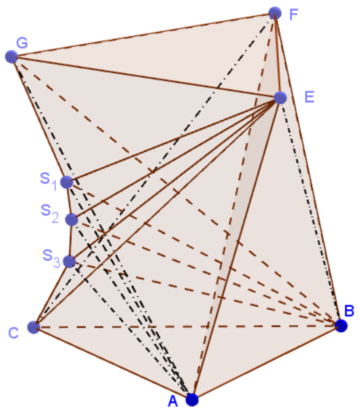

Figure 6. Bagemihl's construction with some exterior diagonals

nonhorisontal egde in the Schönhard's polyhedron with a concave curve and he add new vertices on it (Fig. 5.) In Fig. 6. we can see the exterior diagonals emanating from A, B or C.

Rambau [7] changed the top and the bottom triangles of the Schönhardt's polyhedron for n-gones. Thus he got a twisted prism over n-gon, that is, a nontriangulable polyhedron with n=2k vertices where k is an integer greater than two.

We can find a plenty of examples on nontriangulable polyhedra in [8].

2 NEW RESULT

We give a new method to construct nontriangulable polyhedron.

We modify a polyhedron of Bagemihl. Let ABC be the bottom triangle of the polyhedron and let the EFG be the top triangle of the polyhedron as in Fig. 2. Let S be the Schönhardt polyhedron with vertices $ABCEFG$.

Let H be the plane determined by the points G, E and C. Let H^+ be the halfspace determined by the plane H and containing the point A. Let B_1 be the sphere through the points C, E and G of radius $100diam(S)$ and centre not lying in H^+. Let S_1 be a point on the sphere B_1 and in the hapfspace H^+. Let α_1 be a plane through the points C, G and S_1. Let $A_1 = \alpha_1 \cap B_1 \cap H^+$. We take two points S_2 and S_3 on the arc A_1 as you can see in Fig. 2.

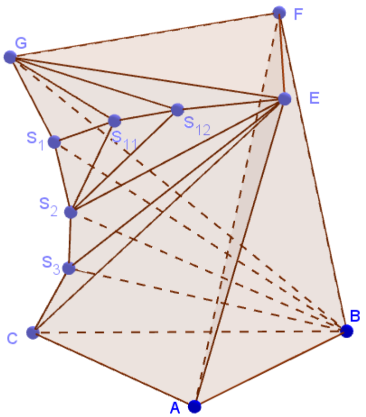

Figure 7. The modified Bagemihl's polyhedron

If the straight segment with endpoints S_3B has nonempty intersection with the plane generated by the vertices A, E and C, then we modify the position of S_1 so that the intersection will be empty. Let the faces of the Bagemihl's polyhedron be ABC, EFG, ACE, AFE, AFB, BGF, GBS_1, S_1S_2B, S_2S_3B, S_3CB, GES_1, S_1S_2E, S_2S_3E, S_3CE. The polyhedron is nontriangulable. Indeed, every tetrahedron with base ABC has an edge outside the polyhedron S.

Our aim is to "partition" the edge S_1E. The faces GS_1E, S_1S_2E, S_2S_3E and S_3CE are not visible from the vertex A. If we modify an edge so that every new vertex is not visible from the vertex A, then the new polyhedron is nontriangulable as the original polyhedron. Let S'_{11} and S'_{12} be the trisection points of the edge S_1E. (We can use the midpoint only or multisection points as well.) Let r_{11} (resp. r_{12}) be the ray emanating from the centre of B_1 and through the point S'_{11} (resp. S'_{12}). Let $S_{11} = B_1 \cap r_{11}$ and $S_{12} = B_1 \cap r_{12}$. Now we omit the faces S_1S_2E, GS_1E and we add the new faces $S_1S_{11}G$, $S_{11}S_{12}G$, $S_{12}EG$, $S_1S_{11}S_2$, $S_{11}S_{12}S_2$ and $S_{12}ES_2$ (Fig. 7.). In Fig. 8. we can see the exterior diagonals emanating from A. The new polyhedron is nontriangulable. We can repeat this method again for the edges GS_{11}, GS_{12}, S_1S_{11}, $S_{11}S_{12}$, $S_{12}E$, S_2S_{11}, S_2S_{12}, S_2E or S_3E. We can repeat the method as many times as we want. Thus we get a nontriangulabe polyhedron with $n > 7$ vertices. This was what we want to show.

3 Conclusion

If we construct a series of the Bagemihl's polyhedron with $n = 6$, $n = 7$, … vertices with the same diameter of Bagemihl's polyhedron, then we get that the minimums of the diameter of the faces of the elements of the series have a positive lover bound. If we construct a series of the Bagemihl's

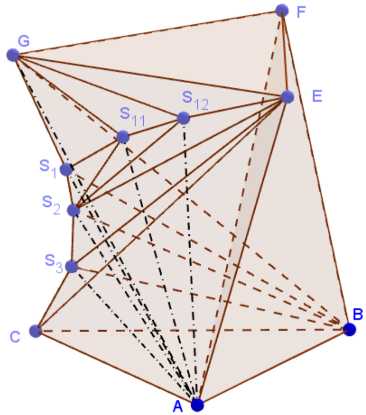

Figure 8. The modified Bagemihl's polyhedron with some exterior diagonals

modified polyhedron with $n = 7$, … vertices with the same diameter of Bagemihl's modified polyhedron, where the partitioning edge is the longest edge among the edges that we could partition, then we get that the minimums of the diameter of the faces of the elements of the series do not have a positive lover bound.

So we get a series where the nontriangulable polyhedra have very “small” faces.

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