On a problem concerning rainbow Ramsey theory

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Abstract:

In this paper we will present a problem from the branch of the so called rainbow Ramsey theory. The problem was posed by András Gyárfás for the annual Schweitzer Miklós Competition in 2009. The solution of the problem is based on the argumentation of Borbély (who submitted a solution at the competition). This solution was refined by the co-author Csala-Takács.

Introduction

Ramsey theory is a part of mathematics that handles problems where one seeks after ordered structures in some great arbitrarily chosen structures. In a more restricted sense the problems of Ramsey theory have abstract or even geometric graphs as their subjects. In this context a Ramsey type theorem includes an allegation, in which one guarantees the existence of some “ordered” subgraphs in every according graph great enough. In order to clarify the nature of problems treated in Ramsey theory, we hereby draft Ramsey’s classical coloring theorem concerning abstract graphs.

Ramsey used this theorem (in another equivalent form, see [2]) to achieve results in the formal logic.

Some Ramsey-type theorems

Here we will present some classical Ramsey-type theorems in order to demonstrate the nature of these problems. First we present Ramsey’s classical result (see [2]):

Theorem 1 (Ramsey, 1930) *For every positive integers k and s there exists a positive integer R(k, s), such that the following is true: if one colors the edges of a complete graph G with at least R(k, s) vertices wit red and blue, then there is a complete subgraph of G with k vertices having all its edges colored by red, or there is a complete subgraph of G with s vertices having all its edges colored by blue.*

Later this theorem was rediscovered and had a many other applications. In a wider sense one can speak about the so called Ramsey principle (Ramsey type results in other structures).

The theorem can be generalized for more colors in an analogous way.

Theorem 2 (Ramsey, generalized form) *For every positive integers s_1, s_2,..., s_k there exists a positive integer*

R(s_1, s_2,..., s_k) so that the following is true: if one colors the edges of a complete graph G having at least R(s_1, s_2,..., s_k) vertices with k colors, then there is a complete subgraph of G with s_i vertices having all its edges colored by color i for for some i.

There exists even an infinite version of Ramsey's theorem.

Theorem 3 (Ramsey, infinite form) *Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X such that the size n subsets of M all have the same colour.*

One of them most substantial theorems of Ramsey theory is Van der Waerden's theorem.

Theorems of the rainbow Ramsey theory have the goal to guarantee great structures colored by distinct colors. This is a developing branch of mathematics with a wide range of problems (see [1]).

Our main theorem

We present the problem of Gyárfás that was posed in 2009 at the Schweitzer competition.

Theorem 4: *If c_1, c_2, ..., c_m are colorings of the edges of a complete graph with 17 vertices (labeled by the numbers 1,2,3,..., 17) by 105 fixed colors in such a way, that for every 15-element subset H of the vertices there is a j so that in the coloring c_j all of the 105 colors occur in the coloring of the edges of H, then 34≤m. Moreover the desired conditions can be fulfilled for m=34.*

Proof:

A complete graph with 17 vertices has exactly $\binom{17}{2}$=136 edges. First we will prove that every complete graph with 17 vertices, whose edges are colored by 105 colors, contains at most 4 complete subgraphs H with 15 vertices whose edges are colored by the 105 colors.

Note that such a complete subgraph H has also the property that every edge of H has a different color (a rainbow property). Let us call such complete subgraphs rainbow-subgraphs. Now, with this terminology, we have to prove that every every complete graph with 17 vertices, whose edges are colored by 105 colors, contains at most 4 rainbow-subgraphs.

Let us take an arbitrary coloring of the edges of the complete graph with 17 vertices by 105 colors.

We will have two different cases.

Case A,

If there is a 16-element subset of the vertices where we can find at least two rainbow-subgraphs, then there is a set F of the vertices with 14 elements and two vertices x and y such that the graph spanned by the vertices of FU{x} and FU{y} are rainbow-subgraphs.

Note that in this case the color classes represented between the sets {x} and F must be the same as the ones represented between {y} and F. Therefore the edges whose endpoints are x or y represent at most 17 color classes (namely there are 31 edges with the endpoints x or y, but there are 14 color classes that are used two times, and 31-14=17).

This means that and y cannot be the vertices of a rainbow-subgraph at the same time. Therefore there are exactly two rainbow-subgraphs spanned by the vertices of the set FU{x}U{y}. Let z be the seventeenth vertex of the complete graph with 17 vertices. If there are another rainbow-subgraphs, then their vertex set must contain z. Assume that there is such a rainbow-subgraph. In this case {z}UF determines a rainbow-subgraph, or there is an element u in F so that (F\u)U{z}U{x} or (F\u)U{z}U{y} determines a rainbow-subgraph.

If {z}UF determines a rainbow-subgraph, then the color classes represented by the edges running between F and z are the same as the ones running between F and x and the ones running between F and y. According to our observations it is clear that no two elements of the set

{x, y, z} can be vertices of a rainbow-subgraph at the same time, therefore there are totally at most 3 rainbow-subgraphs.

If (F\u)U{z}U{y} spans a rainbow-subgraph, then the color classes represented by the edges running between z and (H\u)U{y} are the same as the ones running between u and (H\u)U{y}. Using our argumentations before one can easily verify that the vertices u and z cannot be the vertices of a rainbow-subgraph at the same time.

The same is true for the vertices x and y. Every rainbow-subgraph contains at least two vertices of the set {x,y,z,u}, but not more than two, because the elements of the sets {x, y} and {u, v} exclude each other. Therefore there are at most 4 rainbow-subgraphs in Case A,

Case B,

If every subset of the vertices with 16 elements defines at most one rainbow-subgraph, then let us take a 16-element subset of the vertices where there are 15 vertices determining a rainbow-subgraph.

Let us take another rainbow-subgraph, too (if there is not any more one, then we are ready). In this case there is a subset K of the set of the 17 vertices with |K|=13 and vertices r, s, t, v so that the graphs defined by the sets {r}U{s}UK and {t}U{v}UK are rainbow-subgraphs.

Then the color classes represented by the edges running between the vertex sets {r}U{s} and K are the same as the ones running between {t}U{v} and K (26 colors, totally). The edges that have r, s, t, or v as endpoints, can represent at most 32 color classes, because

$$\binom{4}{2}+26=32.$$

This means that the vertices r, s, t, v cannot be the elements of a rainbow-subgraph at the same time. Therefore there are at most 4 rainbow-subgraphs as we have stated.

Using our result we have the inequality $4m \geq \binom{17}{15}$,

which is equivalent to $34 \leq m$.

Now we will prove that there are colorings $c_1, \ldots, c_{34}$ fulfilling the requirements of the theorem.

Let L be a 13-element subset of the vertices and let a, b, c, d be the other vertices of the graph.

Let us prepare a rainbow-subgraph on the vertex set {a}U{b}UL. Let the color classes between c and L be the same as the ones between a and L, and let the color classes between d and L be the same as the ones between b and L. Let the color of the edge between b and c the same as the one between b and a. Let the color of the edge between a and d the same as the one between a and b.

In this case the vertex sets {a}U{b}UL, {a}U{d}UL, {c}U{b}UL determine rainbow-subgraphs. If we color the edge connecting c and d by the color connecting a and b, then {c}U{d}UL is a rainbow subgraph, too. We will give such colorings in all of the 34 cases. It is enough to choose the ordered quadruples (a, b, c, d) in an appropriate way.

Remember that the vertices of the graph are labeled by the numbers 1,2, …, 17. We will order to every rainbow-subgraph a two element subset in such a way, that we identify the rainbow-subgraph with the two vertices that are not the vertices of it.

With this notation and our construction, for every quadruplet (i, j, k, l) we are able to establish a coloring so that the pairs {k, l}, {j, k}, {i, l}, and {i, j} represent a rainbow-subgraph. In 17 cases we will make the coloring so, that (i, j, k, l) = (x, x+2, x+1, x+4), where x runs through the elements of the residue classes modulo 17.

k-l = -3 (mod17) ,

l-k = 3 (mod17)

j-k = 1 (mod17),

k-j = 1 (mod17)

i-l = -4 (mod17) ,

l-i = 4 (mod17)

i-j = -2 (mod17) ,

j-i = 2 (mod17)

Therefore we get all the pairs of indices that have the difference (depending on the order) 1, 2, 3, 4, -1, -2, -3, -4 modulo 17.

We will make other 17 colorings, using this pattern. In this case we will have (i, j, k, l) = (x, x+6, x+1, x+8), where x runs through the elements of the residue classes modulo 17. In this case we have

k-l = -7 (mod17) ,

l-k = 7 (mod17)

j-k = 5 (mod17),

k-j = -5 (mod17)

i-l = -8 (mod17) ,

l-i = 8 (mod17)

i-j = -6 (mod17) ,

j-i = 6 (mod17)

Therefore we get all the pairs of indices that have the difference (depending on the order) 5, 6, 7, 8, -5, -6, -7, -8 modulo 17.

This completes the proof of our theorem.

With the same method one can prove the following more general result:

Theorem 5: *If c_1, c_2, ..., c_m are colorings of the edges of a complete graph with n=8k+1 vertices (labeled by the numbers 1,2,3,..., 8k+1) by $\binom{n-2}{2}$ fixed colors in such a way, that for every (n-2)-element subset H of the vertices there is a j so that in the coloring c_j all of the colors occur in the coloring of the edges of H, then*

$\frac{1}{4}\cdot\binom{n}{2} \leq m.$

References

[1] R. Graham, B. Rothschild, J. H. Spencer, Ramsey Theory (2nd ed.), New York: John Wiley and Sons, 1990

[2] V. Jungic, J. Nesetril, R. Radoicic, Rainbow Ramsey theory, Integers, 2005, A09, 13 pages

[3] F. P. Ramsey, On a problem of formal logic, Proceedings of the London Mathematical Society, 1930, 264–286.