On some problems of the International Hungarian Mathematical Competition

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***Abstract*—In this paper we present some nice, interesting and unusual problems of the International Hungarian Mathematical Competition. The problems we investigate all need a simple idea but it is not so easy to find the way how to solve them. We tried to select the most uncoventional ones. We give a solution for all these problems but it is highly recommended to the reader searching for other solutions and generalizations.**

Our paper is dedicated to the memory of the excellent mathematics teacher György Oláh (1940-2012).

I. Introduction

In 1991 Mihály Bencze gave the idea of a Hungarian mathematical contest giving the chance to all Middle Hungarian high school students in Carpathian Basin to compete and to solve mathematical problems of higher level. In 1992 György Oláh organized the first time the International Hungarian Mathematical Competition in Révkomárom. Its Hungarian name is Nemzetközi Magyar Matematikaverseny. The competition is held every year in spring. It is a relevant aspect of the organizers to give the chance to all of the Hungarian regions to receive the talented students and their teachers. The programme usually lasts four or five days. The main regions are Hungary and the Hungarian territories before the First World War. The competition plays an essential role in cultural and mathematical aspect. It is an appropriate tool to strengthen the togetherness of Hungarian people and to compare the mathematical culture of the distinct regions. In this paper we present four nice and really hard problems from the history of the competition. The solution of the problems seems to be easy but it is not so simple to find the key idea of them. These problems illustrate the high standard of the Hungarian mathematics.

II. Discussion

In this section we present four problems of the competition. We chose these problems because their solution needs an unusual mathematical idea. We give one detailed solution for each problem. The solutions can be written very short, but the mathematical background of them is deep. The way to solve these problems leads through a lot of "aha-observations".
Problem 1 was suggested in 1995 by Sándor Katz. One can find the following solution in [1].

Problem 1. Let $\{a_1, a_2, \ldots, a_n\}$ a set consisting of positive real numbers. We consider the sums of the elements in all nonempty subsets of the given set. Prove that we can divide these sums in n classes such that in each class the quotient of the maximal and the minimal sum is at most 2.

Proof:

We can assume without the loss of generality that $a_1 \leq a_2 \leq \ldots \leq a_n$. Now we consider the numbers $s_i = \frac{1}{2}(a_1 + a_2 + \ldots + a_i)$ for i=1, 2,…, n. (1)

We will prove that all the sums of the nonempty subsets of the set $\{a_1, a_2, \ldots, a_n\}$ lie on the closed intervals $[s_i, 2s_i]$. Assume indirectly that our statement is false. That means there is a sum s and a positive integer k with property 1≤k≤n such that $2s_k < s < s_{k+1}$. By the inequality $2s_k < s$ (or equivalently $a_1 + a_2 + \ldots + a_k < s$) there is a summand in the representation of s being greater than a_k. Thus we have $a_{k+1} < s$. (2)

Summing up the two inequalities we get

$a_1 + a_2 + \ldots + a_{k+1} < 2s$. (3)

This contradicts the inequality $s < s_{k+1}$.QED

Problem 1 is nice because it states that the behaviour of the given sums cannot be absolutely irregular. The idea how to define the numbers s_i is essential and is not trivial.

Problem 2 was suggested by Sándor Kántor in 2010. One can find essentially the same solution in [3].

Problem 2. n people are standing on a line, where n is a positive integer greater than 2. Their heights given in inches are the terms of a strictly monotone increasing sequence from the left to the right. In every minute every two of the people can change their position, but every person can change his position at most once in a minute. At most how many minutes do they need to stand in alphabetical order from the left to the right,if the names of the persons are different?

Answer: 2

Proof:

It is easy to see that one minute is not always enough. To prove that two minutes are always enough we will use algebraic terminology. The key idea is to use the fact that all elements of the symmetric group S_n are products of disjoint cycles. We label the people in their first position from the left to the right with numbers 1, 2, ..., n. Let p_i denote the position of the ith person in the alphabetical ordering for i=1,2,...n. It is clear that p_1, p_2,…, p_n is a permutation of the numbers 1, 2,..., n. Let us write this composition as a product of disjoint cycles. We can imagine every such cycle as a regular polygon with labeled vertices. It is enough to see how to achieve our goal in a cycle. We have to rotate by $\frac{2\pi}{k}$ where k is the length of the given cycle. It is known that all rotations can be considered as a composition of two appropriate reflections. We can choose two reflections in such a way that they map the polygon into itself and their composition is a rotation by $\frac{2\pi}{k}$. QED

Problem 2 was unusual in many aspects. Our solution clears the algebraic background of the problem (with a clever geometric argument). Otherwise the problem is interesting because the result usually does not depend on n.

Problem 3 was suggested by József Borbély in 2010.

Problem3: Let n be a fixed positive integer and x_1, x_2 ,…, x_n nonnegative real numbers. It is known that $x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2 + x_1 + 2x_2 + \ldots + nx_n = 2012$. Find the minimal possible value of the sum $x_1 + x_2 + \ldots + x_n$.

Answer: $\frac{-n+\sqrt{n^2+4\cdot 2012}}{2}$

Proof:

Let S denote the sum $x_1 + x_2 + \ldots + x_n$. By the nonnegativity of the numbers x_i clearly we have 2012≤S·(S+n). (4)

So we have to solve the quadratic inequality

0≤S^2+nS-2012 in S. (5)

This leads us to $\frac{-n+\sqrt{n^2+4\cdot 2012}}{2}$≤S (6)

(Here we use the fact that all x_i's are nonnegative). The inequality is sharp, because we can choose the numbers such that $x_1 = x_2 = \ldots = x_{n-1} = 0$,

$x_n = \frac{-n+\sqrt{n^2+4\cdot 2012}}{2}$. QED

One can easily understand the solution of Problem 3. But the problem was solved only by two contestants during the competition and noone came up with the idea given in our solution. Both contestants gave a solution via a clever changing argument. The key idea of them was that by changing a positive real number x_i with property

i< n to x_i=0 will decrease S. Thus we can achieve the minimal value of S if we choose $x_1 = x_2 = \ldots = x_{n-1} = 0$, $x_n = \frac{-n+\sqrt{n^2+4\cdot 2012}}{2}$.

One can read both of the given solutions in [3].

Problem 4 was suggested by József Borbély in 2012.

Problem 4. A and B play the following two-player game. At the beginning of the game the numbers 1, 2, 3,..., 2012 are written on the board. In each turn a player deletes exactly one number of these numbers. A begins. The game ends if there are only two numbers on the board. If the absolute value of their difference is a prime number greater than a fixed positive integer k than B wins, otherwise A is the winner. Give the values of k such that player B has a winning strategy.

Answer: If k≤996, than B wins, otherwise A.

Proof:

At first we will prove that for 997≤k A has a winning strategy. The numbers 998, 999, 1000,..., 1006 are not primes. This means that for 997≤k B can only win if the absolute value of the difference of the last two numbers is a prime number greater than 1006. It is easy to verify that B cannot reach such a difference. If A deletes always the minimal number he can find on the board than the absolute value of the difference of the last two numbers is maximum 1006. Thus for 997≤k A has a winning strategy.

Otherwise for k≤996 player B can win the game. To prove our statement we will consider disjoint two-element subsets of the set {1, 2,..., 2012}$ such that the absolute values of the differences in all these pairs are prime numbers greater than 996. For 1≤i≤9 let be the number i+2003 the pair of i. For 10≤ j≤ 1006$ let be the number j+997 the pair of j. So in each turn depending on the choice of A player B has to delete the pair of the number deleted by A in his last step. QED

Noone could solve this problem during the competition. Some contestants gave a winning strategy for A if 997≤ k, but noone could see the "pairing-idea". But even that makes this problem unusual.

Problem 5 was proposed by József Borbély in 2011.

Problem 5. Let t(n) denote the number of the distinct prime divisors of the positive integer n. There are infinitely many positive integers n, such that t(n^2+n) is odd, and there are infinitely many positive integers n, such that t(n^2+n) is even.

Proof:

Let us consider a coloring of the positive integers with the following rules: we color positive integers having an even number of distinct prime divisors by red, the positive integers having an odd number of distinct prime divisors are colored by blue. For every positive integer n the numbers n and n+1 are relatively primes, that means they do not have any common prime divisor. It is straightforward that there are infinitely many red and infinitely many blue numbers. Let us take a red positive integer m. Let k be the minimal positive integer such that m+k is blue. In this case m+k-1 can play the role of the number n, because t$[(m+k-1)(m+k)]$ is odd.

We have to prove that there are infinitely many positive integers n such that t(n^2+n) is even. Consider the numbers 4^k, 4^k+1, …, 3·4^k. The number 4^k is blue, the number 3·4^k is red. The number of the elements in the set

{4^k, 4^k+1, …, 3·4^k} is odd. That means thera are two neighbouring numbers in this set having the same color. If we denote these numbers by n and n+1, then t(n^2+n) is even. The sets {4^k, 4^k+1, …, 3·4^k} are disjoint (k=1, 2, , 3, …). Thus our proof is complete, we proved the required results. QED

Problem 5 was solved by three participiants. Two contestants solved only the case, if t(n^2+n) is odd.

This problem can be generalized. We formulate and prove the general statement in the following

Generalization: For fixed positive integer k there are infinitely many positive integer numbers n such that the sum t(n+1)+t(n+2)+…t(n+k) is odd and there are infinitely many positive integer numbers n such that the sum t(n+1)+t(n+2)+…t(n+k) is even.

Proof of the generalization:

We will prove that for a fixed k there are infinitely many positive integers n such that t(n+k)-t(n) is an odd number.

Let us consider the same coloring as in the proof of Problem 5. Let p be a prime such that k is not divisible by p. In this case the numbers t(k^r) and t(pk^r) do not have the same parity. Consider the set of the positive integers m that are divisible by k and have the property k^r≤m≤ pk^r.

t(k^r) and t(pk^r) do not have the same parity, thus there are two elements in this set having not the same parity, whose difference is equal to k. That means we found a number n such that t(n+k)-t(n) is an odd number. We can choose the number r free and so we have infinitely many positive integers n such that t(n+k)-t(n) is an odd number.

If t(n+k)-t(n) is an odd number, then

the difference

t(n+1)+t(n+2)+…t(n+k)-
-$(\mathrm{t(n)} + \mathrm{t(n+1)} + \cdots \mathrm{t(n+k-1)})$

is odd or equivalently one of the numbers

t(n+1)+t(n+2)+…t(n+k) and

t(n+1)+t(n+2)+…t(n+k) is odd, and the other is even. This completes the proof of the required result.

Problem 6 was proposed by Mihály Bencze in 2010.

The following solution can one find in [3].

Problem 6. The sequence x_n is defined recursively. x_1=1 and $(1+x_n)x_{n+1}$=n, if n≥1. Then for n≥2 hold the following inequalities:

$$\left(\sqrt{\frac{n+1}{2}}-1\right)^2<1+\frac{1}{n}\sum_{k=1}^{n} {x_k}^2<\frac{n^2+n+2}{2n}$$

Proof:

The key of the solution is that one has to use induction in an unusual way. The proof is based on the following lemma:

Lemma: $\sqrt{n}$-1<x_n <$\sqrt{n-1}$, when n≥2 (7)

Proof of the lemma:

We prove our lemma with induction. If n=2 or 3, then the statement of the lemma is true, because $x_2=\frac{1}{2}$ and $x_3=\frac{4}{3}$. If the inequality

$$\sqrt{n}-1<x_n<\sqrt{n-1}$$

holds for a fixed positive integer n, then

$$\frac{n}{1+\sqrt{n-1}}<\frac{n}{1+x_n}<\frac{n}{\sqrt{n}}. \quad (8)$$

Thus we proved the upper bound. We have to prove the lower bound. It is enough to verify the inequality

$$\sqrt{n+1}-1<\frac{n}{1+\sqrt{n-1}}. \quad (9)$$

But this inequality is obviously true, because

$$\sqrt{n+1}-1=\frac{n}{1+\sqrt{n+1}}<\frac{n}{1+\sqrt{n-1}}. \quad (10)$$

So we made the induction step, the proof of the lemma is complete.

With this lemma we can prove the inequalities of the problem easily. By the lemma we have

$k-2\sqrt{k}+1<{x_k}^2<k-1$, if $k\geq 2$. (11)

It is a known fact that $\sum_{k=1}^{n} k=\frac{n(n+1)}{2}$. (12)

Thus we have

$\frac{n(n+1)}{2}-1-2\sum_{k=2}^{n}\sqrt{k}<\sum_{k=2}^{n}(k+1-2\sqrt{k})<\sum_{k=2}^{n}{x_k}^2<\frac{(n-1)n}{2}$. (13)

By the inequality on the right hand side we have

$1+\frac{1}{n}\sum_{k=1}^{n}{x_k}^2<1+\frac{n^2-n+2}{2n}=\frac{n^2+n+2}{2n}$. (14)

We proved one of the given inequalities. We have to prove that

$\left(\sqrt{\frac{n+1}{2}}-1\right)^2<1+\frac{1}{n}\sum_{k=1}^{n}{x_k}^2$. (15)

It is enough to verify that

$\sum_{k=1}^{n}\sqrt{k}<n\sqrt{\frac{n+1}{2}}$. (16)

We will prove this with the Cauchy-Schwarz inequality.

Let a_k be equal to 1, if $1\leq k\leq n$ and let b_k be equal to $\sqrt{k}$, if $1\leq k\leq n$. The Cauchy-Schwarz inequality states that

$\sum_{k=1}^{n} a_k b_k\leq\sqrt{\sum_{k=1}^{n}{a_k}^2}\sqrt{\sum_{k=1}^{n}{b_k}^2}$. (17)

Thus we get the required result. QED

Noone was able to solve Problem 6 during the competition. The critical point of the solution was to find the statement of the lemma. Without this observation the solution of the problem seems to be hopeless. We think that this makes Problem 6 really hard.

III. Remark

After the 21th competition we hope that it will be held even many years later. We want that the Hungarian high school students will train their mathematical abilities solving the problems of the International Hungarian Mathematical Competition. We are sure that the competition must be kept alive because of its high standards in every aspect.

References

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