Óbuda University

Doctoral (Ph.D.) Thesis Summary

Functional Inequalities on Riemann-Finsler Manifolds

by

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Abstract

The theory of functional inequalities has a fundamental role in analysis, with major implications in mathematical physics and nonlinear partial differential equations. One of the main questions within this theory is the study of these inequalities on non-Euclidean spaces, which has as its scope the characterisation of the relationship between the geometry of the studied curved structures and the properties of the corresponding functional inequalities.

The purpose of this work is to demonstrate how certain geometric properties of Riemannian/ Finsler manifolds affect various functional inequalities that hold on these spaces. As a result, the main portion of the thesis is concerned with several Sobolev-type inequalities with or without singular terms which are available on different Riemannian/ Finsler manifolds. The focus of this study is to understand the geometric factors of some nonlinear phenomena which occur on these curved structures. In order to comprehend better the particular geometric/ anisotropic framework of these manifolds, a number of Riemannian and Finsler model spaces are also investigated.

The first part of the thesis is devoted to the study of three Randers-type Finsler manifolds, which serve as model spaces for several examples and counterexamples throughout the dissertation. The next part concerns compact Sobolev embeddings à la Berestycki-Lions on noncompact Riemannian manifolds and Randers spaces. Thirdly, a number of Hardy-type inequalities are studied on Finsler manifolds and, in particular, on Finsler-Hadamard manifolds.

The primary application of these functional inequalities involves the study of elliptic partial differential equations. Accordingly, the last part of the thesis presents the application of the obtained Sobolev embeddings in order to establish a multiplicity result concerning an elliptic problem by using variational methods.

Contents

Chapter 1

Introduction

I. Motivation and background

Functional inequalities and Sobolev spaces play a crucial role in the theory of functional analysis, partial differential equations, mathematical physics, geometric analysis and calculus of variations. Indeed, the modern theory of nonlinear partial differential equations (in short, PDEs) and boundary value problems (in short, BVPs) relies heavily on the theory of Sobolev spaces, since these are the natural function spaces in which one seeks the solution of such problems. Moreover, since several functional inequalities represent the manifestation of certain natural mathematical and physical phenomena, the study of these inequalities is a remarkable area of mathematics in itself. Accordingly, there is a huge body of literature on Sobolev spaces and their applications; for a comprehensive presentation of the topic see the works of Adams and Fournier [8], Brezis [15], Evans [23], Maz'ya [39] and references therein.

Within this theory, a prominent class of Sobolev-type inequalities is provided by the ones defined on curved spaces. Although these non-Euclidean structures are natural extensions of the standard Euclidean space, the theory of Sobolev spaces and functional inequalities on these structures is far from being elementary, as the geometry of the ambient space can have substantial effects on the properties of Sobolev spaces and inequalities in question.

The systematic study of functional inequalities on non-Euclidean spaces originated in the 1970s with the works of Aubin [10, 11] and Cantor [17]. In fact, a particularly important incentive regarding this direction turned out to be the famous AB-program of Aubin [11], which had its objective to determine the best constants within such Sobolev inequalities on complete Riemannian manifolds. Since then, the study of functional inequalities on non-Euclidean structures has become a very active research area of geometric analysis.

It turns out that the properties of such inequalities deeply depend on the geometry of the ambient space, resulting in several surprising phenomena and challenging questions. Nevertheless, in the case of Riemannian manifolds, the theory of Sobolev spaces has undergone great development since the 1970s; for a comprehensive presentation of this topic, see Druet and Hebey [22], Hebey [30] and subsequent references. Moreover, the field has also established the grounds for new, thriving areas of research such as geometric analysis and optimal transport on general metric measure spaces, see Lott and Villani [37], Sturm [49, 50] and Villani [52].

Very recently, there has been a growing effort to extend the theory of Sobolev spaces and functional inequalities to Finsler manifolds, see e.g., Kristály [32], Kristály and Repovš [35], Ohta [42, 43], Ohta and Sturm [44] and Yuan, Zhao, and Shen [54]. However, due to the generally anisotropic nature of the Finsler metric, the adaptation of the standard Riemannian methods to the Finslerian setting requires critical analysis and careful attention, since several Riemannian objects and properties convert to highly nonlinear phenomena in the Finslerian framework, sometimes yielding unexpected results.

Since Riemann-Finsler geometry represents an essential extension of the classical Euclidean geometry, the proper understanding of the geometric aspects of the different phenomena which occur on these spaces represents a fundamental research topic.

II. Scientific objectives

The purpose of this thesis is to present the effects that certain geometric aspects of Riemannian/ Finsler manifolds can have on different functional inequalities available on these spaces. Accordingly, the major part of the dissertation is devoted to various Sobolev-type inequalities on Riemannian/ Finsler manifolds, with an emphasis on the interplay between the different sufficient/ necessary geometric conditions regarding the underlying curved spaces and the corresponding functional inequalities. In addition, we also explore several Riemannian/ Finsler model spaces, and demonstrate the particular geometric phenomena associated with these manifolds by a variety of examples and counterexamples. Since the primary application of functional inequalities involves the study of elliptic PDEs, in the last part of the dissertation we show the applicability of the theoretical results by establishing a multiplicity result considering a nonlinear elliptic PDE by using the techniques of the calculus of variations.

III. Description of own contributions

The thesis is based on the following papers:

[1] C. Farkas, A. Kristály, and Á. Mester. "Compact Sobolev embeddings on non-compact manifolds via orbit expansions of isometry groups". *Calculus of Variations and PDE* 60.4 (2021), Article no: 128.

[2] Á. Mester and A. Kristály. "A bipolar Hardy inequality on Finsler manifolds". *2019 IEEE 13th International Symposium on Applied Computational Intelligence and Informatics (SACI)* (2019), pp. 308–313.

[3] Á. Mester and A. Kristály. "Three isometrically equivalent models of the Finsler-Poincaré disk". *2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics (SACI)* (2021), pp. 403–408.

[4] Á. Mester, I. R. Peter, and C. Varga. "Sufficient Criteria for Obtaining Hardy Inequalities on Finsler Manifolds". *Mediterranean Journal of Mathematics* 18 (2021), Article no: 76.

The thesis contains seven chapters. After the introductory Chapter 1, Chapter 2 of the dissertation provides the preliminaries of Riemann-Finsler geometry, outlining the fundamental analogies and differences between Riemannian and Finsler manifolds.

Chapter 3 is devoted to a specific class of Finsler manifolds called Randers spaces. In this context, we present the isometry between two well-known Randers models, namely the 2-dimensional Funk model and the Finsler-Poincaré disk, while also describing their connections to Riemannian geometry. Then, we introduce a new Randers model in the form of the Finsler-Poincaré upper half plane, and prove the isometrical equivalence of the three Finsler manifolds in question. Finally, we discuss some surprising geometric phenomena which result from the latter isometries. This section is based on Mester and Kristály [3].

Chapter 4 presents compact Sobolev embeddings à la Berestycki-Lions [13] on noncompact Riemannian manifolds and Randers spaces. First, we give a general introduction concerning Sobolev inequalities and continuous and compact embeddings in the Euclidean setting and on complete Riemannian manifolds. Then, given a noncompact complete Riemannian manifold (M, g) with certain curvature restrictions, we introduce a so-called expansion condition concerning a group of isometries G of (M, g) that characterizes the coerciveness of G in the sense of Skrzypczak and Tintarev [46]. Under this particular expansion condition, we prove compact Sobolev embeddings of the form $W^{1,p}_G(M) \hookrightarrow L^q(M)$ for the full range of admissible parameters (p, q), i.e., in the Sobolev, Moser-Trudinger and Morrey case, respectively. After this, we consider the case of noncompact Randers-type Finsler manifolds with finite reversibility constant, which turn out to inherit similar embedding properties as their Riemannian companions; the sharpness of such constructions is shown by means of the Funk model. This chapter is based on Farkas, Kristály, and Mester [1].

In Chapter 5 we establish various Hardy-type inequalities on forward complete Finsler manifolds. Adopting the arguments of D'Ambrosio and Dipierro [20] to the Finslerian context, we prove – among others – a Caccioppoli inequality, a Gagliardo-Nirenberg inequality and a Heisenberg-Pauli-Weyl uncertainty principle. Furthermore, we also obtain some Hardy inequalities on Finsler-Hadamard manifolds with finite reversibility constant. Finally, we study a Hardy inequality with multiple singularities on complete Finsler manifolds, obtaining the anisotropic counterpart of a Riemannian multipolar inequality due to Faraci, Farkas, and Kristály [24]. It turns out that the non-Riemannian properties of the ambient Finsler structure play a critical role in the validity of the studied inequalities, which is manifested by the dependence of the results on the so-called reversibility constant and uniformity constant of the Finsler manifold in question. This chapter is based on Mester, Peter, and Varga [4] and Mester and Kristály [2].

Chapter 6 is devoted to the application of the established functional inequalities in the theory of elliptic PDEs, by means of variational methods. More precisely, we show a multiplicity result concerning a quasilinear PDE involving the p-Finsler-Laplace operator, which is defined on a Randers space satisfying certain geometric assumptions. The proof is based on variational arguments, where the compact Sobolev embedding results established in Chapter 4 provide the means to verify essential properties of the energy functional associated with the studied problem, in order to apply certain minimization arguments. This section elaborates the proof of the multiplicity result given in Farkas, Kristály, and Mester [1].

The present thesis summary follows the structure of the doctoral dissertation, preserving the numbering and titles of the chapters and theorems of the main work, with the exception of Chapter 2, where we now formulate the theses in accordance with the regulation of the Doctoral School of Applied Informatics and Applied Mathematics of Óbuda University.

Chapter 2

Formulation of the theses

In the following we formulate the theses which describe the main contributions of the doctoral dissertation.

1 Three isometrical models of Finsler manifolds

Main result: A new isometric model of the Finsler-Poincaré disk.

1.1 Thesis

The Finsler-Poincaré upper half plane introduced by [3] represents the Finslerian Randers-type generalization of the Riemannian hyperbolic upper half plane.

1.2. Thesis

The Finsler-Poincaré disk model, the 2-dimensional Funk model and the Finsler-Poincaré upper half plane are isometrically equivalent, see [3]. This isometry result represents a natural extension of the isometrical equivalence of the Riemannian hyperbolic model spaces, more precisely, the Riemannian Poincaré disk, the Beltrami-Klein disk and the Poincaré upper half plane model.

1.3. Thesis

In the case of the Finsler-Poincaré disk, the 2-dimensional Funk model and the Finsler-Poincaré upper half plane, the first eigenvalue associated to the Finsler-Laplace operator turns out to be zero, see [3]. This gapless character of the fundamental frequency is in sharp contrast with the Riemannian spectral gap property proved by McKean [40].

2 Sobolev-type inequalities without singular terms

Main results: Compact embeddings on noncompact Riemannian manifolds and Randers spaces.

2.1. Thesis

Given a noncompact complete Riemannian manifold (M, g), we introduce an expansion condition concerning the action of a compact connected group G which represents a subgroup of the isometry group of (M, g), see [1]. Then, in the case when (M, g) is either a Cartan-Hadamard manifold or a manifold with bounded geometry, the previous expansion condition characterizes the coerciveness of G in the sense of Skrzypczak and Tintarev [46].

2.2. Thesis

If (M, g) is either a Cartan-Hadamard manifold or a Riemannian manifold with bounded geometry and G is a compact connected subgroup of the isometry group of (M, g), then compact Berestycki-Lions-type Sobolev embeddings hold for the full range of admissible parameters (i.e., Sobolev, Moser-Trudinger and Morrey case) whenever the above expansion condition is satisfied, see [1].

2.3. Thesis

Randers spaces with finite reversibility constant inherit similar compact embedding properties whenever the underlying Riemannian manifold is either Cartan-Hadamard manifold or has bounded geometry, and the group of isometries G satisfies a similar expansion condition, see [1]. Moreover, the finiteness of the reversibility constant turns out to be an indispensable condition for the validity of the continuous Sobolev embeddings. This property is demonstrated by the counterexample given on the n-dimensional Funk model, where the continuous Sobolev embeddings fail to hold for each admissible parameter pair, rendering the compact embeddings unattainable, see [1].

3 Sobolev-type inequalities with singular terms

Main results: Hardy-type inequalities on Finsler manifolds.

3.1. Thesis

If (M, F) is a forward complete Finsler manifold, then the superharmonicity of a certain nonnegative weight function ρ provides a sufficient condition for several weighted Hardy-type inequalities on (M, F), which form the Finslerian counterparts of the results due to D'Ambrosio and Dipierro [20]. In this setting, we obtain – among others – a weighted Hardy inequality, a weighted Gagliardo-Nirenberg inequality and a weighted Heisenberg-Pauli-Weyl uncertainty principle, see [4].

3.2. Thesis

In the case of a Finsler-Hadamard manifold (M, F) with finite reversibility constant and vanishing mean covariation, [4] establishes the generalization of the classical Euclidean Hardy inequality, which features the Finslerian distance function from an arbitrarily fixed point $x_0 \in M$ and the reversibility constant of (M, F). This construction represents the quantitative version of the inequality given by Zhao [55, Theorem 1.2] and the generalization of the result due to Farkas, Kristály, and Varga [26, Proposition 4.1].

3.3. Thesis

Given a complete Finsler manifold with nonzero uniformity constant (thus finite reversibility constant), in [2] we prove a bipolar Hardy inequality featuring two singularities, which represents the Finslerian generalization of the Riemannian multipolar inequality proved by Faraci, Farkas, and Kristály [24] and the Euclidean version due to Cazacu and Zuazua [18]; the constants appearing in the bipolar Hardy inequality are dependent on the reversibility constant and uniformity constant of (M, F).

4 Application to partial differential equations

Main result: Multiplicity result concerning an elliptic problem defined on a Randers space with finite reversibility constant.

4.1. Thesis

The obtained Sobolev-type inequalities provide means for studying elliptic PDEs via variational methods. In particular, we prove a multiplicity result concerning a quasilinear problem defined on a Randers space with finite reversibility constant, where the leading term is given by the p-Finsler-Laplace operator. More precisely, under certain growth conditions, the studied parameter-dependent quasilinear elliptic PDE has three distinct solutions for parameters small enough, see [1]. The stability of the solutions is ensured by a Ricceri-type critical point theorem due to Bonanno [14].

In the following chapters we summarize the results in precise mathematical formulation. As we indicated before, the elaboration preserves the structure, sectioning and numbering of the doctoral dissertation.

Chapter 3

Three isometrical models of Finsler manifolds

This chapter presents three isometrically equivalent examples of Finsler manifolds, all of which belong to a particular class called Randers spaces. Randers spaces represent one of the most fundamental non-Riemannian family of Finsler manifolds, as they constitute a natural layer of generalization between Riemannian manifolds and general Finsler manifolds.

We say that the pair (M, F) is a Randers space if (M, g) is a Riemannian manifold and the Finsler structure $F : TM \to \mathbb{R}$ is defined by

$$F(x, v) = \sqrt{g_x(v, v)} + \beta_x(v), \quad \forall x \in M,\ v \in T_x M, \tag{3.1}$$

where β_x denotes a 1-form on M such that

$$|\beta_x|_g \coloneqq \sqrt{g_x^*(\beta_x, \beta_x)} < 1, \quad \forall x \in M.$$

In this case F is called a Randers metric.

Evidently, every Riemannian manifold can be regarded as a Randers space with $\beta_x = 0, \forall x \in M$. Furthermore, it can be shown that every Randers space is a Finsler manifold, see Bao, Chern, and Shen [12, Section 1.3 C]. The Finsler metric F from (3.1) is non-symmetric unless $\beta_x = 0, \forall x \in M$, in which case (M, F) coincides with the original Riemannian manifold (M, g).

There exist two fundamental analytical models of Randers spaces in the literature, namely the Finslerian *Funk model* and the *Finsler-Poincaré disk model*.

The Finslerian Funk disk model is given by the pair $(\mathbb{D}, F_F)$, where the Funk metric $F_F : \mathbb{D} \times \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$F_F(x, v) = \frac{\sqrt{(1 - |x|^2)|v|^2 + \langle x, v \rangle^2}}{1 - |x|^2} + \frac{\langle x, v \rangle}{1 - |x|^2}, \tag{3.2}$$

for all $(x, v) \in T\mathbb{D}$, and $\mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ denotes the 2-dimensional Euclidean open unit disk, see Cheng and Shen [19, Example 2.1.2] or Shen [45, Example 1.1.2

& 1.3.4]. It turns out that the Funk model is the Randers-type perturbation of the well known Riemannian *Beltrami-Klein disk*, see Loustau [38, Section 6.2].

The Finsler-Poincaré disk is defined by the pair $(\mathbb{D}, F_P)$, where the Finsler-Poincaré metric $F_P : \mathbb{D} \times \mathbb{R}^2 \to \mathbb{R}$ is given by

$$F_P(x,v) = \frac{2|v|}{1-|x|^2} + \frac{4\langle x, v\rangle}{1-|x|^4}, \tag{3.3}$$

for every $(x, v) \in T\mathbb{D}$, see Bao, Chern, and Shen [12, Section 1.3 E & 12.6]. It can be proved that the manifold $(\mathbb{D}, F_P)$ represents the Finslerian counterpart of the usual Riemannian *Poincaré disk*, see Loustau [38, Section 8.1].

Despite the popularity of these two Finsler models, the relationship among them is rarely discussed. This is even more peculiar if one considers the fact that these two Randers spaces are actually isometrically equivalent, meaning that there exists an isometric diffeomorphism between the two manifolds. Therefore, a side objective of the present chapter is to describe in more detail the isometry between the models $(\mathbb{D}, F_F)$ and $(\mathbb{D}, F_P)$.

Moreover, it turns out that there exists a third model which is isometric to the previous two Randers spaces. This phenomenon can be suspected from the properties of the hyperbolic model spaces, as it is well known that the Riemannian counterparts of the models are all isometric to the *Poincaré upper half plane*, see Cannon et al. [16] or Stahl [47, Chapter 4].

Inspired by this, we introduce the Finslerian counterpart of the Riemannian Poincaré upper half plane. Namely, we define the Finsler structure $F_H : \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$F_H(x,v) = \frac{|v|}{x_2} + \frac{\langle w(x), v\rangle}{x_2(4+|x|^2)}, \tag{3.4}$$

for all $(x, v) \in T\mathbb{H}$, where $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ denotes the Euclidean upper half plane and

$$w(x) = (2x_1x_2,\ x_2^2 - x_1^2 - 4), \quad \forall x = (x_1, x_2) \in \mathbb{H}.$$

We call the pair $(\mathbb{H}, F_H)$ the *Finsler-Poincaré upper half plane*. In this case, it can be proved that $(\mathbb{H}, F_H)$ is a Randers space, which is precisely the Randers-type generalization of the Riemannian hyperbolic upper half plane model, see Mester and Kristály [3].

Considering this analogous construction, the main result of this chapter can be formulated as follows.

Theorem 3.3.1. (Mester and Kristály [3]) *The Funk model* $(\mathbb{D}, F_F)$*, the Finsler-Poincaré disk* $(\mathbb{D}, F_P)$ *and the Finsler-Poincaré upper half plane* $(\mathbb{H}, F_H)$ *are isometrically equivalent.*

We prove Theorem 3.3.1 by giving the explicit form of the isometry functions between the Randers spaces in question, which turn out to be the natural extensions of the isometries of the Riemannian hyperbolic model spaces, i.e., the Beltrami-Klein disk, the usual Poincaré disk and the Poincaré upper half plane. Namely, we have the following results.

Proposition 3.3.1. (Mester and Kristály [3]) *Let us consider the diffeomorphism*

$$f : \mathbb{D} \to \mathbb{D},\ \ f(x) = \frac{2x}{1+|x|^2},$$

and its inverse

$$f^{-1} : \mathbb{D} \to \mathbb{D},\ \ f^{-1}(x) = \frac{x}{1+\sqrt{1-|x|^2}}.$$

Then f is an isometry between the Finsler-Poincaré disk $(\mathbb{D}, F_P)$ and the Funk model $(\mathbb{D}, F_F)$.

Proposition 3.3.2. (Mester and Kristály [3]) *Let us consider the diffeomorphism*

$$g : \mathbb{D} \to \mathbb{H},\ \ g(x) = \left(\frac{2x_2}{1+x_1}, \frac{2\sqrt{1-|x|^2}}{1+x_1}\right)$$

with its inverse function

$$g^{-1} : \mathbb{H} \to \mathbb{D},\ \ g^{-1}(x) = \left(\frac{4-|x|^2}{4+|x|^2}, \frac{4x_1}{4+|x|^2}\right).$$

Then g is an isometry between the Funk model $(\mathbb{D}, F_F)$ and the Finsler-Poincaré upper half plane $(\mathbb{H}, F_H)$.

Proposition 3.3.3. (Mester and Kristály [3]) *Let us consider the diffeomorphism*

$$h : \mathbb{H} \to \mathbb{D},\ \ h(x) = \left(\frac{4-|x|^2}{|x|^2+4x_2+4}, \frac{4x_1}{|x|^2+4x_2+4}\right),$$

and its inverse

$$h^{-1} : \mathbb{D} \to \mathbb{H}, h^{-1}(x) = \left(\frac{4x_2}{|x|^2+2x_1+1}, \frac{2-2|x|^2}{|x|^2+2x_1+1}\right).$$

Then h is an isometry between the Finslerian upper half plane $(\mathbb{H}, F_H)$ and the Finsler-Poincaré disk $(\mathbb{D}, F_P)$.

An important byproduct of the above isometry results is the fact that all the metric related properties which hold on one particular model can be easily transferred to the other two manifolds by the appropriate isometry functions.

To demonstrate this, we state the gapless character of the first Dirichlet eigenvalue associated to the Finsler-Laplace operator on the spaces $(\mathbb{D}, F_P)$ and $(\mathbb{H}, F_H)$, by applying Theorem 3.3.1 to the result of Kristály [33, Theorem 1.3], which provides the nullity of the fundamental frequency on the space $(\mathbb{D}, F_F)$. More precisely, we obtain the following property.

Corollary 3.4.1. (Mester and Kristály [3]) *In case of the Finsler-Poincaré disk $(\mathbb{D}, F_P)$ and the Finsler-Poincaré upper half plane $(\mathbb{H}, F_H)$, we have*

$$\lambda_{1,F_P}(\mathbb{D}) = \lambda_{1,F_H}(\mathbb{H}) = 0.$$

The fact that the first eigenvalues vanish is in sharp contrast with the Riemannian spectral gap property proved by McKean [40]. Considering the fact that the Randers spaces $(\mathbb{D}, F_P)$ and $(\mathbb{H}, F_H)$ represent some of the simplest non-Riemannian Finsler manifolds, this result highlights the anisotropic phenomena that can occur in Finslerian settings.

Chapter 4

Sobolev-type inequalities without singular terms

This chapter focuses on Sobolev-type embeddings à la Berestycki and Lions [13] on noncompact Riemannian manifolds and Randers spaces.

In order to sketch these results, let (M, g) be a complete n-dimensional Riemannian manifold with $n \geq 2$, and let $\text{Isom}_g(M)$ denote the isometry group of (M, g). Note that $\text{Isom}_g(M)$ is a Lie group with respect to the compact open topology and it acts differentiably on M, see Myers and Steenrod [41]. Suppose that G is a compact connected subgroup of $\text{Isom}_g(M)$. For any $x \in M$, let $\mathcal{O}_G^x = \{\xi x : \xi \in G\}$ denote the G-orbit of the point x, where $\xi x \coloneqq \xi(x)$ denotes the action of the element $\xi \in G$ on x.

The purpose of this chapter is to identify suitable general geometric conditions under which the compact embedding

$$W_G^{1,p}(M) \hookrightarrow L^q(M)$$

holds for an appropriate range of parameters p, q, where $W_G^{1,p}(M)$ denotes the subspace of G-invariant functions of the Sobolev space $W_g^{1,p}(M)$, i.e.,

$$W_G^{1,p}(M) = \left\{u \in W_g^{1,p}(M) : u \circ \xi = u, \quad \text{for all } \xi \in G\right\}.$$

Having in mind this objective, for a fixed point $x \in M$ we denote by $m(x, \rho)$ the maximal number of mutually disjoint geodesic balls with radius $\rho > 0$ on the orbit $\mathcal{O}_G^x$, i.e.,

$$m(x, \rho) = \sup\left\{k \in \mathbb{N} : \exists \xi_1, \ldots, \xi_k \in G \text{ such that } B_g(\xi_i x, \rho) \cap B_g(\xi_j x, \rho) = \emptyset, \forall i \neq j\right\}, \quad (4.1)$$

where $B_g(y, \rho) = \{z \in M : d_g(y, z) < \rho\}$ is the usual geodesic ball in M and $d_g : M \times M \to [0, \infty)$ is the distance function induced by the Riemannian metric g.

For $\rho > 0$ and $x_0 \in M$ fixed, in Farkas, Kristály, and Mester [1] we introduce the following *expansion condition*:

$(\mathbf{EC})_G$ $\; m(x, \rho) \to \infty$ as $d_g(x_0, x) \to \infty$.

Clearly, condition $(\mathbf{EC})_G$ is independent of the choice of x_0.

By using the above expansion condition, we are able to give a characterization of the coercive action of the subgroup G described by Skrzypczak and Tintarev [46, Definition 1.2], which in turn yields the compact Sobolev embeddings of the type $W_G^{1,p}(M) \hookrightarrow L^q(M)$ for the full range of *n-admissible* parameter pairs $(p,q) \in (1,\infty) \times (1,\infty]$, i.e., whenever one of the following conditions holds:

(**S**): $1 < p < n$ and $p < q < p^* = \frac{np}{n-p}$ (Sobolev case);

(**MT**): $p = n$ and $p < q < \infty$ (Moser-Trudinger case);

(**M**): $n < p < \infty$ and $q = \infty$ (Morrey case).

We distinguish two cases depending on the curvature of the ambient space.

First, when (M,g) is a Cartan-Hadamard manifold (i.e., simply connected, complete Riemannian manifold with nonpositive sectional curvature), we obtain the following compact Sobolev embedding result.

Theorem 4.3.3. (Farkas, Kristály, and Mester [1]) *Let (M,g) be an n-dimensional Cartan-Hadamard manifold, and let G be a compact connected subgroup of* $\mathrm{Isom}_g(M)$ *such that* $\mathrm{Fix}_{\mathrm{M}}(G) \neq \emptyset$. *Then the following statements are equivalent:*

(i) *G is coercive*;

(ii) $\mathrm{Fix}_M(G)$ *is a singleton*;

(iii) $(\mathbf{EC})_G$ *holds.*

Moreover, from any of the above statements it follows that the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ is compact for every n-admissible pair (p,q).

Note that the equivalence between *(i)* and *(ii)* in Theorem 4.3.3 is proved by Skrzypczak and Tintarev [46, Proposition 3.1], from which they conclude the compactness of the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ for the admissible case (**S**); for a similar result in the case (**MT**), see Kristály [34]. Accordingly, the purpose of Theorem 4.3.3 is to characterize the coerciveness of G by the expansion condition $(\mathbf{EC})_G$, as well as to complement the admissible range of parameters with the Morrey case (**M**).

The counterpart of Theorem 4.3.3 in the case of Riemannian manifolds with bounded geometry (i.e., complete noncompact Riemannian manifolds with Ricci curvature bounded from below and positive injectivity radius) reads as follows.

Theorem 4.4.1. (Farkas, Kristály, and Mester [1]) *Let (M,g) be an n-dimensional Riemannian manifold with bounded geometry, and let G be a compact connected subgroup of* $\mathrm{Isom}_g(M)$. *Then the following statements are equivalent:*

(i) *G is coercive*;

(ii) $(\mathbf{EC})_G$ *holds*;

(iii) the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ is compact for every *n-admissible pair (p,q);*

(iv) the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ is compact for some *n-admissible pair (p,q).*

In Theorem 4.4.1, the equivalence between condition (i) and the compactness of the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ for every n-admissible pair (p,q) in the case $(\mathbf{S})$ is well-known by Tintarev [51, Theorem 7.10.12]; in addition, Górka [29] and Gaczkowski, Górka, and Pons [28] proved that a slightly stronger form of $(\mathbf{EC})_G$ implies (iii) in the admissible case $(\mathbf{S})$ by using a Strauss-type argument. Therefore, the novelty of Theorem 4.4.1 is the equivalence of the expansion condition $(\mathbf{EC})_G$ not only with the coerciveness of G but also with the validity of the compact embeddings for the full range of n-admissible pairs (p,q).

The alternative characterization of the coerciveness condition by the expansion property $(\mathbf{EC})_G$ provides a more intuitive description of the geometric phenomenon that allows Berestycki-Lions-type compactness on noncompact Riemannian manifolds. In addition, $(\mathbf{EC})_G$ connects the coercive property of Skrzypczak and Tintarev [46] to the original approach of Strauss [48].

As a consequence of Theorem 4.4.1, we obtain the following corollary, which is related to the results obtained by Hebey and Vaugon [31] (see also Hebey [30, Theorems 9.5 & 9.6]):

Corollary 4.4.2. (Farkas, Kristály, and Mester [1]) *Let (M,g) be a complete n-dimensional noncompact Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius, and let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$. Assume that there exists $\kappa = \kappa(G,n) > 0$ such that for every $y \in M$ with $d_g(x_0,y) \geq 1$, one has*

$$\mathcal{H}^l(\mathcal{O}_G^y) \geq \kappa \cdot d_g(x_0,y), \tag{$\mathbf{H}$}$$

where $l = l(y) = \dim \mathcal{O}_G^y \geq 1$ and $\mathcal{H}^l$ denotes the l-dimensional Hausdorff measure. Then the embedding $W_G^{1,p}(M) \hookrightarrow L^\infty(M)$ is compact for every $n < p < \infty$.

In the thesis we also provide two explicit examples where hypothesis $(\mathbf{H})$ holds.

Finally, we study similar compact embeddings on noncompact Randers-type Finsler manifolds. In this context, we show that similar compactness results to Theorems 4.3.3 & 4.4.1 can be established on Randers spaces having finite reversibility constant. Nevertheless, it turns out that in more general non-Riemannian Finsler settings the phenomena concerning Sobolev embeddings may change dramatically.

In order to present these results, let (M,F) be an n-dimensional Randers space and $\mathrm{Isom}_F(M)$ be the isometry group of (M,F). In this case, $\mathrm{Isom}_F(M)$ is a closed subgroup of the isometry group $\mathrm{Isom}_g(M)$ of the underlying Riemannian manifold (M,g), see Deng [21, Proposition 7.1]. If G is a subgroup of $\mathrm{Isom}_F(M)$, then $W_{F,G}^{1,p}(M)$ stands for the subspace of G-invariant Sobolev functions of $W_F^{1,p}(M)$, i.e.,

$$W_{F,G}^{1,p}(M) = \left\{ u \in W_F^{1,p}(M) : u \circ \xi = u, \quad \text{for all } \xi \in G \right\}.$$

Furthermore, for any $y \in M$, let $m_F(y,\rho)$ denote the maximal number of mutually disjoint geodesic Finsler balls with radius ρ on the orbit $\mathcal{O}_G^y$. Then, one has the following embedding theorem.

Theorem 4.5.1. (Farkas, Kristály, and Mester [1]) *Let (M,F) be an n-dimensional Randers space endowed with the Finsler metric $F : TM \to \mathbb{R}$,*

$$F(x,v) = \sqrt{g_x(v,v)} + \beta_x(v), \quad \forall (x,v) \in TM, \tag{4.2}$$

such that (M,g) is either a Cartan-Hadamard manifold or a Riemannian manifold with bounded geometry. Suppose that $\sup\limits_{x\in M} |\beta_x|_g < 1$. Then the following results hold:

(i) For every n-admissible pair (p,q) the embedding $W_F^{1,p}(M) \hookrightarrow L^q(M)$ is continuous.

(ii) Let G be a compact connected subgroup of $\mathrm{Isom}_F(M)$ such that $m_F(y,\rho) \to \infty$ as $d_F(x_0,y) \to \infty$ for some $x_0 \in M$ and $\rho > 0$. Then the embedding $W_{F,G}^{1,p}(M) \hookrightarrow L^q(M)$ is compact for any n-admissible pair (p,q).

Note that the assumption $\sup\limits_{x\in M} |\beta_x|_g < 1$ in Theorem 4.5.1 is equivalent to the finiteness of the reversibility constant of (M,F). We demonstrate that this property is in fact indispensable for the validity of the continuous Sobolev embeddings $W_F^{1,p}(M) \hookrightarrow L^q(M)$. Indeed, the following example shows that the continuous (and therefore, compact) Sobolev embeddings do not necessarily hold on Randers spaces having infinite reversibility constant.

Example 4.5.1. (Farkas, Kristály, and Mester [1]) Let $n \geq 2$ and $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the n-dimensional Euclidean open unit ball. Consider the Funk metric $F : \mathbb{B}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$F(x,v) = \frac{\sqrt{(1-|x|^2)|v|^2 + \langle x,v\rangle^2}}{1-|x|^2} + \frac{\langle x,v\rangle}{1-|x|^2},$$

which defines the n-dimensional Finslerian Funk model $(\mathbb{B}^n, F)$, see Cheng and Shen [19, Example 2.1.2], and Shen [45, Example 1.3.4]. In fact, in the particular case $n=2$ we recover the 2-dimensional Funk model $(\mathbb{D}, F_F)$ presented in Chapter 3.

Recall that $(\mathbb{B}^n, F)$ is a noncompact Randers space with constant negative flag curvature $-\frac{1}{4}$, i.e., a Finsler-Hadamard manifold. Furthermore, the underlying Riemannian manifold is the Beltrami-Klein model having constant negative sectional curvature -1, which is a Cartan-Hadamard manifold. Nevertheless, the reversibility constant of $(\mathbb{B}^n, F)$ is $r_F = +\infty$.

Now, let (p,q) be any n-admissible pair and consider the function $u : \mathbb{B}^n \to \mathbb{R}$ defined by

$$u(x) = e^{\frac{d_F(0,x)}{t}} \left(1 - e^{-d_F(0,x)}\right) = \frac{|x|}{(1-|x|)^{\frac{1}{t}}},$$

where $t > 0$ is a parameter and $d_F(0,x) = -\ln(1-|x|)$ denotes the Finslerian distance function from the origin on the Funk model, see Cheng and Shen [19, Example 2.1.2].

In this case, we show that for each admissible case (**S**), (**MT**) and (**M**) there exists a parameter $t > 0$ such that $u \in W_F^{1,p}(\mathbb{B}^n) \setminus L^q(\mathbb{B}^n)$, which yields that the space $W_F^{1,p}(\mathbb{B}^n)$ cannot be continuously embedded into $L^q(\mathbb{B}^n)$ for any n-admissible pair (p, q), thus no further compact embedding can be expected.

Example 4.5.1 demonstrates that the theory of Sobolev spaces on Finsler manifolds cannot be treated analogously to the Riemannian case. Indeed, although the Funk model is a Finsler-Hadamard manifold of Randers-type, none of the continuous Sobolev embeddings are valid on the space because the reversibility constant is infinite. Due to the isometry result proved in Theorem 3.3.1, it follows that these unexpected phenomena are all valid on the Finsler-Poincaré ball and the Finsler-Poincaré upper half plane, all being Finsler-Hadamard manifolds (see Chapter 3). This is in sharp contrast with the Riemannian case, see e.g., Hebey [30].

Chapter 5

Sobolev-type inequalities with singular terms

This chapter concerns Hardy inequalities on Finsler manifolds, which belong to the family of Sobolev-type inequalities having singular terms. Accordingly, the purpose of the present chapter is the study of different Hardy-type inequalities on Finsler manifolds, by exploring the geometric and technical conditions which enable (or, in some cases, inhibit) such investigations.

By expanding the technique applied by D'Ambrosio and Dipierro [20] in the case of Riemannian manifolds, we prove Hardy inequalities involving a weight function on forward complete, not necessarily reversible Finsler manifolds. In addition, we recover and complement some of the results derived by Zhao [55]. In order to avoid technicalities, we consider the case $p = 2$, obtaining L^2-type Hardy inequalities. The results may be extended to any $p > 1$ by applying appropriate changes to the proofs.

In the remainder of this chapter let (M, F) be a forward complete n-dimensional Finsler manifold and let $\Omega \subset M$ be an open set.

We say that a function $\rho \in W^{1,2}_{\text{loc}}(\Omega)$ is p-superharmonic ($p \geq 2$) on Ω in weak sense if

$$\int_\Omega F^*(x, D\rho(x))^{p-2} \cdot D\varphi(x)\big(\nabla_F \rho(x)\big)\, \mathrm{d}v_F(x) \geq 0,$$

for every nonnegative test function $\varphi \in C_0^\infty(\Omega)$. By the divergence theorem, this in turn is equivalent with the fact that $-\Delta_{F,p}\rho \geq 0$ on Ω in weak sense, where $\Delta_{F,p}$ denotes the p-Finsler-Laplace operator. Note that for $p = 2$, we simply say that ρ is superharmonic, meaning that $-\Delta_F \rho \geq 0$ on Ω in the distributional sense. It turns out that this superharmonicity condition provides a sufficient criteria in order to prove several weighted Hardy-type inequalities on (M, F).

The starting point of our results is the following theorem.

Theorem 5.2.2. (Mester, Peter, and Varga [4]) *Let (M, F) be a forward complete Finsler manifold and let $\Omega \subset M$ be an open set. Let $\rho \in W^{1,2}_{\text{loc}}(\Omega)$ be a nonnegative function and $\theta \in \mathbb{R}$ a constant, such that* *(i)* $-(1-\theta)\Delta_F \rho \geq 0$ *on* Ω *in weak sense;*

(ii) $\frac{F^{*2}(D\rho)}{\rho^{2-\theta}}$, $\rho^\theta \in L^1_{\mathrm{loc}}(\Omega)$.

If $\theta \leq 1$, *then*

$$\int_\Omega \rho^\theta F^{*2}(x, Du)\, \mathrm{d}v_F \geq \frac{(1-\theta)^2}{4} \int_\Omega \rho^\theta \frac{u^2}{\rho^2} F^{*2}(x, D\rho)\, \mathrm{d}v_F, \quad \forall u \in C_0^\infty(\Omega),$$

whereas if $\theta > 1$ *and* $r_F < +\infty$, *then*

$$\int_\Omega \rho^\theta F^{*2}(x, Du)\, \mathrm{d}v_F \geq \frac{(1-\theta)^2}{4r_F^2} \int_\Omega \rho^\theta \frac{u^2}{\rho^2} F^{*2}(x, D\rho)\, \mathrm{d}v_F, \quad \forall u \in C_0^\infty(\Omega).$$

On the one hand, in the particular case $\theta = 0$ Theorem 5.2.2 yields the following Hardy inequality.

Corollary 5.2.3. (Mester, Peter, and Varga [4]) *Let* (M, F) *be a forward complete Finsler manifold and let* $\Omega \subset M$ *be an open set. If* $\rho \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a nonnegative function such that* ρ *is superharmonic on* Ω *in weak sense, and* $\frac{F^{*2}(D\rho)}{\rho^2} \in L^1_{\mathrm{loc}}(\Omega)$, *then*

$$\int_\Omega F^{*2}(x, Du)\, \mathrm{d}v_F \geq \frac{1}{4} \int_\Omega \frac{u^2}{\rho^2} F^{*2}(x, D\rho)\, \mathrm{d}v_F, \quad \forall u \in C_0^\infty(\Omega).$$

On the other hand, by choosing $\theta = 2 + q, q > -1$, we obtain the following Caccioppoli inequality.

Corollary 5.2.4. (Mester, Peter, and Varga [4]) *Let* (M, F) *be a complete Finsler manifold with* $r_F < \infty$, *and let* $\Omega \subset M$ *be an open set. If* $\rho \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *is a nonnegative function such that* $\Delta_F \rho \geq 0$ *on* Ω *in weak sense, and* $q > -1$ *such that* $\rho^q F^{*2}(D\rho)$ *and* $\rho^{2+q} \in L^1_{\mathrm{loc}}(\Omega)$, *then*

$$\int_\Omega \rho^{2+q} F^{*2}(x, Du)\, \mathrm{d}v_F \geq \frac{(1+q)^2}{4r_F^2} \int_\Omega u^2 \rho^q F^{*2}(x, D\rho)\, \mathrm{d}v_F, \quad \forall u \in C_0^\infty(\Omega).$$

We also establish weighted Gagliardo-Nirenberg and Heisenberg-Pauli-Weyl inequalities, as follows.

Theorem 5.3.2. (Mester, Peter, and Varga [4]) *Let* (M, F) *be a forward complete Finsler manifold and* $\Omega \subset M$ *an open set. Let* $\rho \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *be a nonnegative function such that* ρ *is superharmonic on* Ω *in weak sense. If* $q \in \mathbb{R}$, $s, z > 0$ *and* $r \in (0,1)$, *then*

$$\left(\int_\Omega |u|^s \frac{F^{*q}(x, D\rho)}{\rho^q}\, \mathrm{d}v_F \right)^{\frac{1}{s}} \leq 2^{\frac{q}{s}} \left(\int_\Omega F^{*2}(x, Du)\, \mathrm{d}v_F \right)^{\frac{r}{2}} \left(\int_\Omega |u|^z\, \mathrm{d}v_F \right)^{\frac{1-r}{z}}$$

for every $u \in C_0^\infty(\Omega)$, *where*

$$\frac{1}{s} = \frac{r}{2} + \frac{1-r}{z} \quad \text{and} \quad \frac{1}{q} = \frac{1}{2} + \frac{1-r}{rz}.$$

Theorem 5.3.3. (Mester, Peter, and Varga [4]) *Let (M, F) be a forward complete Finsler manifold and $\Omega \subset M$ an open set. Let $\rho \in W^{1,2}_{\text{loc}}(\Omega)$ be a nonnegative function such that ρ is superharmonic on Ω in weak sense. Let $s > 0$ and $p, p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then for every $u \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega |u|^s \, \mathrm{d}v_F \leq 4^{\frac{1}{p}} \left(\int_\Omega F^{*2}(x, Du) \, \mathrm{d}v_F \right)^{\frac{1}{p}} \left(\int_\Omega \frac{\rho^{2(p'-1)}}{F^{*2(p'-1)}(x, D\rho)} |u|^{\frac{ps-2}{p-1}} \, \mathrm{d}v_F \right)^{\frac{1}{p'}}.$$

Note that when $(M, F) = (M, g)$ is a Riemannian manifold, the Finsler-Laplace operator Δ_F reduces to the Laplace-Beltrami operator Δ_g. Furthermore, by the Riesz representation theorem, one can identify the tangent space T_xM with its dual space T_x^*M, and the Finsler metrics F and F^* reduce to the norm $|\cdot|_g$ induced by the Riemannian metric g. Therefore, the results presented above extend the functional inequalities obtained by D'Ambrosio and Dipierro [20] to the class of forward complete, not necessarily reversible Finsler manifolds.

Next, we establish Hardy inequalities on Finsler-Hadamard manifolds having finite reversibility constant, by defining the weight function ρ in Theorem 5.2.2 with the help of the Finslerian distance function d_F. For this, let (M, F) be a Finsler-Hadamard manifold with $r_F < \infty$, and let $\mathbf{S}$ denote the mean covariation of (M, F). For an $x_0 \in M$ arbitrarily fixed point let us denote by $r : M \to \mathbb{R}$, $r(x) = d_F(x_0, x)$ the distance function from the point x_0 on M. Note that as (M, F) is a Finsler-Hadamard manifold, we have $\text{Cut}(x_0) = \emptyset$.

By applying Theorem 5.2.2 to a weight function defined with the help of the distance function r, we obtain the following Hardy inequality, which can be considered the quantitative version of the result given by Zhao [55, Theorem 1.2].

Theorem 5.2.5. (Mester, Peter, and Varga [4]) *Let (M, F) be an n-dimensional Finsler-Hadamard manifold with $n \geq 3$, $r_F < \infty$ and $\mathbf{S} = 0$. If $\alpha \in (-\infty, 1)$, then for every $u \in C_0^\infty(M)$ we have*

$$\int_M r^{\alpha(2-n)} F^{*2}(x, Du) \, \mathrm{d}v_F \;\geq\; \frac{(n-2)^2(1-\alpha)^2}{4 r_F^2} \int_M r^{\alpha(2-n)} \frac{u^2}{r^2} \, \mathrm{d}v_F.$$

Note that by choosing $\alpha = 0$ in Theorem 5.2.5, we recover the Hardy inequality obtained by Farkas, Kristály, and Varga [26, Proposition 4.1], namely

$$\int_M F^{*2}(x, Du) \, \mathrm{d}v_F \geq \frac{(n-2)^2}{4 r_F^2} \int_M \frac{u^2}{r^2} \, \mathrm{d}v_F, \quad \forall u \in C_0^\infty(M). \tag{5.1}$$

Theorem 5.2.5, as well as relation (5.1), represents the Finslerian generalization of the classical L^2-Hardy inequality, available on Finsler-Hadamard manifolds. One can see that the obtained relations strongly depend on the geometry of the Finsler structure, which is manifested by the assumption $\mathbf{S} = 0$ and the finite reversibility condition $r_F < \infty$. Moreover, the reversibility constant turns out to be embedded in the constant of the previous Hardy inequalities.

Note that if (M,F) is a reversible Finsler-Hadamard manifold, i.e., $r_F=1$, the constant $\frac{(n-2)^2}{4}$ in (5.1) is sharp and never achieved, see Farkas, Kristály, and Varga [26]. On the other hand, if we let $r_F\to\infty$, inequality (5.1) becomes trivial. The sharpness of the constant $\frac{(n-2)^2}{4r_F^2}$ in the general case $r_F>1$ is an open question.

Finally, we present the following logarithmic Hardy inequality:

Theorem 5.2.7. (Mester, Peter, and Varga [4]) *Let (M,F) be an n-dimensional Finsler-Hadamard manifold with $n\geq 2$, $r_F<\infty$ and $\mathbf{S}=0$, and consider a fixed number $\alpha\in\mathbb{R}\setminus\{1\}$. If $\alpha<1$ define $\Omega:=r^{-1}(0,1)$, while if $\alpha>1$ set $\Omega:=r^{-1}(1,+\infty)$. Then we have*

$$\int_\Omega |\ln r|^\alpha F^{*2}(x,Du)\ \mathrm{d}v_F \geq \frac{(1-\alpha)^2}{4r_F^2}\int_\Omega |\ln r|^\alpha \frac{u^2}{(r\ln r)^2}\ \mathrm{d}v_F,\ \ \forall u\in C_0^\infty(\Omega). \tag{5.2}$$

In the last part of this chapter, we focus on so-called multipolar Hardy inequalities, which represent one of the most challenging directions of extension regarding the classical Hardy inequality.

The optimal multipolar counterpart of the unipolar Hardy inequality on the n-dimensional Euclidean space was given by Cazacu and Zuazua [18], namely

$$\int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x \ \geq\ \frac{(n-2)^2}{m^2}\sum_{1\leq i<j\leq m}\int_{\mathbb{R}^n}\left|\frac{x-x_i}{|x-x_i|^2}-\frac{x-x_j}{|x-x_j|^2}\right|^2 u^2\mathrm{d}x, \quad \forall u\in C_0^\infty(\mathbb{R}^n), \tag{5.3}$$

where $x_1,\ldots,x_m\in\mathbb{R}^n$ represent pairwise distinct poles, $m\geq 2$, $n\geq 3$, and the constant $\frac{(n-2)^2}{m^2}$ is sharp.

This result was extended to complete Riemannian manifolds by Faraci, Farkas, and Kristály [24], as follows. If (M,g) is an n-dimensional complete Riemannian manifold with $n\geq 3$, and $\{x_1,\ldots,x_m\}\subset M$ is a set of pairwise distinct poles with $m\geq 2$, then the following multipolar Hardy inequality holds:

$$\begin{aligned}\int_M |\nabla_g u|_g^2\mathrm{d}v_g \ &\geq\ \frac{(n-2)^2}{m^2}\sum_{1\leq i<j\leq m}\int_M\left|\frac{\nabla_g d_i}{d_i}-\frac{\nabla_g d_j}{d_j}\right|_g^2 u^2\mathrm{d}v_g\\ &+\frac{n-2}{m}\sum_{i=1}^m\int_M\frac{d_i\Delta_g d_i-(n-1)}{d_i^2}u^2\mathrm{d}v_g, \quad \forall u\in C_0^\infty(M),\end{aligned} \tag{5.4}$$

where $d_i:=d_g(x_i,\cdot)$ denotes the Riemannian distance from the pole $x_i\in M$, $i=\overline{1,m}$. In addition, the constant $\frac{(n-2)^2}{m^2}$ is sharp in the bipolar case, i.e., when $m=2$.

In the spirit of these studies, the purpose of this section is to investigate multipolar Hardy inequalities on complete, not necessarily reversible Finsler manifolds. Once again, we arrive to the conclusion that the obtained results are strongly influenced by the geometric properties of the given Finsler structure, expressed in terms of the reversibility constant r_F and uniformity constant l_F. Our first result reads as follows.

Theorem 5.4.1. (Mester and Kristály [2]) *Let (M, F) be a complete n-dimensional Finsler manifold with $n \geq 3$ and $l_F > 0$, and consider the set of pairwise distinct poles $\{x_1, \dots, x_m\} \subset M$, where $m \geq 2$. Then*

$$\left(2 - \frac{l_F^2}{r_F^2}\right) \int_M F^{*2}(Du)\mathrm{d}v_F \;\geq\; (l_F - 2)\frac{(n-2)^2}{m^2} \int_M F^{*2}\Big(\sum_{i=1}^{m} \frac{Dd_i}{d_i}\Big)u^2\mathrm{d}v_F + l_F \frac{n-2}{m} \int_M \mathrm{div}\Big(J^*\Big(\sum_{i=1}^{m} \frac{Dd_i}{d_i}\Big)\Big)u^2\mathrm{d}v_F \tag{5.5}$$

holds for every nonnegative function $u \in C_0^\infty(M)$, where $d_i(x) := d_F(x, x_i)$ denotes the Finslerian distance from the point x to the pole x_i, $i = \overline{1,m}$.

First, let us remark that the condition $l_F > 0$ implies the fact that $r_F < \infty$.

Furthermore, we prove that Theorem 5.4.1 represents exactly the Finslerian counterpart of the Riemannian inequality (5.4). Indeed, when $(M, F) = (M, g)$ is a Riemannian manifold, it follows that $l_F = r_F = 1$, and it can be shown that (5.5) becomes equivalent to the inequality (5.4). Therefore, Theorem 5.4.1 extends the multipolar Hardy inequality obtained by Faraci, Farkas, and Kristály [24, Theorem 1.1] to the class of complete Finsler manifolds, provided that the uniformity constant l_F is nonzero.

Next, applying Theorem 5.4.1 by choosing $m = 2$ results in the following bipolar Hardy inequality:

Theorem 5.4.2. (Mester and Kristály [2]) *Let (M, F) be a complete n-dimensional Finsler manifold with $n \geq 3$ and $l_F > 0$. If $x_1, x_2 \in M, x_1 \neq x_2$, then*

$$\begin{aligned}\int_M F^{*2}(Du)\mathrm{d}v_F \;\geq\;& \frac{l_F(2-l_F)}{2-\left(\frac{l_F}{r_F}\right)^2}\frac{(n-2)^2}{4}\int_M F^{*2}\Big(\frac{Dd_2}{d_2} - \frac{Dd_1}{d_1}\Big)u^2\mathrm{d}v_F \\ &+ \frac{l_F}{2-\left(\frac{l_F}{r_F}\right)^2}\frac{n-2}{2}\int_M \mathrm{div}\Big(J^*\Big(\frac{Dd_1}{d_1} + \frac{Dd_2}{d_2}\Big)\Big)u^2\mathrm{d}v_F \\ &- \frac{2-l_F}{2-\left(\frac{l_F}{r_F}\right)^2}\frac{(n-2)^2}{2}\int_M \Big(\frac{1}{d_1^2} + \frac{1}{d_2^2}\Big)u^2\mathrm{d}v_F \end{aligned} \tag{5.6}$$

holds for every nonnegative function $u \in C_0^\infty(M)$.

The proof of an expanded form of inequality (5.5) appears to be a difficult problem to solve. This can be contributed to the fact that the Legendre transform J^* associated to the Finsler metric F is usually not linear. Furthermore, the 'expansion of the square' method cannot be applied due to the lack of an appropriate inner product. Therefore, the sensible approach is to use suitable estimates, but such approximations do not produce the desired results in the multipolar case. Nevertheless, to our knowledge Theorems 5.4.1 and 5.4.2 seem to be the first contributions considering multipolar Hardy inequalities in the Finslerian setting.

Regarding the role of the constants l_F and r_F in the previous inequalities, we remark the following example.

Example 5.4.1. (Mester and Kristály [2]) Let $(\mathbb{B}^n, F)$ be the n-dimensional Euclidean open unit ball endowed with the Funk metric, see Example 4.5.1. In this case, we have that $r_F = +\infty$ and $l_F = 0$, thus both inequalities (5.5) and (5.6) reduce to trivial statements. This particular example indicates the importance of the condition $l_F > 0$ in Theorems 5.4.1 and 5.4.2.

Chapter 6

Application to partial differential equations

The primary application of functional inequalities manifests in the theory of partial differential equations. When studying different elliptic PDEs and the associated BVPs via variational methods, the appropriate Sobolev inequalities and embedding results provide a tool to analyze the energy functional associated with the given problem. This way, one can verify essential properties of the energy functional such as sequential lower semicontinuity or the Palais-Smale condition. These conditions in turn enable us to prove existence/uniqueness/multiplicity results by applying certain minimization and/or minimax arguments, see e.g., Willem [53].

Due to the unusual phenomena which can result from the anisotropic nature of the Finsler metric, the adaptation of the standard variational methods in the case of Finsler manifolds requires careful analysis and increased attention. Various elliptic PDEs associated with the Finsler-Laplace operator have been studied on Minkowski spaces, see Alvino et al. [9], Farkas, Fodor, and Kristály [25], Ferone and Kawohl [27], as well as on more general Finsler manifolds, see Farkas, Kristály, and Varga [26], Kristály and Rudas [36] and Ohta and Sturm [44].

Accordingly, this chapter offers a demonstration of the application of the Sobolev inequalities and embeddings proved in Chapter 4, by presenting a multiplicity result concerning an elliptic problem defined on a Randers space (M, F).

In order to present the results of this chapter, let (M, F) be a complete n-dimensional Randers space with the Finsler structure $F : TM \to \mathbb{R}$,

$$F(x, v) = \sqrt{g_x(v, v)} + \beta_x(v), \quad (x, v) \in TM, \tag{6.1}$$

where g is a Riemannian metric and β_x is a 1-form on M. Recall that $|\beta_x|_g = \sqrt{g^*_x(\beta_x, \beta_x)} < 1$, for every $x \in M$, where g^* is the co-metric of g.

We consider the following parameter-dependent elliptic problem, where the leading term is given by the p-Finsler-Laplace operator $\Delta_{F,p}$, i.e.,

$$\begin{cases} -\Delta_{F,p} u(x) = \lambda \alpha(x) h(u(x)), & x \in M, \\ u \in W^{1,p}_F(M), \end{cases} \tag{$\mathcal{P}_\lambda$}$$

where $n < p < \infty$, λ is a positive parameter, $\alpha \in L^1(M) \cap L^\infty(M)$, and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function. For each $s \in \mathbb{R}$, let $H(s) = \int_0^s h(t)\,\mathrm{d}t$. We assume the following properties:

(A_1) there exists $s_0 > 0$ such that $H(s) > 0$, $\forall s \in (0, s_0]$;

(A_2) there exist $C > 0$ and $1 < w < p$ such that $|h(s)| \leq C(1 + |s|^{w-1})$, $\forall s \in \mathbb{R}$;

(A_3) there exists $q > p$ such that

$$\limsup_{s \to 0} \frac{H(s)}{|s|^q} < \infty.$$

Then we can prove the following multiplicity result regarding problem $(\mathcal{P}_\lambda)$:

Theorem 6.2.1. (Farkas, Kristály, and Mester [1]) *Let (M, F) be an n-dimensional Randers space endowed with the Finsler metric (6.1) such that $a := \sup_{x \in M} |\beta_x|_g < 1$ and g is a Riemannian metric, where (M, g) is a Hadamard manifold with sectional curvature bounded above by $-\kappa^2$, $\kappa > 0$. Suppose that G is a compact connected subgroup of $\mathrm{Isom}_F(M)$ such that $\mathrm{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$. Let $n < p < \infty$ and $\lambda > 0$ a parameter. If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying $(A_1) - (A_3)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ is a nonzero, nonnegative function which depends on $d_F(x_0, \cdot)$ and satisfies*

$$\sup_{R>0} \operatorname*{essinf}_{d_F(x_0,x) \leq R} \alpha(x) > 0,$$

then there exists an open interval $\Lambda \subset [0, \lambda^*]$ and a number $\mu > 0$, such that for every $\lambda \in \Lambda$, problem $(\mathcal{P}_\lambda)$ admits at least three distinct solutions in $W^{1,p}_{F,G}(M)$ having $W^{1,p}_F(M)$-norms less than μ.

The proof is based on variational arguments, combining the compact embedding from Theorem 4.5.1 with the Ricceri-type critical point result of Bonanno [14].

Chapter 7

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Chapter 8

List of publications

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