Óbuda University

Doctoral (PhD) Thesis

Advanced Application of the Catenary and the Parabola for Mathematical Modelling of the Conductor and Sag Curves in the Span of an Overhead Line

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INTRODUCTION

Motivation of the Work

My interest toward electrical gadgets and electrical energy comes from my childhood; it was evident for me that I was going to be an electrical engineer. Thus, I attended and finished Electrical Engineering High School and graduated from the Faculty of Electrical Engineering.

Having a university degree I got a job in a company whose main activities were designing and constructing electrical distribution network. Working as an electrical designer engineer I faced some generally used approximate methods and calculations for OHL design which go after the determination of the maximum sag of the parabolic conductor curve and the parameter of the catenary. In fact, I have concluded that the sag–tension calculation is appropriate and well described in literature, but new mathematical equations related to the conductor curve and the sag are needed for an easy and correct determination of the conductor height and the sag at any point of the span, without any limitations caused by the span inclination or the span type. This is of a particularly high importance for the accurate clearance calculation. Hence, I had to make a decision soon, whether to accept and use the approximate methods or to develop mathematically exact new ones on my own. This way, come–by–chance, I found a field of my future research. The motivation was double, providing novel results and taking entire responsibility for planned network. Thus, I started to create new algorithms in order to complement the existing OHL design.

At the beginning of my research, implementing strictly mathematical approach for solving actual problems, I widely upgraded the parabola based calculation for OHL design providing several new equations for a direct use in practice. The company, where I worked at that time, accepted my methods and implemented them in OHL design process after I had trained the other designers in the company. Representing the previous company, my project named *Designer Programme* successfully entered the *19th Hungarian Innovation Award Competition* and as a recognised innovation got into the *Innovation Award 2010* book. This success enthused me a lot and inspired me to continue my research.

Having found the solutions for several parabola problems from the aspect of OHL design, I started solving the catenary's actual problems as well. It was clear that much more complicated algorithms have to be developed than in the case of the parabola, but at the same time the challenge was also bigger. Continuing my research and providing new methods, equations and relations concerned to the catenary, necessary conditions have also been achieved for the mathematical comparison of the catenary and the parabola, and then for finding out their specific similarities and differences which are relevant for OHL design.

I have been doing my research in the field of OHL design for more than 20 years. Presenting my new results in full, both in scientific journals and on professional conferences is a particular satisfaction to me. In order to reach more readers and listeners interested in my work, I use more languages alike, English, Hungarian, Croatian, Bosnian and Serbian.

Structure of the Dissertation and Research Objectives

As I spent many years doing different activities connected to OHL design, first as a design engineer and later also as a plan supervisor, I had the opportunity to recognize the special problems, which electrical design engineers face in practice when planning OHL, but literature does not give adequate solutions. In accordance to that, my aim was to complement the OHL design by special new equations for an easy application in practice. On the other hand, I wanted to provide a mathematical background, which helps to understand not only the derivation of the new equations but also the behaviour of the parabola and the catenary when the span inclination changes, as well as a mathematical connection between the two latter curves. These are important conditions for a conscious design. The main objectives are the following:

- The aim of Chapter 1 was to present the drone and its wide usability for overhead lines inspection. The necessity of the mathematical algorithms (derived in Chapters 2–5) is highlighted for planning the appropriate flight path of an autonomous drone applied for inspection of overhead lines.

- Considering the conductor curve as a catenary, the goal of Chapter 2 was to derive universal equations for determining the conductor height and the sag at any point of the span, usable in all span types, applying the coordinate system in a new way, which is uniformly used through the whole dissertation.

- The aim of Chapter 3 was to create a method for modelling an inclined span by known data of a level span, when the span length and the catenary parameter are common data in both spans. Evaluating the use of $1/\cos\psi$ multiplier in the case of the catenary was also targeted.

- In Chapter 4 the goal was to derive universal parabolic equations for the conductor and the sag curves, and to create a mathematical parabolic approximation of the catenary in an inclined span and then to evaluate the application of 1/cosψ in the case of the parabola.

- Chapter 5 deals with the conductor length calculation separately for the cases of the parabola and the catenary, and also gives the comparison of the lengths of the catenary and its approximation by a parabola, both the basic and the modified ones, i.e. first without and then also with the use of 1/cosψ multiplier.
- The aim of Chapter 6 was to introduce the extension of the new methods shown in Chapters 2–5 in the entire section of OHL consisting of several support spans between the two dead–end structures.

It is important to emphasize that the mathematical background has been provided in such a way that the frequently applied conveniences for simplification of calculations, as for instance considering that the maximum sag of the catenary in an inclined span is located at a mid–span or using 1/cosψ multiplier in the case of the catenary, are absolutely avoided. All new equations are mathematically exact ones without simplifications and are accompanied by appropriate explanations. The main steps of the derivations have been given within the body of the chapters, but deductions of significant lengths have been presented separately in Appendices, in order to read the dissertation easily.

Chapters 2–5 have a very similar structure. Each one starts with an introduction of the actual subject and gives a literature overview highlighting those with a particular importance or uniqueness in the field of the actual research. The biggest part of the chapter is about the achievement of the new results and their explanation in details. A practical application has been shown through numerical examples, which are commonly used also for analysis of results and drawing important conclusions. Ending a chapter, the new results are summarised.

Overhead Lines and their Design

An overhead power line (overhead line, OHL) is a structure used in electric power transmission and distribution to transmit electrical energy along large distances. It consists of one or more conductors (commonly multiples of three) suspended by towers or poles. Since most of the insulations are provided by air, overhead lines are generally the lowest–cost method of power transmission for large quantities of electrical energy [1].

Overhead lines are classified in the electrical power industry by the range of voltages as follows, according to [2 p. 7–8]:

- *Low Voltage – less than 1 kV, used for connection between a residential or a small commercial customer and the utility*
- *Medium Voltage (distribution) – between 1 kV and about 33 kV, used for distribution in urban and rural areas*
- *High Voltage (subtransmission if 33–115 kV and transmission if >115 kV) – between 33 kV and about 230 kV, used for subtransmission and transmission of bulk quantities of electrical power and a connection to very large consumers*
- *Extra High Voltage (transmission) – over 230 kV, up to about 800 kV, used for long distance, very high power transmission*
- *Ultra High Voltage – higher than 800 kV.*

Lines classified as "High Voltage" are quite hazardous. A direct contact with (touching) energized conductors still presents a risk of electrocution.

Overhead lines have an important role in electric power transmission, a process in the delivery of electricity to consumers. As it is shown in Fig. 1, *a power transmission network typically connects power plants to multiple substations near a populated area. The wiring from substations to customers is referred to as electricity distribution, following the historical business model separating the wholesale electricity transmission business from distributors who deliver the electricity to the homes* [2 p. 5].

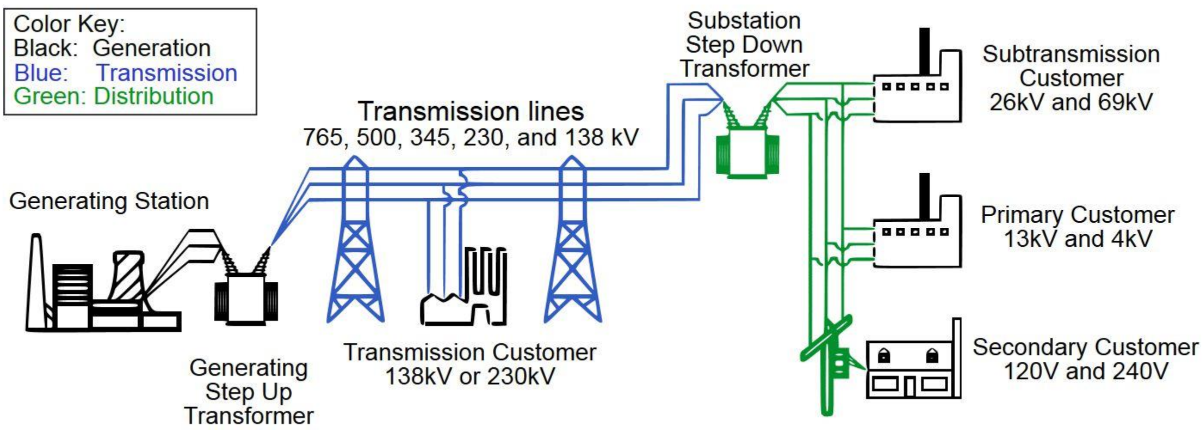


Fig. 1: The split of elements of electric power global arrangement [2]

Figs. 2, 3 and 4 show extra high voltage (EHV), medium voltage (MV) and low voltage (LV) OHL. Thus, the first one is a transmission OHL, while the other two ones are distribution OHL.

Fig. 2: EHV OHL

Fig. 3: MV OHL

Fig. 4: LV OHL

The major components of OHL are the towers (supports), insulators and conductors. *The towers for support of the lines are made of wood (as–grown or laminated), steel (either lattice structures or tubular poles), concrete, aluminum, and occasionally reinforced plastics*. The insulators are used to separate the bare conductors from the tower structure. *They must support the conductors and withstand both the normal operating voltage and surges due to switching and lightning. The insulators are broadly classified as either the pin–type, which supports the conductor above the structure, or the suspension type, where the conductor hangs below the structure. The bare wire conductors on the line are generally made of aluminum (either plain or reinforced with steel or sometimes composite materials). Bundled conductors are applied for voltages over 200 kV to avoid corona losses and audible noise. They consist of several subconductors which are connected by non–conducting spacers. Overhead lines are often equipped with a ground conductor (a shield wire or an overhead earth wire).* It *is a conductor that is usually grounded (earthed) at the top of the supporting structure to minimise the likelihood of direct lightning strikes to the phase conductors.* The *shield wires on transmission lines may include optical fibers (OPGW), used for communication and control of the power system* [2 p. 8–9].

According to [3], *classification of OHL towers is given below from the point of view of the tower's function. It means whether the tower is a suspension type, an angle type or a dead–end type. Depending on the deviation angle of the line, the respective tower is chosen. The suspension type of the towers* (shown in Fig. 2) *carries the load of the conductor in a normal situation only. However, suspension towers are usually designed to work satisfactorily for very small angular deviation of the line. The standard code of practice of different countries has specified the maximum deviation angle for the use of the suspension towers. The angle* *towers are used when the line route deviates more than this specified maximum angle. The angle towers can again be sub–grouped for different ranges of angular deviation. So the towers can be categorized as small angle, medium angle or large angle towers. The towers used at the termination point of the line are dead–end towers and are designed to carry large unbalanced load. The dead–end towers are the strongest and heaviest ones. In practice large angle towers are designed so that they can be used as dead end towers. Doing so will eliminate the need for designing one more tower type that is dead–end. The angle towers and dead end towers use tension insulator strings* (see in Fig. 5), *whilst the suspension towers are provided with the suspension insulator strings* [3].

Fig. 5: The angle tower of transmission line

The energized conductors of transmission and distribution lines must be installed in a manner that minimizes the possibility of injury to people, flashovers to other conductors, and to inanimate objects such as buildings, whether below or adjacent to the line. Self–supporting overhead conductors elongate with time, with increasing temperature and with ice and wind loads; any such conductor elongation increases the sag of the conductor which may decrease the clearance to objects or people. Under all foreseeable conditions, despite the effects of weather and variations in electrical loading, the line's conductors must remain at safe *distances from people, other conductors, vehicles, buildings, and any other reasonably anticipated activities. To ensure safe minimum electrical clearances under all conditions, the height and lateral position of the conductor between support points must be calculated for all wind, ice and temperature conditions which the conductor may experience. These calculations are commonly referred to as sag–tension calculations.* [4 p. 1]. These give the necessary data for obtaining the equation for the conductor curve which is needed for a clearance calculation.

A major goal of OHL design is to maintain adequate clearance between energized conductors and the ground or objects so as to prevent dangerous contact with the line. *This is extremely dependent on the voltage the line is running at*. [2 p. 8]

Methodology of the Dissertation

In order to describe the conductor curve when planning overhead lines, the parabola or the catenary model is used. Thus, the calculations can be parabola or catenary based. The last one is known as exact and complicated, while the first one as approximate and simple. In practice it is a well–known fact that when the spans are large (for instance over 400 metres) the conductor curve cannot be considered as a parabola, since the difference in comparison to the catenary is then not negligible. According to this work, when the conductor curve is considered as a catenary, then the main datum, which has to be taken from the sag–tension calculation, is a parameter of the catenary, while in the case of the parabola it is the maximum sag. Since both cases have been discussed separately, the main input data (all given in metres) are grouped as follows, while their usage is detailed in Chapters 2 and 4.

Catenary	***Parabola***
S – span length	S – span length
h_1 – height of the left–hand side support point	h_1 – height of the left–hand side support point
h_2 – height of the right–hand side support point	h_2 – height of the right–hand side support point
c – catenary parameter	D_{max} – maximum sag of the parabola

The first three data listed above are considered as known or given ones, while the fourth datum is taken from the sag–tension calculation. The latter calculation is not the subject of this work, as it is widely available and well explained in literature [4]. The focus is placed on deriving new equations for the conductor curve and the sag, and also defining the length formulas. In connection with the above listed input data for the case of the parabola only, there is one part of Chapter 4 where the maximum sag is not an input datum, and hence the results of the sag–tension calculation are not necessary, but one coordinate of the vertex point (either x or y) has to be determined and then applied as the fourth input datum.

Generally, each calculation is referred to one temperature of the conductor, and it is the one which the catenary parameter or the parabola's maximum sag datum is related to. The change in temperature causes the change of the two latter data, and hence the conductor curve is different at each temperature, as well as the conductor sag and the length. In accordance with that, the minimum vertical clearances have to be checked for the most unfavourable conditions (worst–case scenario), i.e. when the sag is maximum (see in Fig. 6). On a warm summer day, the conductor will sag more than on a cooler winter day, making the lowest point of the conductor much closer to the ground or objects. The conductor will also sag more as the electricity load increases. A highly loaded transmission line in the summer can sag more metres than in the winter carrying the same amount of electricity [5]. *The maximum allowable sag (for which the minimum ground clearance is maintained) determines the maximum allowable conductor temperature* [6]. However, it is worth mentioning that ice load can also be the cause of the maximum sag.

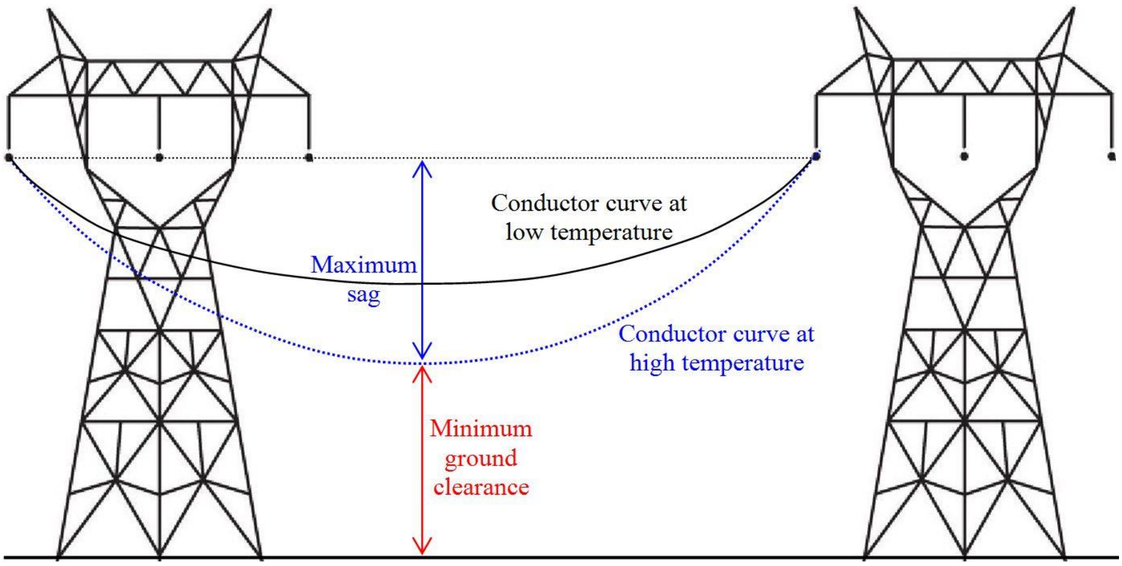


Fig. 6: Conductor curve, maximum sag and minimum ground clearance

Determining the equation for the conductor curve is of high importance, because the conductor height then can be calculated at any point of the span. It is necessary for instance to calculate the conductor clearance when some objects are placed under the conductors in a span. Fig. 6 shows a simple example with the supports on the same elevation. It is a level span. When the supports are on the different elevations (inclined span) each calculation becomes more difficult. In that case the lowest point of the conductor is not located at a mid–

span, but it is removed. This work has targeted inclined spans, but level ones have been discussed as well.

In practice, the sag–tension–temperature tables [7] are frequently used when planning distribution OHL by the parabola method. These are created by sag–tension calculations and are available for many different types of the conductors. Such tables contain the mid–span sags in level spans for different temperatures. This work also shows how to use it in an inclined span, when the span length is a common datum in both spans, by applying the special new equations. Regarding to the catenary a method for modelling an inclined span is provided, which is applicable when the span length and the catenary parameter are common data in level and inclined spans. Proposed methods can be used in all spans with line post and dead–end insulators as well. In spans with suspension insulators, which can move freely in the direction of the line, in some cases the horizontal distance between the conductor's attachment points can differ from the span length. In these cases the techniques described in [8–12] are recommended. However, as long as the suspension insulators stay vertically (Fig. 6), the methods and the equations shown in this work are all usable by applying the conductor's attachment point's data instead of the support point's data, in order to take the length of the insulators into account.

All new equations and relations in this work have been derived analytically and most of them have directly been checked in practice and by practical numerical examples as well. Knowing that the parabola, the catenary and the square of hyperbolic sine are all even functions, they are suitably applied for creating the *mirror image* examples, which made the proposed methods and also the correctness of the obtained results very clear. There are practical examples in each chapter used also for drawing important conclusions, which cannot be drawn analytically. Considering the fact that the parabola is an algebraic function, while the catenary is a transcendental one, therefore solutions of both algebraic and transcendental equations are included. The parabola based calculation is improved by algebraic transformations and matrix calculus, while the catenary based calculation is widen by the application of hyperbolic and their inverse functions, as well as related identities. Basic mathematical techniques for finding the first derivative and the maximum of the curve have been applied in both cases. The conductor length calculation is uniformly improved by the use of the integral calculus. New equations have been derived for the use in inclined spans, whereas the adequate equations related to level spans have been obtained as the simplifications of the first ones.

1 DRONE USAGE IN OHLs ENVIRONMENT

1.1 A Drone (unmanned aerial vehicle, UAV)

In a technological context a drone [13,14] is an unmanned aircraft. Drones are more formally known as unmanned aerial vehicles (UAVs) or unmanned aircraft systems (UASes). In fact, a drone is a flying robot. Generally the aircrafts may be remotely controlled or can fly autonomously through software–controlled flight plans in their embedded systems working in conjunction with onboard sensors and a GPS module. In recent past, UAVs were most often associated with the military, where they were used initially for anti–aircraft target practice, intelligence gathering and then, more controversially, as weapons platforms. Drones are now also used in a wide range of civilian roles ranging from search and rescue, surveillance, traffic monitoring, weather monitoring and fire–fighting to personal drones and business drone–based photography, as well as videography, agriculture and even delivery services [15].

Fig. 1.1: Drone [16]

The basic components of a drone are a frame, propellers, motors, battery (Li–Po), camera, gimbal (for image stabilization), GPS module, speed controller, receiver, flight controller, landing gear and other sensors.

The frame is generally made of lightweight material such as plastic, fibreglass while in the case of expensive models aluminum or titanium. Propellers are responsible for drone movements and they are made of plastic or carbon fibre.

1.2 Overhead Lines Inspection by Drones

In this section the principle of autonomous drone implementation in overhead lines inspection will be shown. Fig. 1.2 presents the trajectory of drone flight following the conductors in spans. The inclined spans located in a hilly terrain are selected here in order to show that a drone has to fly differently forward, down and up in each span of an overhead line to be inspected. As it can be seen the drone trajectory has been built by catenary curves in actual spans. These catenaries are the conductor curves. Thus, for programming the flight path of a drone, it is needed to know the equations for the conductor curves in actual spans of an overhead line.

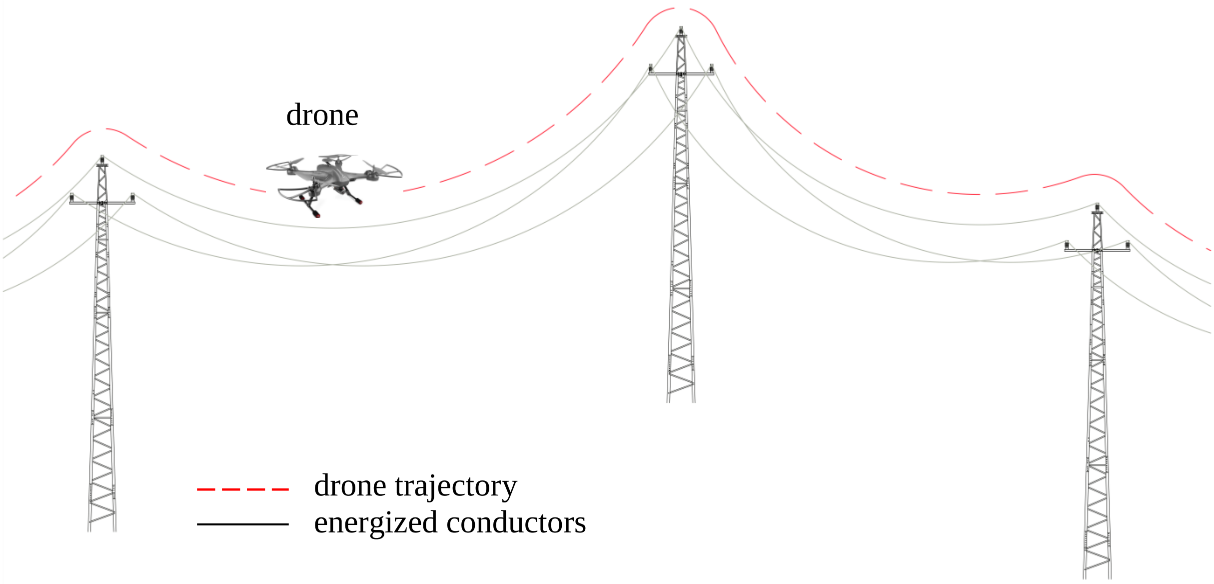


Fig. 1.2: Principle of overhead line inspection by a drone

The inspection of long overhead line by a remote controlled drone would be quite difficult in practice as it is necessary to permanently and precisely guide and control the drone. There must be a person who is responsible for this demanding task. Therefore it is more advisable to use an autonomous drone but in this case the drone needs a pre–calculated flight path. In order to plan an appropriate flight path [17] it is recommended to combine an offline and online planning. Using obtained online data from sensors mounted on the conductors or poles of overhead lines, the offline pre–calculated flight path has to be suitably modified. The corresponding principle scheme is given on Fig. 1.3.

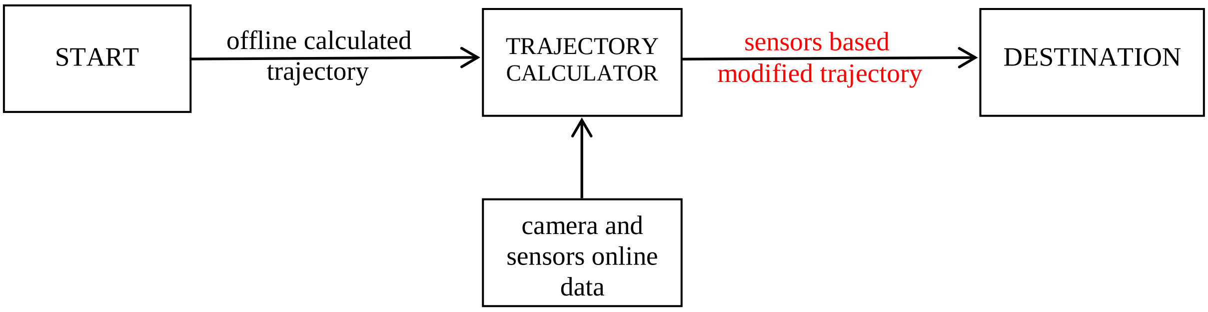


Fig. 1.3: Combining offline and online planning of a drone trajectory

Regardless to what extent the drone trajectory [18,19] within one span has to be modified, it will still be a catenary curve. As it is explained in details in Chapter 2, in fact the parameter of the catenary (c) has to be appropriately changed since it determines the shape of the catenary. In the case of overhead lines with shorter spans, when the parabola model is in use, instead of the catenary model, the leading coefficient (a) of the parabola has to be suitably modified as it determines the shape of the parabolic curve. The parabola and also its special features as the conductor curve are widely described in Chapter 4.

Regarding to the above explained modification of the drone trajectory Fig. 1.4 presents both the *offline calculated trajectory* (soft black line) and the *sensors based modified trajectory* (dashed red line) within one span. If the drone trajectory in a span is the catenary, then the parameter c is different in two cases. Depending on the online data received from the camera and sensors, the modified trajectory may be either under or up to the offline calculated trajectory. Evidently, the special case is also possible when two trajectories are quite identical.

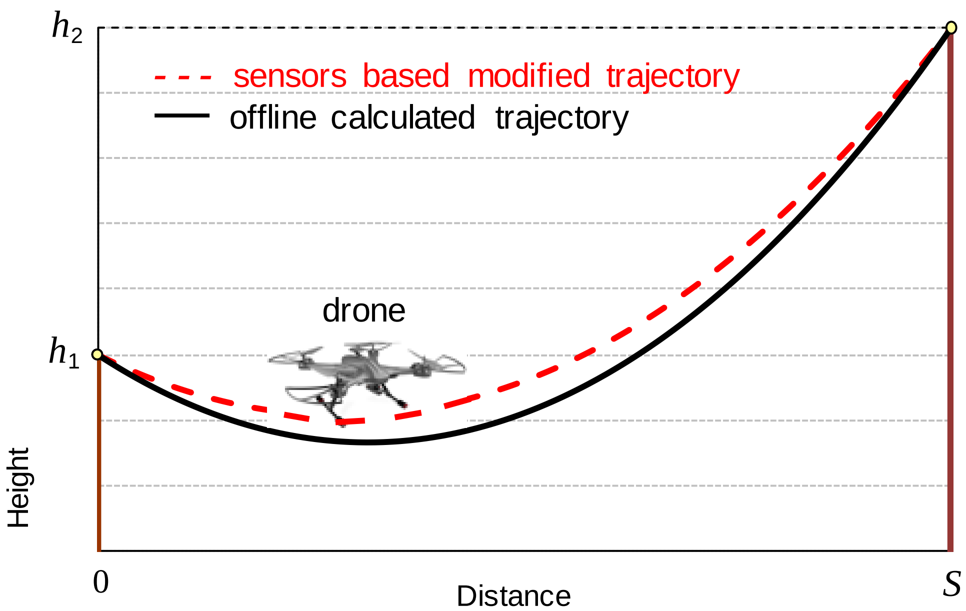


Fig. 1.4: Drone trajectories within one span

1.3 Importance of Overhead Lines Inspection

For uninterrupted power supply it is necessary to keep the electric grid (both overhead lines and underground cables) in a good condition. For this purpose their periodical inspection is essential. Presenting some possible damages in practice, the following pictures show well the importance of overhead lines inspection. In Fig. 1.5a, down on the left–hand side, it can be seen that the vibration damper (mass drooping) has significantly slipped down from its original place. It is enlarged in Fig. 1.5b.

a.) **b.)**

Fig. 1.5: Slipped vibration dumper mounted on an overhead line

In this case a double problem is present. In one hand the vibration dumper is not positioned on the right place (notice that the other dumpers are all put near the insulators) and so cannot work properly, on the other hand it causes abrasion (Fig. 1.6) of the conductor because of its loose fixation.

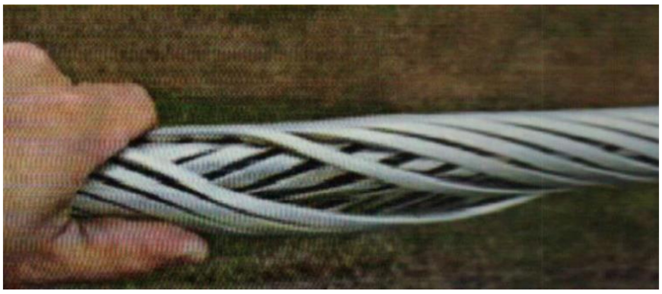
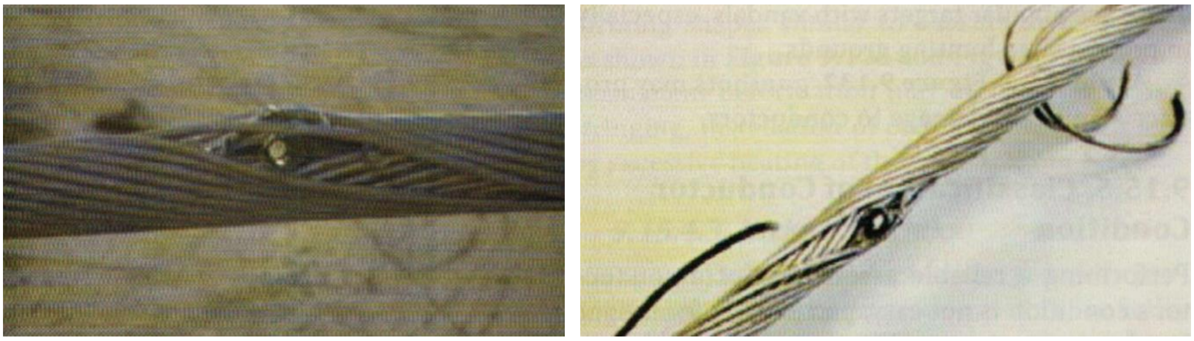


Fig. 1.6: Abrasion [20] **Fig. 1.7: Bird–caging [20]**

Fig. 1.7 shows bird–caging (or basketing) in a conductor. It occurs when conductor wires are loaded in compression, causing them to buckle and in severe cases, flare outward radially, forming shapes which are similar to baskets or birdcages. This permanent deformation can be induced during stringing, installation of compression fittings, or by excessive heating of the conductor in service [20].

Fig. 1.8 presents gunshot damages. It may be observed on conductors, aerial warning markers, and insulators as well. These are popular targets with vandals, especially in proximity to hunting grounds. Obviously, gunshots may cause significant damage to conductors [20].



a.) **b.)**

Fig. 1.8: Gunshot damages [20]

Due to technological improvements all over the world, the electric energy consumption is continuously raising. As a consequence, bigger requirements related to transmission grids have appeared. In recent years, large wind parks and solar parks have been connected and thus, they have also contributed to this fact. The capacity of the grid has to be expanded in order to be able to accommodate the new renewable energy projects.

Taking into consideration the above mentioned facts, it can be generally stated that the increasing capacity of electric energy transmission is becoming more and more important. One possibility of increasing the capacity of existing power lines is the application of the *dynamic line rating* method, which is becoming widely used nowadays.

The target of dynamic line rating is to safely utilise the transmission capacity of the existing transmission lines, which is based on real conditions in which overhead lines operate. The most important difference between static and dynamic line rating is that ‘static current’ is determined based on mainly conventional atmospheric conditions, while dynamic line rating takes into consideration the actual atmospheric conditions, which in most of the cases offer better cooling and thus, allow higher ‘dynamic’ current, contributing to improving safety [21]. Computing the dynamic line rating of an overhead transmission line is a difficult and complex task because it has to inherently solve two problems:

- determining the thermal current limit for a particular span, which can involve different measurements and different calculation methods
- determining the weakest span i.e. the span, which gives a limit for the entire transmission line, which presumes that determination of thermal current for all spans has been performed; therefore the weakest span may change in consequence of different atmospheric conditions and span features (tension, clearance margin, etc.) [21].

There are different monitoring technologies existing on the market that are designed to measure certain line parameters such as conductor tension, temperature (Fig. 1.9) or sag.

Fig. 1.9: Conductor temperature sensor attached to the energized conductor [22]

These measurements are useful for calculating line ratings but they are not sufficient for dynamic line rating. It is necessary to take into account the atmospheric conditions, because, for instance, higher wind speeds and lower ambient temperatures improve conductor cooling [21]. It is important to highlight that the line load capacity may vary even 20 – 30 %. For example, on a windy night at –20 ˚C the line can be much more loaded than in sunny windless weather at +40 ˚C. The sensors are not mounted in each span only in critic spans, for instance on a mountaintop and in a valley. They are usually attached to a place which is hard to reach. Attaching, maintaining and replacing of these *smart gauges*, which have to be mounted on the conductor or pole of overhead lines, should be done by drones instead of high–cost human power. This will be one of the advantageous usages of drones in overhead lines environment.

The conductors of overhead lines are regularly endangered by aeolian vibrations, it may cause fatigue failure of conductor strands (Fig. 1.10). The most critical locations are at suspension clamps, dead–end clamps, splices, clamp of spacers and vibration dampers [20]. Lightning discharges on overhead lines can also cause strand failure (Fig. 1.11), but in this case its location can be at any point of the span.

Fig. 1.10: Strand failure caused by aeolian vibrations [20]

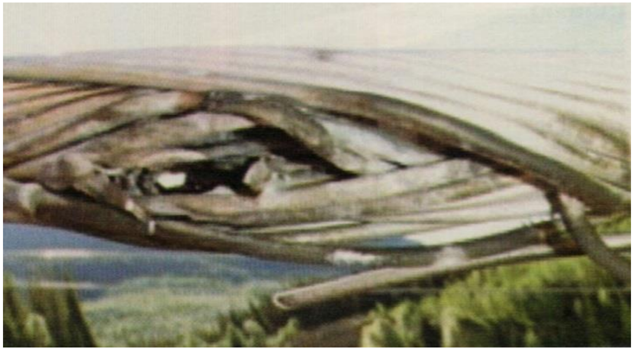


Fig. 1.11: Strand failure caused by lightning discharge [20]

Detection of these problems in time is of particular importance, as strand failures result in reduction of conductor’s cross–section. In some cases it can lead to a complete breakage of the conductor. A strand failure can be identified only by a close–up or recording, but not by helicopter or with binoculars from the ground. Therefore, it is of great importance for the inspection by drones, preferably by autonomous ones. Due to these day–to–day tasks, besides planning and building overhead lines, it is necessary to know the exact spatial location of the conductors in spans in order to plan an appropriate drone flight path.

Modern drones have been installed with special sensors which continuously ensure the safety distance between a drone and the objects in its close area to prevent collision. Evidently, besides the drone flying over a pre–calculated flight path, board sensors and human intervention must also be taken into account, but it is not being dealt here.

For planning a precise drone flight path, it is essential to know the mathematic equation for the conductor curve in a span. Depending on the given data the parabola model or the catenary model is in use but it is very important to know what the specific differences of the parabola and catenary curves are. It is also necessary to clearly understand in which cases the multiplier $1/\cos\psi$ can be applied in the calculations and when cannot. The answers on these questions are explained in details in the following chapters with a complete mathematic background and practical examples. The length of the drone trajectory can be calculated by universal formulas for the conductor length, presented in Chapter 5.

The new mathematical derivations and equations presented in Chapters 2–5 can also be used to complement the results produced by the sensor (Fig. 1.11) mounted on an overhead line, which measures the fundamental frequency of the conductor vibrating under natural turbulent wind [23].

Fig. 1.12: Sensor for measuring the frequency of the conductor vibrating [24]

This sensor provides the maximum sag datum. Knowing the span length and the coordinates of the support points of the conductor in a span, the equations for the conductor curve and the sag can be determined. Then the conductor height and the sag value can be calculated at any point of the span. Furthermore, the coordinates of the lowest point as the most critical point of the conductor in a span can also be obtained. Finally the drone flight path can be made. These facts clearly justified the importance and necessity of the new equations from Chapters 2–5.

1.4 Extensive Applicability of Drones

Utilities have to gain a lot by drone adoptions. Kilometres of electric lines, both transmission and distribution ones, have to be periodically inspected [25–28]. The use of manned aircraft

for line inspections is significantly more expensive in comparison to the use of drones for doing the task. What is more, drones are generally able to collect more data [29].
Fig. 1.13 shows an overhead line inspection by a drone. In Fig 1.14 many inspection points can be seen in a single close photo taken by a drone camera.

Fig. 1.13: Overhead line drone inspection [30]

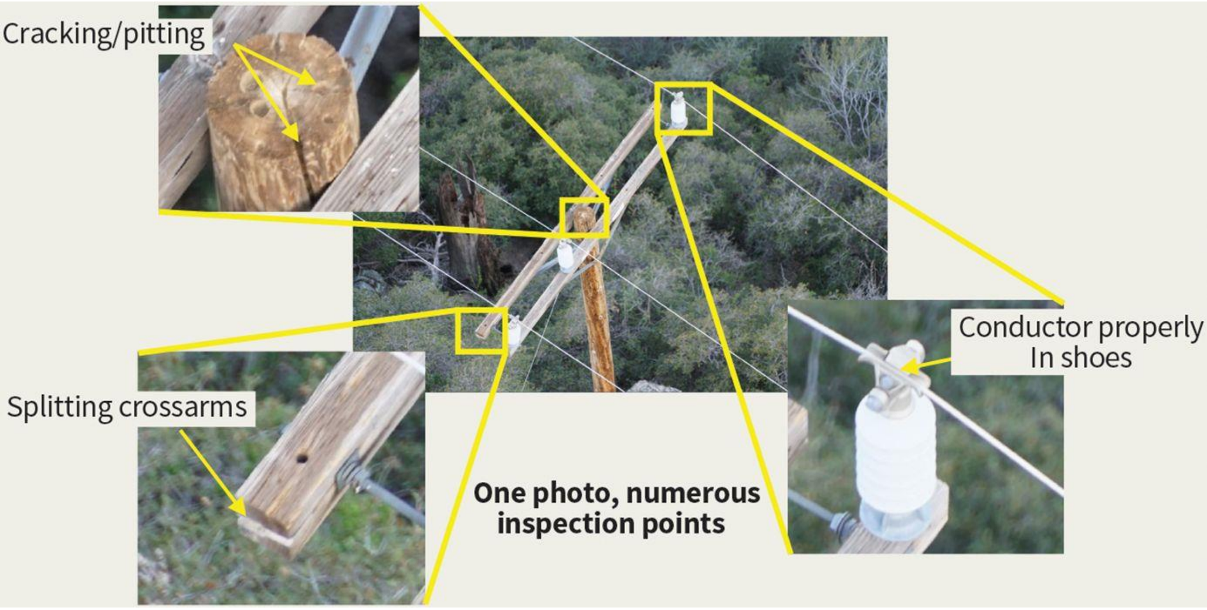


Fig. 1.14: Inspection points in a photo taken by a drone camera [31]

Due to their numerous benefits, drones have been more frequently applied in several areas of everyday life and work. In article [29] the author lists 25 ways how drones are lowering worker and business risk and improving operational efficiency for energy companies, electric, gas, solar, water and wind utilities.

1. Vegetation management inspections for transmission and distribution lines and water pipeline rights–of–way
2. Inspection of transmission and distribution lines for equipment wear, corrosion, leaning, sagging wires, broken insulators or stay wires and real–time looks during and after emergencies
3. Survey–grade maps for siting transmission lines, pipelines, dams, solar farms and wind farms
4. Construction site monitoring and reporting (counting numbers of rigs, documenting avoidance of endangered species set–aside areas)
5. Line of sight analysis
6. Interactive visual simulations, like transmission line tower heights, for stakeholder engagement
7. Substation equipment inspections
8. Gas pipeline inspections and leak detection
9. Pinpointing malfunctioning solar panels
10. Mapping ideal orientation of solar panels to maximize energy output
11. Inspection of underwater intake pipes
12. Leak detection in water pipelines
13. Hydroelectric dam inspections, including fish ladders on older dam systems
14. Aqueduct and canal inspections
15. Reservoir monitoring, including water level trends related to climate change
16. Landslide documentation
17. Wind turbine preventive maintenance inspections
18. Surveys and documentation of bird mortality at wind farms
19. Discovery of damaged fencing or anti–climb guards from vandalism
20. Monitoring for potential terrorist security threats
21. Monitoring for criminal activity in remote areas (illegal drug labs/grows)
22. Remediation site monitoring
23. Coal stockpile volume calculation
24. Inspections of ash ponds
25. Smokestack inspections

Unmanned Aerial Vehicles are robots that are aerial–based and designed to perform visual inspection. Such robots are increasing in interest to electric power utilities because they automate the inspection of transmission line assets. Work can be performed while the transmission lines are energized. Currently, electric power utilities are interested in investigating the technology of unmanned aerial vehicles (UAV’s) since they give clear images and unique inspection view when they fly near the transmission lines. In a fully autonomous mode the complete flight is performed without human help or interactions. Thus, neither an operator nor supervision by humans is needed. However, knowledge for planning the flight and the analysis of the results are required. Currently, an autonomous mode is not yet allowed in most countries or it needs a special permission by authorities. Special navigation aids and emergency procedures are also essential for this type of flight [32].

1.5 Summary of Drones Usage

To conclude this chapter the following facts can be highlighted in connection with the drone usage:

For planning the flight path for autonomous drone used for the inspection of overhead lines, it is necessary to know the equation for the conductor curve.

The future application of autonomous drones will definitely be mounting the sensors (as temperature and sag sensors, etc.) on the conductors of overhead lines. The drone will carry it out without necessity of the disconnection of the power supply since it can work on energized lines. This will be a huge benefit of drones in comparison to present practice when it is done with costly human power and the disconnection of the power supply. The use of drones for overhead lines inspection provides a significant reduction of safety risks and costs.

2 APPLICATION OF A CATENARY MODEL

2.1 Introduction

Conductors in spans of overhead lines take the shape of a catenary, a curve of the hyperbolic cosine function. The word catenary comes from the Latin word *catena*, meaning chain [33].

The mathematical derivation of the catenary equation is shown below, according to a procedure presented in [34,35], it is the same as shown in numerous related literature. In order to simplify the derivation, the curve is replaced in the coordinate system in a way that its vertex point is set at the origin. Notice, that the catenary's vertex is originally located on the y–axis, at the point $(0;c)$, where $c = H/w$ is the catenary parameter (see in Section 2.3).

Fig. 2.1 shows the conductor APB suspended between the two support points, *A* and *B*. The following symbols are used:

- w – weight of the conductor per unit length
- T – tension at any point P of the conductor
- T_0 – tension at the point O of the conductor
- l – length of the span
- L – total length of the conductor
- s – length of the conductor OP.

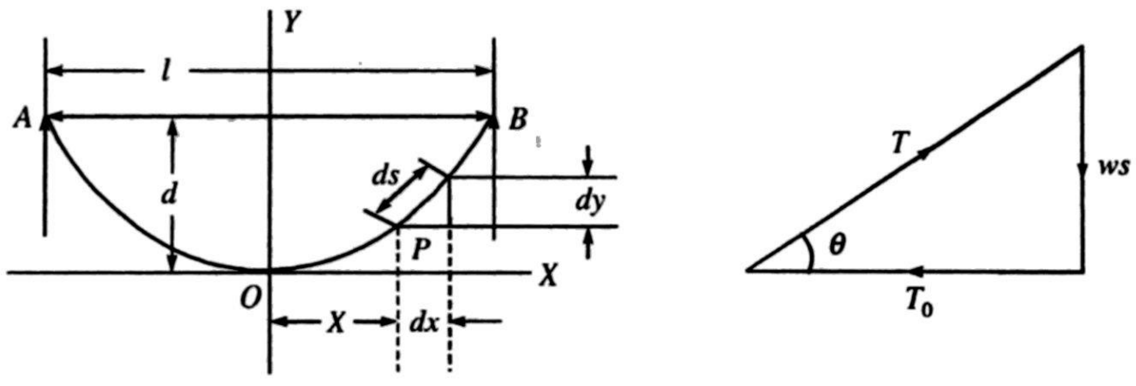


Fig. 2.1: Conductor curve of overhead transmission line [35]

At point P, the weight of the conductor is ws acting along the y–axis and tension T_0 is acting along the x–axis. Hence,

$$\tan\theta = \frac{ws}{T_0} \tag{2.1}$$

From the triangle, shown in Fig. 2.1, we can also write the following relations

$$\tan\theta = \frac{dy}{dx} \tag{2.2}$$

and
$$\frac{dy}{dx} = \frac{T_y}{T_0} = \frac{ws}{H} \quad \text{as} \quad T_0 = H \tag{2.3}$$

where H is a horizontal tension. Let us suppose ds is a small portion along the conductor from the point P, and dx and dy are the components along x and y. Then

$$(ds)^2 = (dx)^2 + (dy)^2 \tag{2.4}$$

or
$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{w^2 s^2}{H^2}} \tag{2.5}$$

Thus,
$$dx = \frac{ds}{\sqrt{1+\frac{w^2 s^2}{H^2}}} \tag{2.6}$$

Integrating (2.6) with respect to s yields (2.7)

$$x = \frac{H}{w}\operatorname{arcsinh}\frac{ws}{H} + K \tag{2.7}$$

where K is a constant of the integration. From the initial conditions, at s=0, x=0, then we get K=0 (at point O). Therefore, (2.7) becomes

$$x = \frac{H}{w}\operatorname{arcsinh}\frac{ws}{H} \tag{2.8}$$

or
$$s = \frac{H}{w}\sinh\frac{wx}{H} \tag{2.9}$$

From (2.3), dy/dx can be expressed as

$$\frac{dy}{dx} = \frac{ws}{H} \tag{2.10}$$

Substituting (2.9) into (2.10) gives (2.11)

$$\frac{dy}{dx} = \frac{w}{H}\cdot\frac{H}{w}\sinh\frac{wx}{H} = \sinh\frac{wx}{H} \tag{2.11}$$

Integrating (2.11) with reference to x, we get

$$y = \frac{H}{w}\cosh\frac{wx}{H} + B \tag{2.12}$$

where B is the constant of the integration. From the initial conditions, when x=0, y=0, so that

$$0 = \frac{H}{w} + B \quad \Rightarrow \quad B = -\frac{H}{w} \tag{2.13}$$

Thus, the equation of the curve in Fig 2.1 is given by (2.14).

$$y = \frac{H}{w}\cosh\frac{wx}{H} - \frac{H}{w} \tag{2.14}$$

After replacing the curve in a way that its vertex is at the point $(0; H/w)=(0; c)$, the final equation for the catenary curve is obtained.

$$y = \frac{H}{w}\cosh\frac{wx}{H} \tag{2.15}$$

The equation (2.15) is widely applied in the following sections for deriving the new expressions applicable in OHL design. The curve of (2.15) is shown in Section 2.3, as well as its replacement in order to define the equation for the conductor curve, which is usable for the practical determination of the conductor height at any point of the span.
Regarding to a sag–tension calculation of the conductor [4], professional literature gives a well explained procedure. However, for describing the conductor curve the coordinate system is generally used in a way that the origin is set at the vertex point of the conductor curve, so the distance toward the left–hand or right–hand side support of the span is measured from the vertex, in both directions with a positive sign. This method is not suitable for extensive mathematical analysis of the conductor curve, and also not optimal for OHL design, for instance, because of the fact that the height of the vertex of the conductor curve is never zero. For these reasons this work proposes a strictly mathematical approach, applying a coordinate system in a way that an origin is set on the line which goes through the left–hand side support, on the elevation of the bottom of the lower–positioned support of the span. The latter condition prevents the height of any support point from being negative, even in sharply inclined spans. The *y*–coordinate of the conductor curve presents the conductor height related to the *x*–axis, but the *x*–coordinate presents a horizontal distance from the left–hand side support. The heights of the support points are not generally the support heights (but they can be), it means that the proposed method also bears the use of the supports of different heights. Such an approach makes the application of many mathematical techniques easy, which can contribute to a more accurate OHL design.

This chapter is built as follows. After an overview of related research, which is briefly given in Section 2.2, the equation for the catenary conductor curve is derived using the coordinate system as it is explained above. Section 2.4 deals with the sag equation and the determination of the maximum sag's location, and also provides special formulas for computing the characteristic sags of the catenary. Section 2.5 introduces the special cases of inclined spans, while Section 2.6 shows equations for the conductor and the sag curves in a simplified case of an inclined span called levelled. A practical application of the developed equations and an analysis of the catenary on a rising span inclination are presented in Sections 2.7 and 2.8 respectively. The last section in this chapter gives a short conclusion and summarizes the novel results.

2.2 Related Research

A catenary based calculation for OHL design is widely available in professional literature. The basis of the proposed procedure is generally the same in most publications, but there are also some ones which use an uncommon approach. In the following, some studies are briefly cited from both mentioned categories.

Deriving the equation of a catenary is well available and frequent in literature; for instance in [36–38]. When planning overhead lines the mentioned equation has to be appropriately used in order to define the equation for the conductor curve. Generally, the catenary curve is replaced so that its vertex is located at the origin (explained in Section 2.3) and the equation of that curve is in use, as it is applied in [35,39]. It is very practical in the case of a level span since the maximum sag then can be obtained without the sag equation, by substituting $x=S/2$ into the equation of the replaced curve. Computing the conductor sag at any point within the span is also a relatively simple task when the support points are on the same elevation. On the other hand, in the case of an inclined span the determination of the equations for the conductor and the sag curves is more complicated. Some studies which discuss the spans with the support points on the different elevation are [40–42]. These publications present the completion of an inclined span to a longer level one, which consists of two inclined sub–spans. Both curves in these sub–spans are parts of the same catenary; therefore their parameters are the same. This way the initial task is transformed into a much simpler one. An important result obtained by this method is the horizontal distance of the catenary's vertex point from the lower support point. It can be used for defining the equation for the conductor curve.

Paper [43] is also one of those concerned to inclined spans. The author presents the so called *Newton–Raphson sag method* for calculating the sag and the horizontal tension at the lowest point of a conductor supported at unequal heights. The method is based on an iterative procedure and is also applicable to a conductor supported at equal heights.

The instructions for determining the sag in an inclined span from the given sag in a level one by the use of 1/cosψ multiplier can be found inter alia in [44–46]. The publication [47] mentions that the use of this multiplier results an approximate calculation.

In comparison to the methods mentioned above, the new approach for determining the equations for the conductor and the sag curves, which is detailed in this chapter, avoids the use of 1/cosψ and the presumption that the maximum sag of the catenary in an inclined span is located at a mid–span, as it is in the case of a level span, and provides a universal sag equation for the sag calculation at any point of the span. Another essential difference is in the way of the replacement of the catenary curve in the coordinate system. According to the new approach, the origin is never located at the catenary's vertex point.

The following two papers concerned to the catenary conductor equation should also be mentioned as these are unique each in their own way.

Paper [28] presents the determination of the equation for the catenary conductor curve from the survey data. The method described employs the least–squares criterion for curve fitting, and uses the iterative algorithm to solve for the required parameters. Input data points are (0;0), $(x_1;y_1)$, $(x_2;y_2)$,…, $(x_n;y_n)$ where (0;0) is the left–hand side support point and $(x_n;y_n)$ is the right–hand side support point. Thus, the two support points and a number of the conductor's points along the span are necessary to obtain the catenary equation. The author explains the usability of the presented method from both the line design engineer's point of view and the surveyor's point of view. The origin of the coordinate system is set at the left–hand side support point.

Paper [49] uses a specific approach, which differs from the previous ones mentioned above, providing a transcendental equation and proposes its solving by the use of a scientific calculator because it cannot be solved analytically. The author also mentions the catenary approximation by a parabola, which is very frequent in the existing literature: „*when the supports of a catenary are at different elevations, the mathematical complexity precludes a theoretically correct solution, and a parabolic approximation is the recommended approach*".

Finally, [50] has to be mentioned. This book widely discusses overhead lines, but also deals with sag–tension calculation, and numerous other topics in connection with electrical conductors.

2.3 Equation for the Catenary Conductor Curve

The main goal of this section is the determination of the equation for the catenary conductor curve using the following input data: span length, heights of the support points related to *x*–axis and the catenary parameter. The shape of the catenary depends on this parameter $c > 0$. An increase in the catenary parameter causes the catenary curve to become shallower and the sag to decrease [4]. For a conductor with no wind and no ice, the catenary parameter is the ratio of the horizontal tension H (given in N/mm^2 or daN/mm^2) to unite the mass of the cable w (given in N/mm^3 or daN/m,mm^2), i.e. $c = H/w$ (given in metres) [51]. *The catenary parameter typically has a value in the range of 500 – 2000 metres for most transmission lines under most conditions* [4].

2.3.1 Basic Equation for the Conductor Curve

The basic catenary equation is expressed by (2.16) and its curve, y_1, is illustrated in Fig. 2.2.

$$y_1 = c \cdot \cosh \frac{x}{c} \qquad x \in (-\infty, +\infty) \tag{2.16}$$

It can be seen that the vertex of the catenary curve y_1 is located at point $(0;c)$. If y_1 curve is replaced so that its vertex is set at the origin, the equation of the replaced curve, y_2, is then given by (2.17).

$$y_2 = c \cdot \cosh \frac{x}{c} - c \qquad x \in (-\infty, +\infty) \tag{2.17}$$

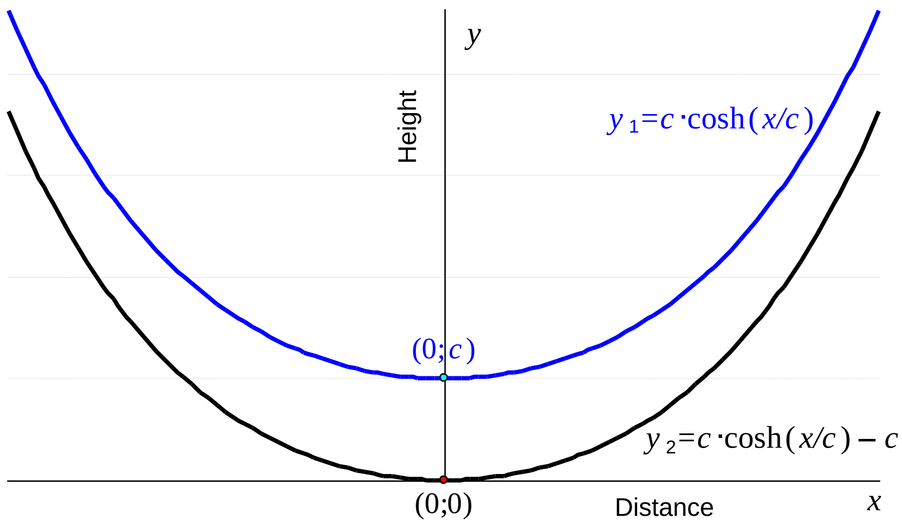


Fig. 2.2: Catenary curves

In order to present the conductor curve in a mathematically convenient coordinate system for OHL design, the catenary curve y_2 has to be appropriately replaced both horizontally and vertically, as it is shown in Fig. 2.3. The inclined span has been deliberately chosen instead of the level one, so the derived equation will be universal.

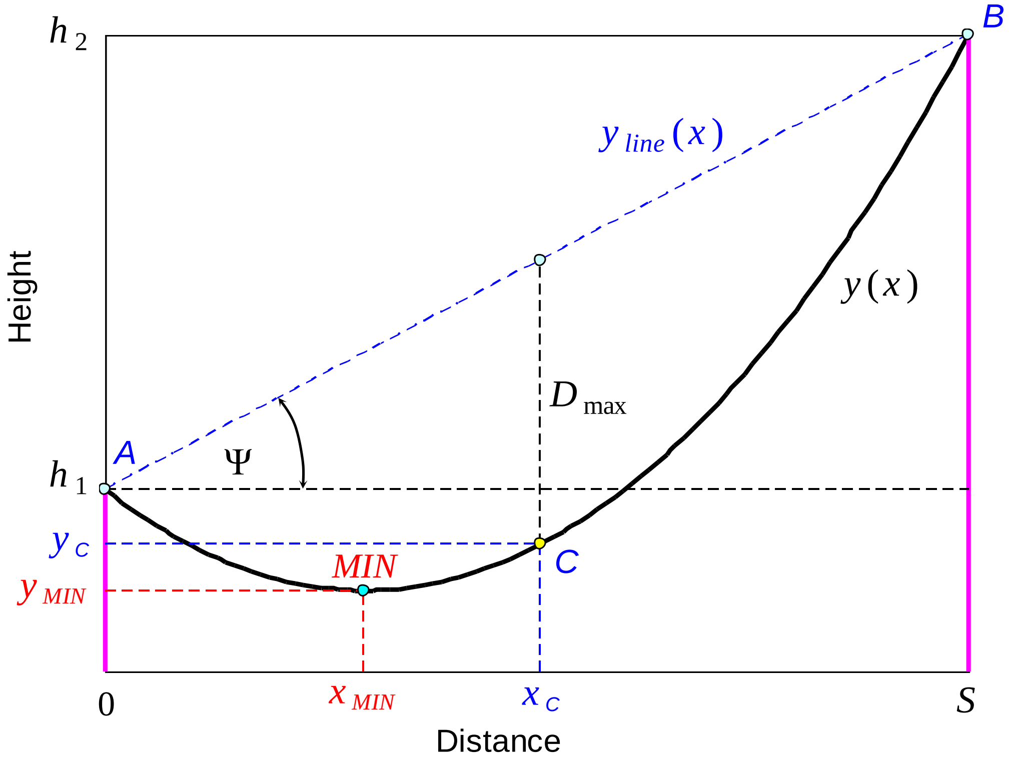


Fig. 2.3: Catenary conductor curve in an inclined span

The following symbols are used in Figure 2.3:

$A(0;h_1)$ – left–hand side support point

$B(S;h_2)$ – right–hand side support point

$MIN(x_{MIN};y_{MIN})$ – catenary's low point

$C(x_c;y_c)$ – conductor's point with a maximum sag

S – span length

D_{max} – maximum sag

$y(x)$ – conductor curve (catenary)

$y_{line}(x)$ – chord, straight line between the support points

ψ – angle of the span inclination.

The basic equation for the conductor curve on the interval [0,S] shown in Fig. 2.3 is the following:

$$y(x) = c \cdot \cosh \frac{x - x_{MIN}}{c} - c + y_{MIN} \qquad x \in [0, S] \tag{2.18}$$

This equation is also expressible by square of the hyperbolic sine function as (2.19).

$$y(x) = 2c \cdot \sinh^2 \frac{x - x_{MIN}}{2c} + y_{MIN} \qquad x \in [0, S] \tag{2.19}$$

Notice that cosh(x) and sinh^2(x) are both even functions, while sinh(x) is an odd one. Hence, cosh(–x)=cosh(x) and also sinh^2(–x)=sinh^2(x), but sinh(–x)$\neq$sinh(x). The basic expressions for functions cosh(x) and sinh^2(x) are as follows:

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \tag{2.20}$$

$$\sinh^2(x) = \left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]^2 \tag{2.21}$$

Thus, equations (2.18) and (2.19) have their equivalent versions in exponential form given by (2.22) and (2.23).

$$y(x) = \frac{c}{2}\left(e^{\frac{x - x_{MIN}}{c}} + e^{-\frac{x - x_{MIN}}{c}}\right) - c + y_{MIN} \qquad x \in [0, S] \tag{2.22}$$

$$y(x) = \frac{c}{2}\left(e^{\frac{x - x_{MIN}}{2c}} - e^{-\frac{x - x_{MIN}}{2c}}\right)^2 + y_{MIN} \qquad x \in [0, S] \tag{2.23}$$

All four equations (2.18), (2.19), (2.22) and (2.23) are universal, i.e. they can be applied in inclined and level spans as well, but for their concrete usage the vertex point (marked with *MIN*) has to be determined previously. In Fig. 2.3 it is the lowest point of the curve. In comparison to the parabolic (quadratic) equation for the conductor curve there is a significant difference, since it can be defined even without knowing the vertex point of the conductor curve when the maximum sag of the parabola is given (see Chapter 4).

2.3.2 Determining the Vertex Point and the Final Catenary Equation

The coordinates of the catenary's vertex point can be determined on the basis of the following input data: S, h_1, h_2, c. By points A and B, we can write two equations (2.24) and (2.25) in two unknowns, then the first equation has to be subtracted from the second one.

$$h_1 = c \cdot \cosh \frac{-x_{MIN}}{c} - c + y_{MIN} \tag{2.24}$$

$$h_2 = c \cdot \cosh \frac{S - x_{MIN}}{c} - c + y_{MIN} \tag{2.25}$$

$$h_2 - h_1 = c \cdot \left(\cosh \frac{S - x_{MIN}}{c} - \cosh \frac{-x_{MIN}}{c} \right) \tag{2.26}$$

By the application of identity (2.27) [52] x_{MIN} can be defined as (2.29) [4,53–56].

$$\cosh(x) - \cosh(y) = 2 \sinh \frac{x+y}{2} \cdot \sinh \frac{x-y}{2} \tag{2.27}$$

$$h_2 - h_1 = 2c \cdot \sinh \frac{S - 2x_{MIN}}{2c} \sinh \frac{S}{2c} \tag{2.28}$$

$$x_{MIN} = \frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \left(S / 2c \right)} \tag{2.29}$$

Using (2.29), y_{MIN} is obtained from (2.24) and transformed into (2.32).

$$y_{MIN} = h_1 - c \cdot \left(\cosh \frac{-x_{MIN}}{c} - 1 \right) \tag{2.30}$$

$$y_{MIN} = h_1 - 2c \cdot \sinh^2 \frac{x_{MIN}}{2c} \tag{2.31}$$

$$y_{MIN} = h_1 - 2c \cdot \sinh^2 \left[\frac{S}{4c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \left(S / 2c \right)} \right] \tag{2.32}$$

Since x_{MIN} and y_{MIN} are determined, the basic equation (2.19) for the conductor curve can be completed to (2.33), whereas the final catenary equation is given by expression (2.34). The actual interval is [0,S].

$$y(x) = 2c \cdot \sinh^2 \frac{x - x_{MIN}}{2c} + h_1 - 2c \cdot \sinh^2 \frac{x_{MIN}}{2c} \qquad x \in [0, S] \tag{2.33}$$

$$\begin{aligned} y(x) = 2c \cdot \Bigg\{ & \sinh^2 \left[\frac{S}{4c} - \frac{x}{2c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \left(S / 2c \right)} \right] - \\ & - \sinh^2 \left[\frac{S}{4c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \left(S / 2c \right)} \right] \Bigg\} + h_1 \qquad x \in [0, S] \end{aligned} \tag{2.34}$$

With the help of the previous equation the conductor height related to x–axis can be computed at any point of the span. Since the terrain within the span frequently differs from x–axis, the height of the terrain related to x–axis has to be taken into account when computing the ground clearance of OHL. Another important usage of equation (2.34) is drawing the conductor curve.

The vertex point of the catenary is generally the lowest point of the conductor. However, there are special cases of inclined spans when the vertex is out of the span and hence differs from the lowest point of the conductor, in their location. The latter point is then positioned at the lower support point of the span, but the coordinates of the catenary's vertex are still given by (2.29), (2.32). However, the equation (2.34) is applicable in any case, so it proves its universal usability.

2.4 Sag Equation and its Use

The conductor sag is the distance measured vertically from the conductor to the straight line (chord) joining two support points of a span. Actually the sag varies on the interval of the span, i.e. increases from zero to maximum, then decreases to zero, going from the left–hand side support to the right–hand side one. It can be appropriately described by the equation for the sag, $D(x)$, as the function of x, where x varies from zero to the span length, $x \in [0,S]$. The curve of $D(x)$ is called here a sag curve and is shown in Fig. 2.4. The sag value at some point within the span is a vertical distance from the x–axis to the sag curve.

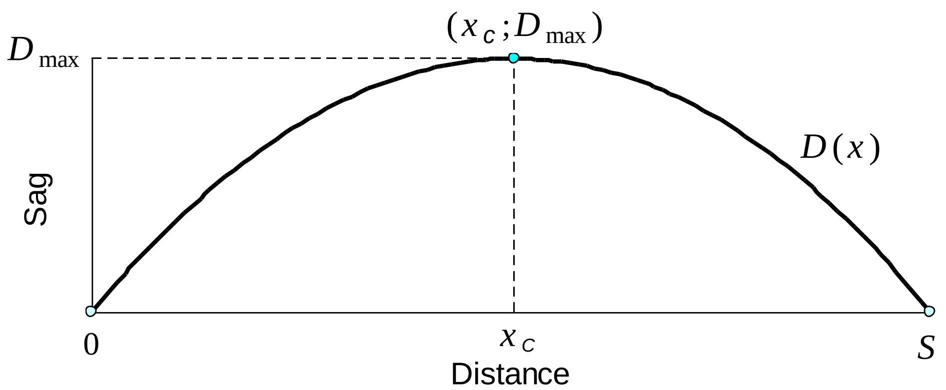


Fig. 2.4: Sag curve

Differently to the conductor curve, which has the low point, the sag curve has the maximum point. The coordinates of the latter are x_c and D_{max}.

Some literature identifies the sag curve with the conductor curve even though they have own separate equations. It is mathematically incorrect according to the explanation given above. However, the vertical distance measured from the straight line to the conductor curve at some point of the span is equal with the vertical distance measured from the x–axis to the sag curve at the same point of the span. The actual use of the sag equation is a calculation of the sag at an arbitrary point of the span. It is necessary for example, to obtain the clearance over the conductors at some points within the span.

2.4.1 Derivation of the Sag Equation

The equation for the sag curve (shortly called as *sag equation*), $D(x)$, can be derived by the use of the equation for the conductor curve. For this purpose, firstly the equation for the straight line, $y_{line}(x)$, passing through the support points A and B has to be defined on the interval $[0,S]$, then subtract (2.34) according to (2.36). The result provided is the sag equation, which is usable for the sag calculation at any point of the span.

$$y_{line}(x)=\frac{h_2-h_1}{S}x+h_1 \qquad x\in[0,S] \tag{2.35}$$

$$D(x)=y_{line}(x)-y(x) \qquad x\in[0,S] \tag{2.36}$$

$$\begin{aligned}D(x)=\frac{h_2-h_1}{S}x-2c\cdot\Bigg\{&\sinh^2\left[\frac{S}{4c}-\frac{x}{2c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh(S/2c)}\right]-\\ &-\sinh^2\left[\frac{S}{4c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh(S/2c)}\right]\Bigg\} \qquad x\in[0,S]\end{aligned} \tag{2.37}$$

Besides the sag calculation, (2.37) can be applied for determining the location of the maximum sag in a span, and also for defining special formulas for the following characteristic sags of the catenary conductor curve:

- Maximum sag $D_{\max}$
- Mid–span sag $D(S/2)$
- Low point sag $D(x_{MIN})$, where $0 \le x_{MIN} \le S$.

2.4.2 Location of the Maximum Sag in a Span

Finding the first derivative of (2.37) and considering $2\sinh x\cdot\cosh x=\sinh 2x$ [52], then solving equation (2.39), the location of the maximum sag in a span is obtained and given by (2.42).

$$\frac{d}{dx}D(x)=\frac{h_2-h_1}{S}+\sinh\left[\frac{S}{2c}-\frac{x}{c}-\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh(S/2c)}\right] \tag{2.38}$$

$$\frac{d}{dx}D(x)=0 \qquad \Rightarrow \qquad x_C \tag{2.39}$$

$$\frac{h_2-h_1}{S}+\sinh\left[\frac{S}{2c}-\frac{x_C}{c}-\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh(S/2c)}\right]=0 \tag{2.40}$$

$$\frac{S}{2c}-\frac{x_C}{c}-\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh(S/2c)}=-\operatorname{arcsinh}\frac{h_2-h_1}{S} \tag{2.41}$$

$$x_C=\frac{S}{2}+c\cdot\left[\operatorname{arcsinh}\frac{h_2-h_1}{S}-\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh(S/2c)}\right] \tag{2.42}$$

From (2.42) it is obvious that the maximum sag of the catenary in an inclined span is not located at a mid–span, but it is moved toward one of the two support points. Now there is a question whether it is moved toward the higher or the lower one. The answer to this question is given below without the use of numerical examples, but strictly analytically.
Denoting the second summand in (2.42) with q yields expression (2.43):

$$x_C = \frac{S}{2} + q \tag{2.43}$$

Now let us assume that the maximum sag is moved from the mid–span toward the higher support point and that the right–hand side support point is higher than the left–hand side one, i.e. assume that relation (2.44) is valid and then check it mathematically step by step.

$$\text{If} \quad h_1 < h_2 \quad \Rightarrow \quad q > 0 \tag{2.44}$$

The initial conditions are given: $S>0$, $c>0$, $h_1>0$, $h_2>0$. The main steps for checking the validity of the assumption given by (2.44) are shown in the following lines:

$$c \cdot \left[\operatorname{arcsinh} \frac{h_2 - h_1}{S} - \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right] > 0 \tag{2.45}$$

$$\operatorname{arcsinh} \frac{h_2 - h_1}{S} > \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \tag{2.46}$$

The inverse hyperbolic sine (see Fig. 2.5) is a monotonic, strictly increasing function [57], so

$$\text{if} \quad x_2 > x_1 \quad \Rightarrow \quad \operatorname{arcsinh}(x_2) > \operatorname{arcsinh}(x_1) \tag{2.47}$$

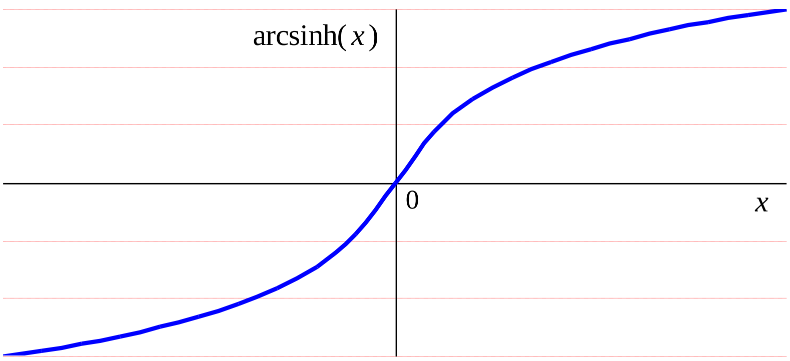


Fig. 2.5: Curve of arcsinh(*x*)

Applying (2.47) in (2.46) gives (2.48), which can deduce (2.50)

$$\frac{h_2 - h_1}{S} > \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \tag{2.48}$$

$$\frac{1}{S} > \frac{1}{2c \cdot \sinh\left(S/2c\right)} \tag{2.49}$$

$$S/2c < \sinh\left(S/2c\right) \tag{2.50}$$

Taking into consideration relations (2.51) and (2.52), it can be stated that the previous one is valid.

$$S/2c > 0 \tag{2.51}$$

$$\text{if} \quad x > 0 \qquad \Rightarrow \qquad x < \sinh\left(x\right) \tag{2.52}$$

This way the validity of the assumption (2.44) is also proved. The same process applied for cases $h_1 > h_2$ and $h_1 = h_2$ gives further two relations:

$$\text{if} \quad h_1 > h_2 \qquad \Rightarrow \qquad q < 0 \tag{2.53}$$

$$\text{if} \quad h_1 = h_2 \qquad \Rightarrow \qquad q = 0 \tag{2.54}$$

Thus, the above question of the movement of $D_{\max}$ has been satisfactory answered.

Relation (2.54) refers to a level span when there is no movement of $D_{\max}$. Summarizing (2.44), (2.53) and (2.54) the final conclusion in connection with the location of $D_{\max}$ related to the mid–span, proved analytically here, is the following:

The maximum sag of the catenary conductor curve in a level span is located at a mid–span, but in an inclined span it is moved from a mid–span toward a higher suspension point.

This is an essential difference in comparison to the parabola, since the maximum sag of the parabolic conductor curve is always located at a mid–span, in level and inclined spans as well. This feature effectively simplifies the parabola based algorithms for overhead line design.

2.4.3 Characteristic Sags

Since x_C is obtained, it can be used to determine the maximum sag (2.57). The main steps of the deduction are the following:

$$D_{\max} = D(x_C) = y_{line}(x_C) - y(x_C) \tag{2.55}$$

$$D_{\max} = \frac{h_2 - h_1}{S} x_C - 2c \cdot \left(\sinh^2 \frac{x_C - x_{MIN}}{2c} - \sinh^2 \frac{x_{MIN}}{2c} \right) \tag{2.56}$$

$$\begin{aligned} D_{\max} = 2c \cdot \Bigg\{ & \frac{h_2 - h_1}{2S} \left[\frac{S}{2c} - \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} + \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right] - \\ & - \sinh^2 \left(\frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) + \sinh^2 \left[\frac{S}{4c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right] \Bigg\} \end{aligned} \tag{2.57}$$

The previous expression is a formula for calculating the maximum sag of the catenary conductor curve in an inclined span. Similarly, formulas (2.59) and (2.61) for the other characteristic sags can be defined as follows.

Mid–span sag:

$$D(S/2)=\frac{h_2-h_1}{S}\cdot\frac{S}{2}-2c\cdot\left(\sinh^2\frac{S/2-x_{MIN}}{2c}-\sinh^2\frac{x_{MIN}}{2c}\right) \tag{2.58}$$

$$\begin{aligned}D(S/2)=\frac{h_2-h_1}{2}-2c\cdot\Bigg\{&\sinh^2\left[\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]-\\&-\sinh^2\left[\frac{S}{4c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]\Bigg\}\end{aligned} \tag{2.59}$$

Low point sag (sag at the lowest point of the conductor):

$$D(x_{MIN})=\frac{h_2-h_1}{S}x_{MIN}-2c\cdot\left(\sinh^2\frac{x_{MIN}-x_{MIN}}{2c}-\sinh^2\frac{x_{MIN}}{2c}\right)\quad\forall\quad 0\le x_{MIN}\le S \tag{2.60}$$

$$\begin{aligned}D(x_{MIN})=2c\cdot\Bigg\{&\frac{h_2-h_1}{2S}\left[\frac{S}{2c}-\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]+\\&+\sinh^2\left[\frac{S}{4c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]\Bigg\}\quad\forall\quad 0\le x_{MIN}\le S\end{aligned} \tag{2.61}$$

Considering the fact that the conductor sag is defined only within the span, the previous formula concerns to all spans where $x_{MIN}\in[0,S]$; the lowest point of the conductor and the vertex point have then the same location. In these cases the low point sag is in fact identical with the vertex point sag. The term *low point sag* is frequent in literature and that is why it is also used here, instead of the term *vertex point sag*. The adequate clarification can be done with the following five relations taking into consideration that in some cases x_{MIN} as x–coordinate of the vertex point may be outside the span.

$$\text{If}\quad x_{MIN}<0\quad\Rightarrow\quad D(x_{MIN})\text{ is not defined} \tag{2.62}$$

$$\text{If}\quad 0\le x_{MIN}<S/2\quad\Rightarrow\quad 0\le D(x_{MIN})<D(S/2)<D_{\max} \tag{2.63}$$

$$\text{If}\quad x_{MIN}=S/2\quad\Rightarrow\quad D(x_{MIN})=D_{\max} \tag{2.64}$$

$$\text{If}\quad S/2<x_{MIN}\le S\quad\Rightarrow\quad D_{\max}>D(S/2)>D(x_{MIN})\ge 0 \tag{2.65}$$

$$\text{If}\quad x_{MIN}>S\quad\Rightarrow\quad D(x_{MIN})\text{ is not defined} \tag{2.66}$$

Relations (2.62) and (2.66) correspond to special cases of inclined spans (explained in Section 2.5) where the vertex point is out of the span.

2.5 Special Cases of Inclined Spans

In practice, in most spans the vertex point of the catenary curve and the lowest point of the conductor are the same (like in Fig. 2.3), and in that case $x_{MIN} \in [0,S]$. It has to be mentioned that the conductor curve is in fact part of the mathematical catenary curve. They both have the same equation, only their domains differ. While the conductor curve is defined on the interval $[0,S]$, the domain of the mathematical catenary curve is an open interval $(-\infty,+\infty)$. Notice that the lowest point of the conductor cannot be out of the span, but the vertex point can. Thus, the locations of these two points may differ. Fig. 2.6 shows such a special case of an inclined span with $x_{MIN} < 0$.

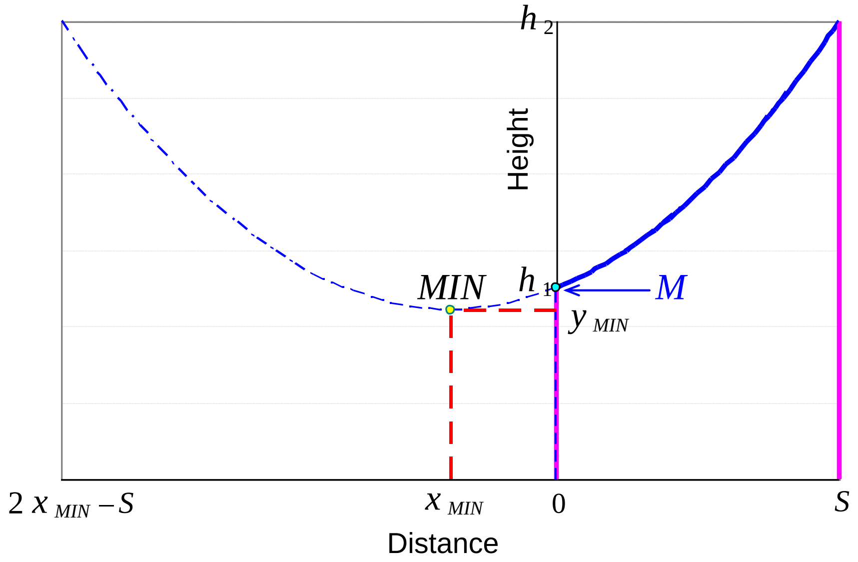


Fig. 2.6: Special inclined span with $x_{MIN} < 0$

In this case the lowest point of the conductor (here denoted by M) is identical with the left–hand side support point of the span. In order to present this rare case appropriately, the catenary curve is shown on the interval $[2x_{MIN} - S, S]$; it is drawn by a dashed line out of the span (i.e. on the interval $[2x_{MIN} - S, 0]$), while the conductor curve is drawn by a soft line on the interval $[0, S]$. In this case $x_{MIN} < 0$ and $h_1 < h_2$. There is also another type of the special case of an inclined span, when $x_{MIN} > S$ and $h_1 > h_2$. It is shown in Fig. 2.7, where the catenary curve is shown on the interval $[0, 2x_{MIN}]$ and it is drawn by a dashed line out of the span (i.e. on the interval $[S, 2x_{MIN}]$). On the other hand, the conductor curve is drawn within the span, by a soft line.

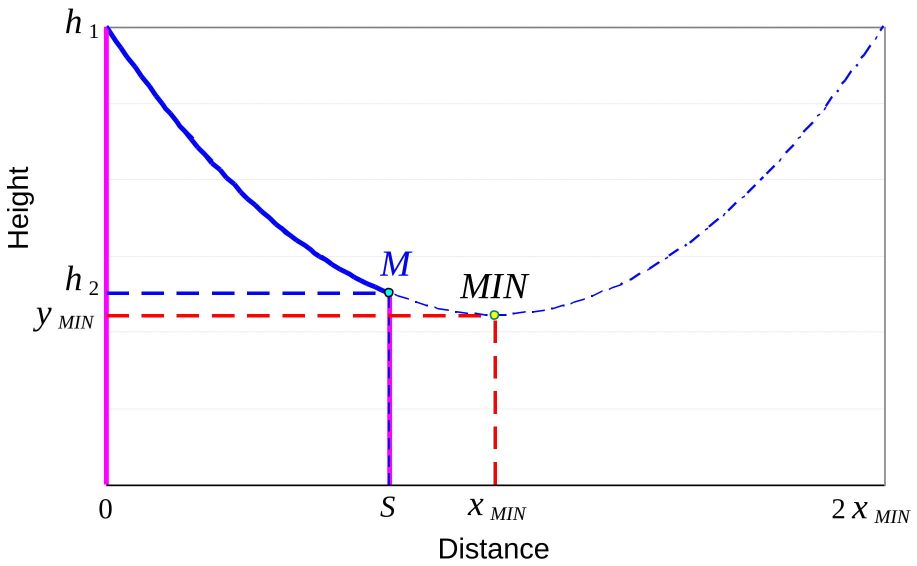


Fig. 2.7: Special inclined span with $x_{MIN} > S$

It is important to emphasize that the coordinates of the vertex point defined by expressions (2.29) and (2.32) are usable in all possible cases, i.e. in both special and classical inclined spans, but also in level ones. This proves the universality of the mathematical approach applied in this work which sets the origin of the coordinate system on the line of the left–hand side support.

The common characteristic of the introduced special inclined spans is the following: when the vertex point is located out of the span, the location of the lowest point of the conductor is at the lower support point of the span. The sag value equals zero at that point.

Taking into consideration the above discussed, the inclined spans can be divided into two groups:

- *Classical inclined spans* where the vertex point and the low point have the same location
- *Special inclined spans* where the vertex point and the low point differ in their location.

Logically, each of two groups has two types of the span, $h_1 < h_2$ and $h_1 > h_2$.

2.6 Equations for Conductor and Sag Curves in a Level Span

In a level span the support points are on the same elevation (h_1=h_2=h). Actually, it is a simplification of an inclined span. In this special case the equations for the conductor curve (2.67) – (2.70) are much simpler than the adequate ones in an inclined span, since the lowest point (2.71) of the conductor is located at a mid–span.

$$y(x) = c \cdot \left(\cosh \frac{x - S/2}{c} - \cosh \frac{S}{2c} \right) + h \qquad x \in [0, S] \tag{2.67}$$

$$y(x)=2c\cdot\left(\sinh^2\frac{x-S/2}{2c}-\sinh^2\frac{S}{4c}\right)+h \qquad x\in[0,S] \tag{2.68}$$

$$y(x)=\frac{c}{2}\cdot\left(e^{\frac{x-S/2}{c}}+e^{-\frac{x-S/2}{c}}-e^{\frac{S}{2c}}-e^{-\frac{S}{2c}}\right)+h \qquad x\in[0,S] \tag{2.69}$$

$$y(x)=\frac{c}{2}\cdot\left[\left(e^{\frac{x-S/2}{2c}}-e^{-\frac{x-S/2}{2c}}\right)^2-\left(e^{\frac{S}{4c}}-e^{-\frac{S}{4c}}\right)^2\right]+h \qquad x\in[0,S] \tag{2.70}$$

$$MIN\left(\frac{S}{2};h-2c\cdot\sinh^2\frac{S}{4c}\right) \tag{2.71}$$

The equation for the conductor sag in a level span is given by expressions (2.72) – (2.75), while the formula for the calculation of the maximum sag is (2.76).

$$D(x)=c\cdot\left(\cosh\frac{S}{2c}-\cosh\frac{x-S/2}{c}\right) \qquad x\in[0,S] \tag{2.72}$$

$$D(x)=2c\cdot\left(\sinh^2\frac{S}{4c}-\sinh^2\frac{x-S/2}{2c}\right) \qquad x\in[0,S] \tag{2.73}$$

$$D(x)=\frac{c}{2}\cdot\left(e^{\frac{S}{2c}}+e^{-\frac{S}{2c}}-e^{\frac{x-S/2}{c}}-e^{-\frac{x-S/2}{c}}\right) \qquad x\in[0,S] \tag{2.74}$$

$$D(x)=\frac{c}{2}\cdot\left[\left(e^{\frac{S}{4c}}-e^{-\frac{S}{4c}}\right)^2-\left(e^{\frac{x-S/2}{2c}}-e^{-\frac{x-S/2}{2c}}\right)^2\right] \qquad x\in[0,S] \tag{2.75}$$

$$D_{\max}=D(S/2)=D(x_{MIN})=2c\cdot\sinh^2\frac{S}{4c} \tag{2.76}$$

2.7 Practical Application of the Derived Equations

This section deals with a practical use of the new equations for the conductor and the sag curves, and also the special formulas for characteristic sags of the catenary. Two numerical examples of high voltage OHL have been shown for this purpose. Inclined spans have been applied in order to find out the direction of the movement of the maximum sag related to the mid–span, i.e. to its location in a level span. This way the result provided analytically in Section 2.4.2 can be proved numerically. Besides, two examples also show how to calculate the conductor height related to some point, which does not lie on the *x*–axis. The input data in the first of the two examples have been taken from [41], but the height of the left–hand side support point is added as an additional datum which is necessary for calculations by the use of the mathematical approach shown in this work. The second example is a *mirror image* of the

first one; together they present the logic and usability of the new method well. The height difference between the support points is significant (200 metres), in one example the left–hand side support point is lower, while in the other example it is higher than the right–hand side support point. In practice the inclined span is a more complicated task, while the level span is considered to be a simple one.
Using the given input data the following equations, coordinates, heights and distances have been determined in both examples:

- Equation for the straight line connecting the support points, $y_{line}(x)$
- Equation for the conductor curve, $y(x)$
- Equation for the sag curve, $D(x)$
- Conductor height related to the given point $K(x_K\,;\,y_K)$
- Location of the maximum sag related to a mid–span
- x–coordinate of the vertex point, x_{MIN}
- y–coordinate of the vertex point, y_{MIN}
- Distance between the vertex point and the right–hand side support point, $S - x_{MIN}$
- Conductor sag at the vertex point, $D(x_{MIN})$
- Location of the maximum sag, x_{C}
- Maximum sag, $D_{\max}$
- Distance $x_{\mathrm{C}} - S/2$.

The results and the graphs of the two examples have been presented side by side in order to show well that the two graphs are mirror images of each other, and also to confirm the usability of the developed new equations for the conductor and the sag curves in different span types.

Example 2.1: Inclined span ($h_1 < h_2$)

Table 2.1: Input data in Example 2.1

S [m]	h_1 [m]	h_2 [m]	c [m]	x_K [m]	y_K [m]
700	50	250	1000	550	105

The equation for the straight line connecting the support points:

$$y_{line}(x) = 0.285714 \cdot x + 50 \qquad x \in [0,700] \tag{2.77}$$

The equation for the conductor curve:

$$y(x) = 2 \cdot 10^3 \cdot \sinh^2 \frac{x - 73.570691}{2 \cdot 10^3} + 47.292456 \qquad x \in [0,700] \tag{2.78}$$

The equation for the sag curve:

$$D(x) = 0.285714 \cdot x - 2 \cdot 10^3 \cdot \sinh^2 \frac{x - 73.570691}{2 \cdot 10^3} + 2.707544 \qquad x \in [0,700] \tag{2.79}$$

The conductor height related to point K:

$$h_K = y(x_K) - y_K = y(550) - 105 = 57.947963\,\mathrm{m}$$

The location of the maximum sag related to a mid–span: $x_C > S/2$.

Table 2.2: Results of Example 2.1

x_{MIN} [m]	73.570691	y_{MIN} [m]	47.292456
$S - x_{MIN}$ [m]	626.429309	$D(x_{MIN})$ [m]	23.727742
x_C [m]	355.533930	$D_{\max}$ [m]	64.272969
$x_C - S/2$ [m]	5.533930	h_K [m]	57.947963

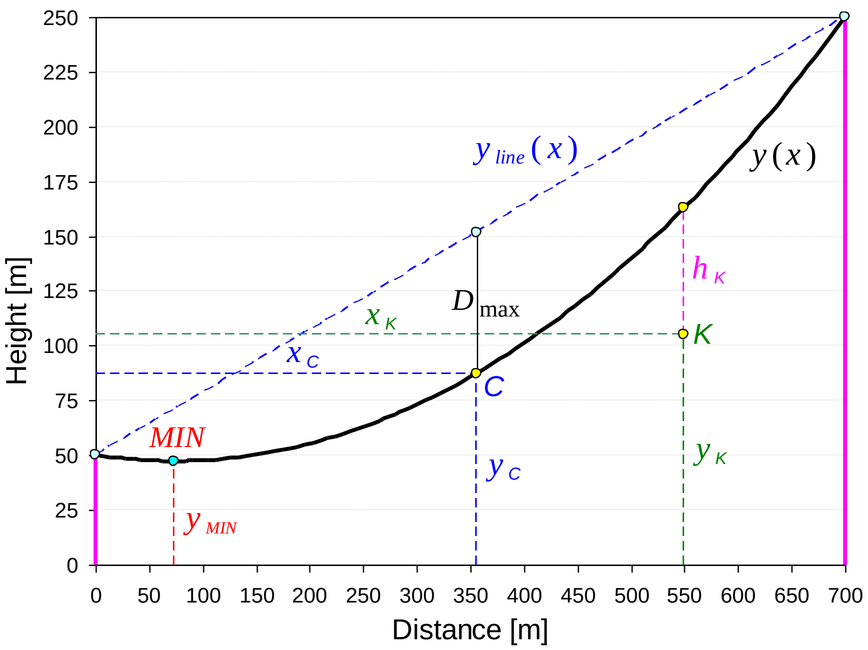


Fig. 2.8: Conductor curve in Example 2.1

Example 2.2: Inclined span ($h_1 > h_2$)

Table 2.3: Input data in Example 2.2

S [m]	h_1 [m]	h_2 [m]	c [m]	x_K [m]	y_K [m]
700	250	50	1000	150	105

The equation for the straight line connecting the support points:

$$y_{line}(x) = -0.285714 \cdot x + 250 \qquad x \in [0,700] \tag{2.80}$$

The equation for the conductor curve:

$$y(x) = 2 \cdot 10^3 \cdot \sinh^2 \frac{x - 626.429309}{2 \cdot 10^3} + 47.292456 \qquad x \in [0,700] \tag{2.81}$$

The equation for the sag curve:

$$D(x) = -0.285714 \cdot x - 2 \cdot 10^3 \cdot \sinh^2 \frac{x - 626.429309}{2 \cdot 10^3} + 202.707544 \qquad x \in [0,700] \tag{2.82}$$

The conductor height related to point K:

$$h_K = y(x_K) - y_K = y(150) - 105 = 57.947963\,\mathrm{m}$$

The location of the maximum sag related to a mid–span: $x_C < S/2$.

Table 2.4: Results of Example 2.2

x_{MIN} [m]	626.429309	y_{MIN} [m]	47.292456
$S - x_{MIN}$ [m]	73.570691	$D(x_{MIN})$ [m]	23.727742
x_C [m]	344.466100	$D_{\max}$ [m]	64.272969
$x_C - S/2$ [m]	–5.533930	h_K [m]	57.947963

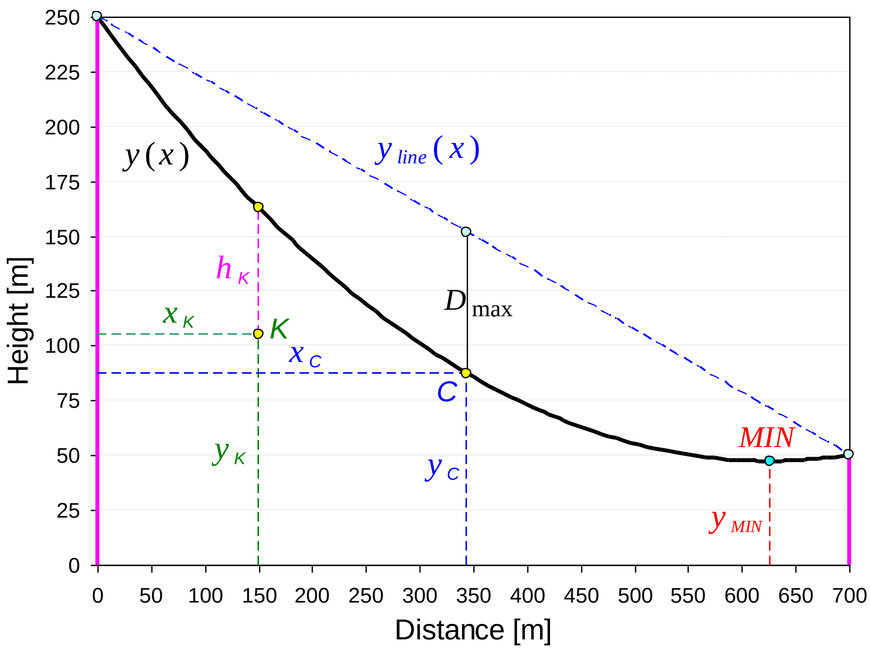


Fig. 2.9: Conductor curve in Example 2.2

Considering the above results the maximum sag in inclined span is located near the mid–span and is closer to the higher support point. This way the analytically provided conclusion from Section 2.4.2 has been confirmed numerically here. $|x_C - S/2| = 5.53$ metres in both examples.

2.8 Analysis of the Catenary on Rising Span Inclination

Special features of the catenary conductor curve and also some differences in comparison to the parabolic conductor curve can be provided by analysing the catenary when the span inclination increases, while the span length remains unchanged. The span inclination changes with $\Delta h = h_2 - h_1$. In order to perform the mentioned analysis, a practical numerical example is shown with one level span and two inclined ones. The catenary parameter, c, is common datum in all spans. The height of the left–hand side support point, h_1, remains unchanged, but the height of the right–hand side support point, h_2, is different in each case (see Table 2.5). This way the same catenary is presented with different span inclinations (i.e. with different Δh). In other words, all three conductor curves are parts of the same catenary curve.

Example 2.3: One level span and two inclined spans

Table 2.5: Input data in Example 2.3

Input data	Case 1 $h_2 = h_1$	Case 2 $h_2 = 2h_1$	Case 3 $h_2 = 3h_1$
S [m]	700	700	700
h_1 [m]	100	100	100
h_2 [m]	100	200	300
c [m]	1000	1000	1000

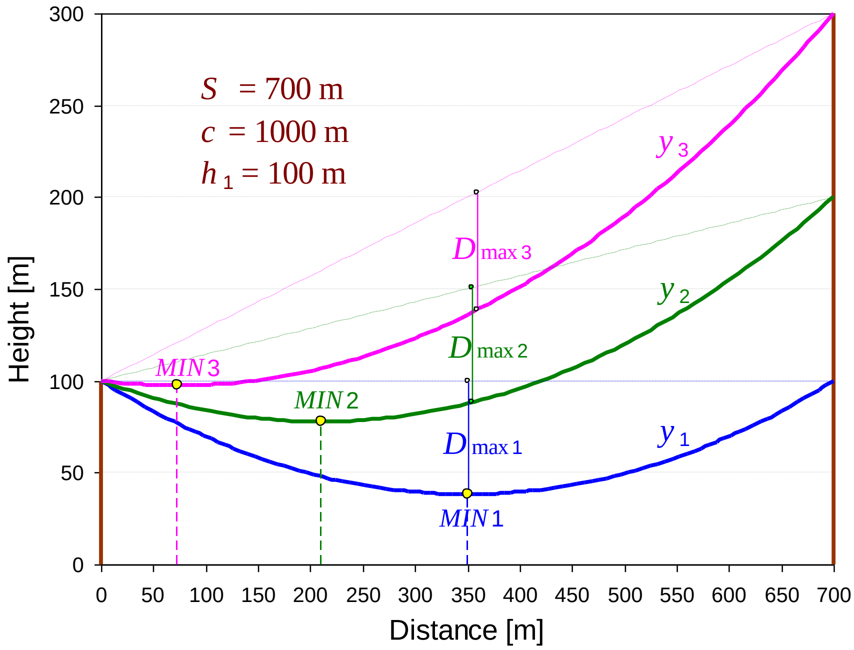


Fig. 2.10: Conductor curves in Example 2.3

Table 2.6: Calculated maximum sags and their locations in Example 2.3

Results	Case 1	Case 2	Case 3
x_C [m]	350	352.84718	355.53393
D_{max} [m]	61.87782	62.48522	64.27297

Based on Fig. 2.10, it is evident that while the *MIN* point moves toward the lower support point, the maximum sag moves toward the higher one when the span inclination (or $\Delta h = h_2 - h_1$) increases. It is also seen from Table 2.6 that the maximum sag value is higher in an inclined span than in a level one, and grows on rising Δh. This is a significant difference in comparison to the parabola because its maximum sag does not vary when Δh changes. For this reason $1/\cos\psi$ multiplier is used to increase the parabola sag in an inclined span. It is discussed in details in Chapter 4, while the use of $1/\cos\psi$ in the case of the catenary is examined and evaluated in Chapter 3.

Further conclusions in connection with the catenary sag can be drawn by using the sag equations (2.83), (2.84) and (2.85). These are obtained by (2.73) and (2.37), and concern to the three conductor curves shown in Fig. 2.10.

$$D_1(x) = -2\cdot 10^3 \cdot \sinh^2 \frac{x-350}{2\cdot 10^3} + 61.877819 \qquad x \in [0,700] \tag{2.83}$$

$$D_2(x) = 0.142857 \cdot x - 2\cdot 10^3 \cdot \sinh^2 \frac{x-210.471540}{2\cdot 10^3} + 22.231019 \qquad x \in [0,700] \tag{2.84}$$

$$D_3(x) = 0.285714 \cdot x - 2\cdot 10^3 \cdot \sinh^2 \frac{x - 73.570691}{2\cdot 10^3} + 2.707544 \qquad x \in [0,700] \qquad (2.85)$$

The following tables are created applying the previous equations and are constructed for making a comparison of $D(x)$ and $D(S-x)$ when $0 \le x \le S/2$, in three actual cases with different Δh. In fact the results present the sags on the same distance measured in the positive direction from the left–hand side support and also in the negative direction from the right–hand side support. The tables have been structured in a bit unusual way in order to compare the obtained results easily.

Table 2.7: $D_1(x)$ and $D_1(S-x)$

x [m]	$D_1(x)$ [m]	$S-x$ [m]	$D_1(S-x)$ [m]
0	0	700	0
50	16.539	650	16.539
100	30.465	600	30.465
150	41.811	550	41.811
200	50.607	500	50.607
250	56.874	450	56.874
300	60.628	400	60.628
350	61.878	350	61.878

Table 2.8: $D_2(x)$ and $D_2(S-x)$

x [m]	$D_2(x)$ [m]	$S-x$ [m]	$D_2(S-x)$ [m]
0	0	700	0
50	16.471	650	16.930
100	30.409	600	31.115
150	41.831	550	42.607
200	50.748	500	51.453
250	57.164	450	57.692
300	61.078	400	61.360
350	62.481	350	62.481

Table 2.9: $D_3(x)$ and $D_3(S-x)$

x [m]	$D_3(x)$ [m]	$S-x$ [m]	$D_3(S-x)$ [m]
0	0	700	0
50	16.715	650	17.635
100	30.930	600	32.343
150	42.643	550	44.195
200	51.848	500	53.258
250	58.532	450	59.589
300	62.677	400	63.240
350	64.257	350	64.257

Analysing the previous results, it is obvious that identity $D(x) = D(S-x)$ when $0 \le x \le S/2$ is valid only in the first case of the three ones shown above, but in other two cases it is not, i.e. $D(x) \ne D(S-x)$ when $0 < x < S/2$. The first case is a level span, while the others are inclined ones. This way the following very important feature of the catenary is identified: the sag function of its curve replaced from interval [0, *S*] to [–*S*/2, *S*/2] is an *even* function in the case of a level span, but in an inclined span it is neither an *even* nor an *odd* function. It is expressed mathematically as follows:

$$D_{lev}^{(\text{cat})}\left(-x+\frac{S}{2}\right)=D_{lev}^{(\text{cat})}\left(x+\frac{S}{2}\right) \quad \Rightarrow \quad D_{lev}^{(\text{cat})}\left(x+\frac{S}{2}\right) \ \underline{\text{is an } \textit{even} \text{ function}} \tag{2.86}$$

$$D_{inc}^{(\text{cat})}\left(-x+\frac{S}{2}\right)\neq D_{inc}^{(\text{cat})}\left(x+\frac{S}{2}\right) \quad \Rightarrow \quad D_{inc}^{(\text{cat})}\left(x+\frac{S}{2}\right) \ \underline{\text{is not an } \textit{even} \text{ function}} \tag{2.87}$$

Furthermore, due to (2.88) the latter <u>is not an *odd* function</u> either.

$$D_{inc}^{(\text{cat})}\left(-x+\frac{S}{2}\right)\neq -D_{inc}^{(\text{cat})}\left(x+\frac{S}{2}\right) \tag{2.88}$$

Sag curve *D*(*x*) from Fig. 2.4 and its replacement to interval [–*S*/2, *S*/2] are shown in Fig. 2.11. The replaced curve is denoted by *D*(*x*+*S*/2).

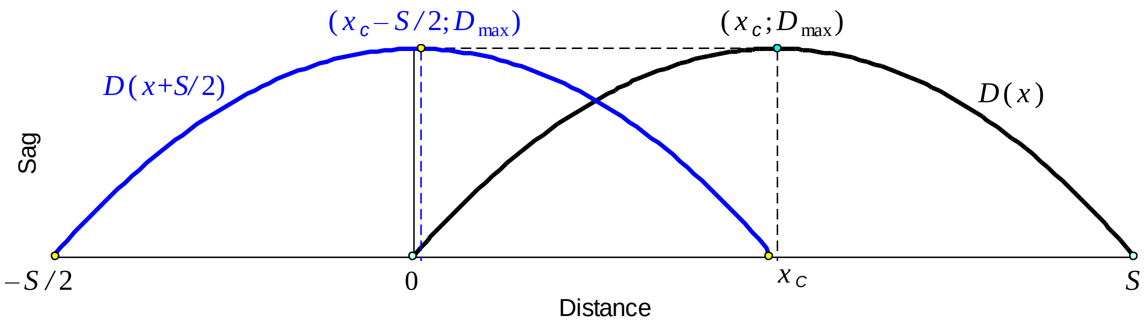


Fig. 2.11: Curves *D*(*x*) and *D*(*x*+*S*/2)

Taking into consideration the above discussed, it can be concluded that the sag curve in a level span has the exact shape of an inverted catenary, while in an inclined span it is very similar to an inverted catenary. When the span inclination (or the difference in elevation of the support points, Δh) increases, then the sag curve differs better from an inverted catenary. The difference between the sag curves in inclined and level spans is not big enough to be seen well on a common diagram, but it is also not negligible from the aspect of OHL design, especially when Δh is significant. The comparison of the sag curves in inclined and level spans is discussed in details in Chapter 3.

2.9 Summary of the Chapter

Chapter 2 widens the catenary based calculation using a given catenary parameter, *c*, as one of the input data besides the span length, *S*, and the heights of the support points related to *x*–axis, h_1 and h_2. The conductor curve is presented in the coordinate system in such a way which is much more practical than the way that it is generally used in literature. Setting the origin on the line of the left–hand side support instead of the catenary's vertex point is more natural for OHL designers since the distances in a span are usually measured from the left–hand side support, when planning OHL, but not from the vertex point. Due to the new approach, a universal equation for the conductor curve has been derived, which efficiently covers all types of the spans ensuring the determination of the conductor height and also drawing the conductor curve easily, with no limitations caused by the span inclination.

A universal sag equation derived by applying the new equation for the conductor curve is usable not only for determining the maximum sag, but also the sags at the low point and the mid–span, or any other point of the span. On the basis of the provided special sag formulas, the mentioned three characteristic sags of the catenary are different in an inclined span, but they are all equal in a level span.

The new approach introduced in this chapter also gives an opportunity for a qualitative analysing of the catenary conductor curve on rising span inclination (or difference in the support points elevation, Δh). This way some specific differences between the catenary and the parabola, important from the aspect of OHL design, can be adequately discussed.

The practical applications of the new results are presented widely in Section 2.7 and partly in Section 2.8. The catenary based calculation ensures an exact determination of the conductor height and the sag, avoiding errors caused by the approximation of the catenary by a parabola.

3 INCLINED SPAN MODELLING BY A GIVEN LEVEL SPAN

3.1 Introduction

As the support points in a span can be located on the same or different elevation, all spans are divided into two basic groups, level [58–60] and inclined spans. In some literature the inclined spans are called *non level* spans [58] or *sloping* spans [47]. Taking into consideration that the supports (or towers) used for constructing OHL usually have the same height, level spans are commonly present in a flat terrain (see in Fig. 3.1), while inclined spans can be found in a hilly terrain. However, inclined spans also occur in a flat terrain when two adjacent towers have different heights. The latter case is shown in Fig. 3.2, where the height difference between the support points is visibly significant. Considering the elevation of the support points, the basic difference between level and inclined spans is well seen in these two figures. An example of an inclined span in a hilly terrain is shown in Fig. 3.3.

Fig. 3.1: Level spans in a flat terrain

Figure 3.2: Inclined span in a flat terrain

Fig. 3.3: Inclined span in a hilly terrain [61]

The equations for the conductor curve and the sag in a level span can be obtained if the following input data are known: span length, S, catenary constant, c and height of the support points, h_1. Besides the freely chosen vertical distance (Δh) between the support points, the listed data are sufficient for determining the equations for the conductor curve and the sag concerned to a formed inclined span. It is shown by using the new method called *inclined span modelling by a given levelled span* (or shortly *inclined span modelling*). *A given level span* here means that S, c and h_1 data are given. While modelling, these data have to remain unchanged. The conductor curve is considered here as a catenary and the following conditions have to be fulfilled:

- Equal catenary parameter, c, in both level and inclined spans, i.e. $c_{lev} = c_{inc}$,
- Equal span length, S, in both level and inclined spans, i.e. $S_{lev} = S_{inc}$.

These conditions practically mean that the conductor curves in level and inclined spans are in fact different parts of the same catenary, but the span length is equal in these two cases.

Because of the specificity of the actual theme, the following symbols are used in this chapter:

$y_{lev}(x)$ – equation for the conductor curve in a level span

$y_{inc}(x)$ – equation for the conductor curve in an inclined span

$y_{line\ lev}(x)$ – equation for the straight line connecting the support points in a level span

$y_{line\ inc}(x)$ – equation for the straight line connecting the support points in an inclined span

$D_{lev}(x)$ – sag equation in a level span

$D_{inc}(x)$ – sag equation in an inclined span

$D_{lev}(S/2)$ – mid–span sag in a level span

$D_{inc}(S/2)$ – mid–span sag in an inclined span

$D_{lev\,\max}$ – maximum sag in a level span

$D_{inc\,\max}$ – maximum sag in an inclined span

$\Delta D(x)$ – equation for the difference between the sags in inclined and level spans

$\Delta D(S/2)$ – difference between the mid–span sags in inclined and level spans

$\Delta D_{\max}$ – difference between the maximum sags in inclined and level spans.

The structure of this chapter is as follows. After a short overview of related research, which is given in Section 3.2, the *inclined span modelling* method is presented in details in Section 3.3. Deriving the difference between $D_{inc}(x)$ and $D_{lev}(x)$, its analysis and practical application are all shown in Section 3.4 as well as the examination of the use of $1/\cos\psi$ multiplier. Section 3.5 deals with the analysis of the existing formula for the maximum sag available in some earlier literature, then the relation between the maximum sags of the catenary in inclined and level spans is derived. The latter section also shows a practical application of the new relation. Section 3.6 gives a short conclusion and a summary of the novel results.

3.2 Related Research

Since the main application of the *inclined span modelling* method presented in this chapter is derivation of the relation between the conductor sags in inclined and level spans, therefore the related literature is which deals with this relation.

Regarding to the conductor sag in two span types the publications [62–65] contain the following or the corresponding statement:

The midpoint sag in inclined span is approximately equal to the sag in a horizontal span equal in length to the inclined span.

The horizontal span means here a level span, while the midpoint is a mid–span point. However, the above statement is not an exact relation, but an approximate one which is concerned to a mid–span sag only.

Another relation between the sags in inclined and level spans can be found inter alia in [41,42,44,45,66] publications which state that the mid–span sag in a level span multiplied by the reciprocal of the cosine of the chord’s inclination is equal to the mid–span sag in an inclined span. Thus, this relation is also regarded exclusively to a mid–span sag.

According to paper [67] the sag in an inclined span is approximately equal to the sag of a level span of the same length multiplied by the secant of the inclination of the chord. The author provides an expression which relates to a mid–span sag and mentions that the proposed expression is an approximate one. Considering the fact that the secant is the reciprocal of the cosine function, the actual statement in fact does not differ from the previous one which does not use the secant function, but the reciprocal of the cosine one.

The author of the publication [41] widens the previous statement and alleges that $1/\cos\psi$ multiplier used for the determination of the mid–span sag in an inclined span from the given mid–span sag in a level span can be applied not only at a mid–span point but at any other point of the span. In other words, this means that the sag equation in a level span multiplied by $1/\cos\psi$ (or $\sec\psi$) gives the sag equation in an inclined span.

Summarising the all above mentioned, it is evident that the relation between the sags in two span types in some publications is considered as an exact relation, but in some others as an approximate one. Furthermore, in some papers it regards only to mid–span sag, but in some others to a sag at any point of the span. The adequate mathematical background in connection with the use of $1/\cos\psi$ multiplier is not available in literature, thus the question of the real relation is still open. That is the reason why this chapter discusses this topic in details providing an exact mathematical relation and containing the examination and evaluation of the mentioned multiplier's application.

The expression for computing the maximum sag of the catenary in an inclined span, given in [41,68] has to be mentioned as well. Having analysed the proposed expression, it can be concluded that in fact it is the product of $1/\cos\psi$ and the maximum sag of the catenary in a level span. Thus, the actual expression can be considered as the relation between the maximum sags in inclined and level spans. It is appropriately discussed in Section 3.5 and also the exact mathematical relation is derived.

3.3 Description of the Inclined Span Modelling Method

In order to explain adequately the method for modelling an inclined span, four curves with common S and c for each one are drawn in Fig. 3.4. The angle of the span inclination is marked as ψ.

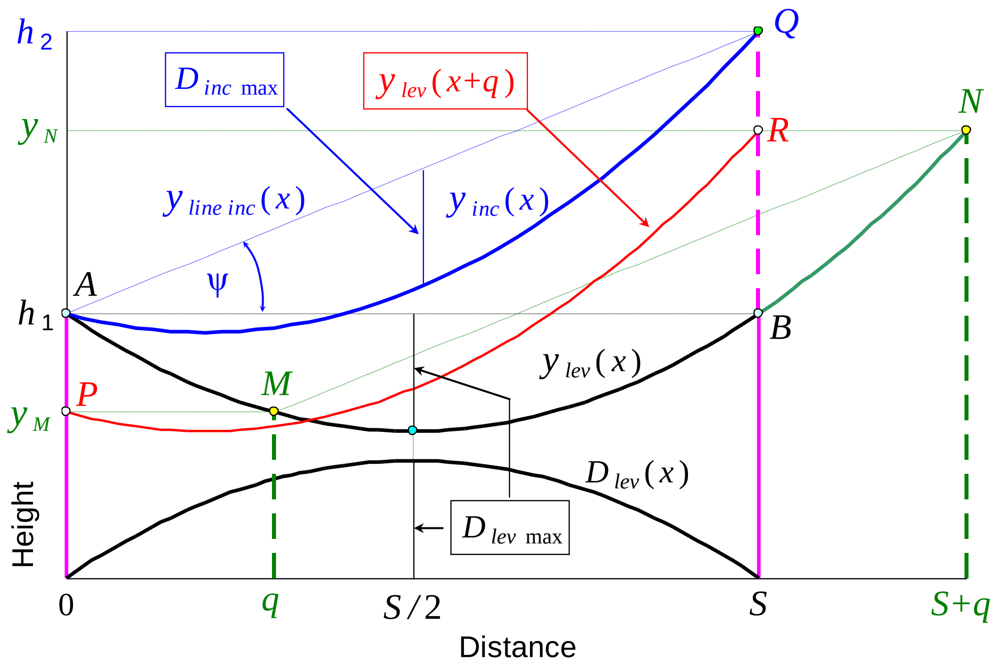


Fig. 3.4: Curves for explaining the *inclined span modelling by a given level span*

The initial curve is the one in a level span, drawn from point $A(0;h_1)$ to point $B(S;h_1)$. The equation of this curve is given by (3.1) or (3.2) and is defined on the interval $[0,S]$.

$$y_{lev}(x) = c \cdot \cosh\frac{x - S/2}{c} - c \cdot \cosh\frac{S}{2c} + h_1 \tag{3.1}$$

$$y_{lev}(x) = c \cdot \cosh\frac{x - S/2}{c} - c + h_1 - D_{lev\,\max} \tag{3.2}$$

Using the previous equations, two versions of the sag equation regarded to a level span can be obtained in the following way:

$$D_{lev}(x) = h_1 - y_{lev}(x) = c \cdot \cosh\frac{S}{2c} - c \cdot \cosh\frac{x - S/2}{c} \tag{3.3}$$

$$D_{lev}(x) = D_{lev\,\max} - c \cdot \cosh\frac{x - S/2}{c} + c \tag{3.4}$$

Each of four preceding equations is a function of x, so it can be applied at any point of the span.

Extending the curve of $y_{lev}(x)$ on the interval $(S,S+q]$ and omitting its part on the interval $[0,q)$ creates the conductor curve in an inclined span drawn from point $M(q;y_M)$ to point $N(S+q;y_N)$, where $0 < q < S$, $y_M = y_{lev}(q)$ and $y_N = y_{lev}(S+q)$. This way the created curve has the same equation as the initial one does, but it is defined on the interval $[q,S+q]$:

$$(AB)_{\text{curve}} = y_{lev}(x) \qquad x \in [0,S] \tag{3.5}$$

$$(MN)_{\text{curve}} = y_{lev}(x) \qquad x \in [q, S+q] \tag{3.6}$$

Note that both curves, *AB* and *MN*, are the parts of the same catenary. Thus, the catenary parameter is equal in two cases. Furthermore, the span length remains unchanged.

The following step is the displacement of the curve *MN* so that point *M* is set at point *A*. To reach that, the actual curve has to be translated horizontally q units to the left, and vertically $h_1 - y_M$ units upward. The horizontal translation produces curve *PR* and then the performed vertical translation gives the final curve *AQ*:

$$(PR)_{\text{curve}} = y_{lev}(x+q) \qquad x \in [0, S] \tag{3.7}$$

$$(AQ)_{\text{curve}} = y_{lev}(x+q) + h_1 - y_{lev}(q) \qquad x \in [0, S] \tag{3.8}$$

Both translations are made by an appropriate use of the equation concerned to a level span, $y_{lev}(x)$. The equation of the final curve drawn from point *A* to point *Q* is marked below as $y_{inc}(x)$, and is defined on the interval $[0,S]$. Considering (3.9) it is expressed by (3.10).

$$h_1 - y_{lev}(q) = c \cdot \cosh \frac{S}{2c} - c \cdot \cosh \frac{q - S/2}{c} \tag{3.9}$$

$$y_{inc}(x) = c \cdot \cosh \frac{x - S/2 + q}{c} + h_1 - c \cdot \cosh \frac{q - S/2}{c} \tag{3.10}$$

The rearrange of (3.10) yields (3.11). (The complete deduction is given in Appendix 1.)

$$y_{inc}(x) = 2c \cdot \sinh \frac{x}{2c} \cdot \sinh\left(\frac{x-S}{2c} + \frac{q}{c}\right) + h_1 \tag{3.11}$$

The next step is the determination of $q=q(S,c,h_1,h_2)$, i.e. expressing q by S, c, h_1 and h_2. The quotient q/c can be obtained by using identity (3.12), see deduction in Appendix 2.

$$\Delta h = h_2 - h_1 = y_N - y_M = y_{lev}(S+q) - y_{lev}(q) \tag{3.12}$$

$$\frac{q}{c} = \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} \tag{3.13}$$

Substituting (3.13) into (3.11) gives the final equation (3.14) for the conductor curve in a modelled inclined span, expressed by the given data (S, c, h_1) for a level span and h_2 obtained by $h_2 = h_1 + \Delta h$, where Δh is freely chosen.

$$y_{inc}(x) = 2c \cdot \sinh \frac{x}{2c} \cdot \sinh\left(\frac{x-S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}}\right) + h_1 \tag{3.14}$$

Since h_1 is fixed, then considering (3.12), either h_2 or Δh can be freely chosen for an inclined span. In other words, the span inclination is defined by the choice of either h_2 or Δh in our case. Let us mention that h_1 and h_2 data are considered as the heights of the support points related to x–axis, but not as the tower heights.

Using (3.14), the sag equation concerned to an inclined span, $D_{inc}(x)$, can be obtained as shown below:

$$D_{inc}(x) = \frac{h_2 - h_1}{S} x + h_1 - y_{inc}(x) \tag{3.15}$$

$$D_{inc}(x) = \frac{h_2 - h_1}{S} x - 2c \cdot \sinh \frac{x}{2c} \cdot \sinh\left(\frac{x - S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} \right) \tag{3.16}$$

Having obtained (3.14) and (3.16), the inclined span modelling is finished. Both $y_{inc}(x)$ and $D_{inc}(x)$ are defined on the interval $[0,S]$.

As a particular form of the wide usability of the *inclined span modelling* method, the further sections of this chapter also show how it can be applied for derivation of the following unique relations:

- Relation between the catenary sags in inclined and level spans,
- Relation between the maximum sags of the catenary in inclined and level spans.

3.4 Difference between $D_{inc}(x)$ and $D_{lev}(x)$

It is well known that the sag–tension calculation in inclined spans is more difficult than it is in level spans. Nowadays the sag–tension tables [7] are available for many different types of conductors, and contain the mid–span sags in level spans in dependence of the temperature. It causes a problem that the sag in an inclined span differs from the sag in a level span. This statement concerns not only to a mid–span sag but to the sag at any point of the span excluding the start and end points, i.e. $D_{inc}(x) \neq D_{lev}(x)$, $0 < x < S$. That is why the available sag–tension tables cannot be directly applied in inclined spans. Thus, it would be very useful to have an exact relation between the sags in inclined and level spans in order to be able to calculate the first from the latter given one. This opportunity may be reached when the difference between $D_{inc}(x)$ and $D_{lev}(x)$ is known. Derivation, analysis and practical application of the mentioned difference are the main goals of this section.

3.4.1 Deriving the Equation $\Delta D(x) = D_{inc}(x) - D_{lev}(x)$

Since the sag equations in level and inclined spans have been determined in the previous section, these can be used now to define their difference, denoted by $\Delta D(x)$. Based on (3.17) and considering (3.16) and (3.3), it is expressed by (3.18). (See deduction in Appendix 3.)

$$\Delta D(x) = D_{inc}(x) - D_{lev}(x) \tag{3.17}$$

$$\Delta D(x) = \frac{h_2 - h_1}{S} x - 2c \cdot \sinh\frac{x}{2c} \cdot \left(\sinh\left(\frac{x-S}{2c} + \operatorname{arcsinh}\frac{h_2 - h_1}{2c \cdot \sinh\frac{S}{2c}} \right) + \sinh\frac{S-x}{2c} \right) \tag{3.18}$$

Applying identity for hyperbolic sine function given by (3.19) [69], then considering (3.20), the previous equation changes into (3.21).

$$\sinh(x) + \sinh(y) = 2\sinh\frac{x+y}{2} \cdot \cosh\frac{x-y}{2} \tag{3.19}$$

$$\cosh(-x) = \cosh(x) \tag{3.20}$$

$$\Delta D(x) = \frac{h_2 - h_1}{S} x - 4c \cdot \sinh\frac{x}{2c} \cdot \sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{h_2 - h_1}{2c \cdot \sinh\frac{S}{2c}} \right) \cdot \cosh\left(\frac{S-x}{2c} - \frac{1}{2}\operatorname{arcsinh}\frac{h_2 - h_1}{2c \cdot \sinh\frac{S}{2c}} \right) \tag{3.21}$$

Using (3.21) it is possible to compute the difference between the catenary sags in inclined and level spans at any point of the span, when the catenary parameter is equal in both spans, as well as the span length.

Let us mention that if (3.17) is transformed into (3.22) then $D_{inc}(x)$ can be obtained from the given $D_{lev}(x)$ by computing $\Delta D(x)$ with the use of (3.21).

$$D_{inc}(x) = D_{lev}(x) + \Delta D(x) \tag{3.22}$$

The previous expression can be considered as the relation between the catenary sags in inclined and level spans. As each part in (3.22) is the function of x, it can be applied at any point of the span, not only at a mid–span. This justifies the completeness of the shown relation.

The special version of (3.22) concerns to a mid–span, $x=S/2$. According to (3.23) it is given by (3.24). Notice that $D_{lev}(S/2)=D_{lev\,\max}$, but $D_{inc}(S/2)\neq D_{inc\,\max}$, and so $\Delta D(S/2)\neq(\Delta D)_{\max}$. That is why the actual symbols are used in (3.23), but not $D_{lev\,\max}$, $D_{inc\,\max}$ and $(\Delta D)_{\max}$.

$$\Delta D(S/2) = D_{inc}(S/2) - D_{lev}(S/2) \tag{3.23}$$

$$\Delta D(S/2)=\frac{h_2-h_1}{2}-4c\cdot\sinh\frac{S}{4c}\cdot\sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\cdot\cosh\left(\frac{S}{4c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right) \tag{3.24}$$

Expression (3.23) can be simply transformed for calculating $D_{lev}(S/2)$ from the given $D_{inc}(S/2)$, and also (3.22) can be transformed for calculating $D_{lev}(x)$ from the given $D_{inc}(x)$, but these are infrequent tasks in practice. The conditions listed above (c_{lev} = c_{inc} and S_{lev} = S_{inc}) have to be fulfilled in this case as well.

3.4.2 Analysis of $\Delta D(x)$

This section deals with the analysis of $\Delta D(x)$. Drawing the graph of $\Delta D(x)$ when the span inclination (or Δh) changes, it shows how the sag difference, its minimum and maximum vary. It is presented in a following numerical Examples 3.1 and 3.2. The first example has one conductor curve (y_1) in a level span and four other ones (y_2, y_3, y_4, y_5) in inclined spans. The height of the left–hand side support point is identical in each of the five spans, but the height of the right–hand side support point is higher in each following span than in the previous one, as the span inclination increases. The span length and the catenary parameter remain unchanged in all spans. The curve of $\Delta D(x)$ has been drawn in four different cases: $\Delta D_1(x)=D_2(x)–D_1(x)$,….., $\Delta D_4(x)=D_5(x)–D_1(x)$. As it can be seen in Fig. 3.6, $\Delta D(x)$ has three roots (0, r and S) within the span when the span inclination is low, and $\Delta D(x)$ changes sign. As the span inclination increases, the root r moves toward the next nearer root, afterwards it does not exist. Then $\Delta D(x)$ does not change sign any more and $D_{inc}(x)>D_{lev}(x)$. It can be observed from all curves in Fig. 3.6 that the value of $|(\Delta D)_{\min}|$, if it exists on the interval (0,S), is not significant at all, while $(\Delta D)_{\max}$ can be significant in the case of high inclination.

Example 3.2 is deliberately chosen as a *mirror image* of the Example 3.1 in order to demonstrate that the developed method is also usable in the opposite case, i.e. when the height of the right–hand side support point is fixed in each span, but the height of the left–hand side one increases. Four curves of $\Delta D(x)$ drawn in Example 3.2 are the following: $\Delta D_{\mathrm{I}}(x)=D_{\mathrm{II}}(x)–D_{\mathrm{I}}(x)$,….., $\Delta D_{\mathrm{IV}}(x)=D_{\mathrm{V}}(x)–D_{\mathrm{I}}(x)$. The input data for the level spans in both examples are identical, thus $D_1(x)\equiv D_{\mathrm{I}}(x)$.

Example 3.1

Table 3.1: Input data in Example 3.1

Data	**1**	**2**	**3**	**4**	**5**
S [m]	800	800	800	800	800
h_1 [m]	100	100	100	100	100
h_2 [m]	100	160	220	280	340
c [m]	1200	1200	1200	1200	1200

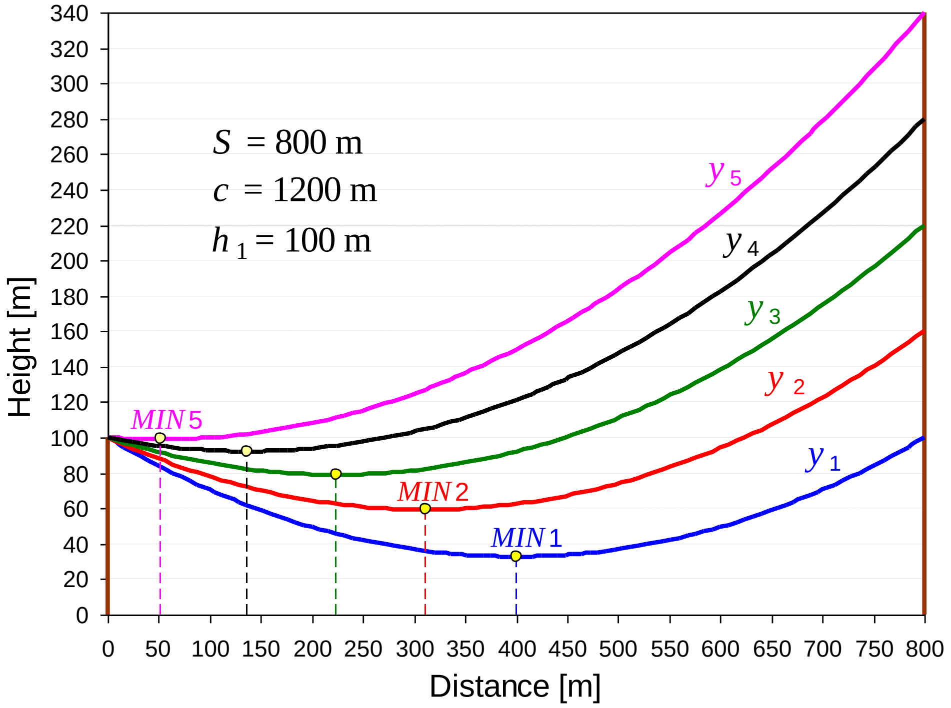


Fig. 3.5: Conductor curves in Example 3.1

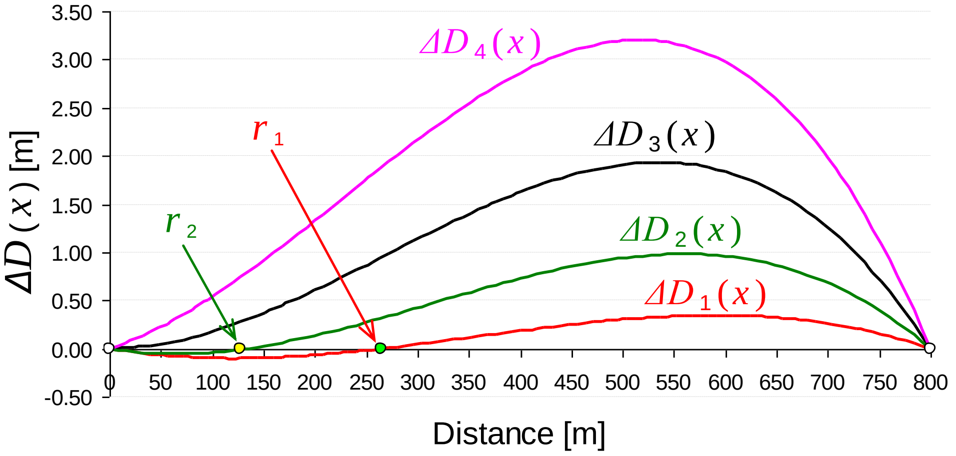


Fig. 3.6: $\Delta D(x)$ curves in Example 3.1

Example 3.2

Table 3.2: Input data in Example 3.2

Data	I	II	III	IV	V
S [m]	800	800	800	800	800
h_1 [m]	100	160	220	280	340
h_2 [m]	100	100	100	100	100
c [m]	1200	1200	1200	1200	1200

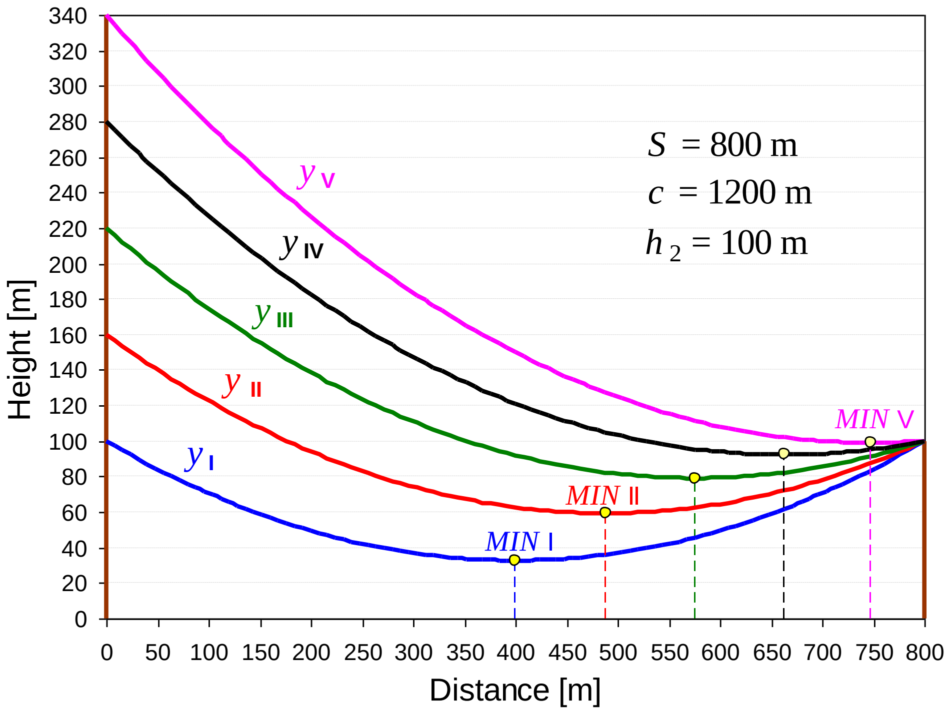


Fig. 3.7: Conductor curves in Example 3.2

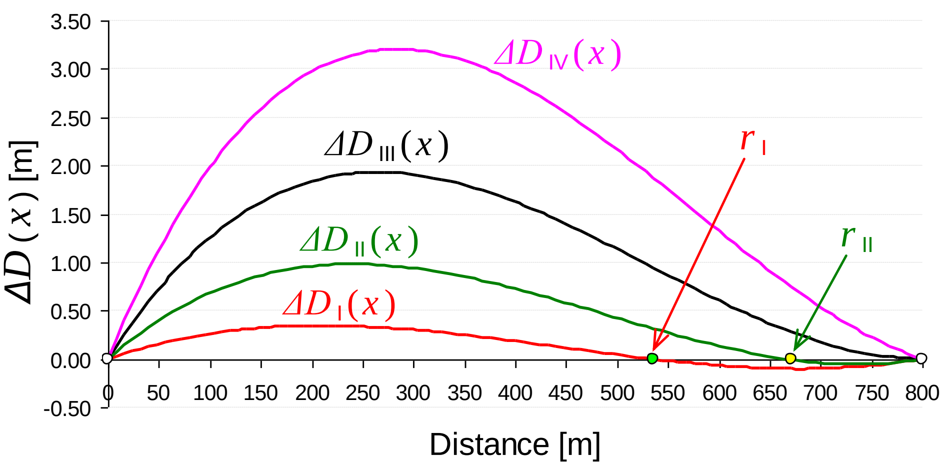


Fig. 3.8: $\Delta D(x)$ curves in Example 3.2

Now some important conclusions can be drawn from $\Delta D(x)$. First, let us mention the following evident identity:

$$D_{lev}(0) = D_{lev}(S) = D_{inc}(0) = D_{inc}(S) = \Delta D(0) = \Delta D(S) = 0 \tag{3.25}$$

Thus, $\Delta D(x)$ always has at least two roots, $x_1=0$ and $x_2=S$, on the interval $[0,S]$, but taking into consideration Figs. 3.6 and 3.8, the third one also exists when the span inclination is not significant. Denoting the third root as r, we can write the following relevant relations:

<u>*Case 1 (three roots):*</u>

If $h_1<h_2$ then:

$$\Delta D(x) < 0 \;\Rightarrow\; D_{inc}(x) < D_{lev}(x) \quad \forall \quad x \in (0,r) \tag{3.26}$$

$$\Delta D(x) > 0 \;\Rightarrow\; D_{inc}(x) > D_{lev}(x) \quad \forall \quad x \in (r,S) \tag{3.27}$$

$$(\Delta D)_{\max} > |(\Delta D)_{\min}| \tag{3.28}$$

If $h_1>h_2$ then:

$$\Delta D(x) > 0 \;\Rightarrow\; D_{inc}(x) > D_{lev}(x) \quad \forall \quad x \in (0,r) \tag{3.29}$$

$$\Delta D(x) < 0 \;\Rightarrow\; D_{inc}(x) < D_{lev}(x) \quad \forall \quad x \in (r,S) \tag{3.30}$$

$$(\Delta D)_{\max} > |(\Delta D)_{\min}| \tag{3.31}$$

<u>*Case 2 (two roots):*</u>

If $h_1<h_2$ or $h_1>h_2$ then:

$$\Delta D(x) > 0 \;\Rightarrow\; D_{inc}(x) > D_{lev}(x) \quad \forall \quad x \in (0,S) \tag{3.32}$$

Returning to *case 1*, it is worth emphasizing that relation $D_{inc}(x)>D_{lev}(x)$ is more dominant than its opposite one. Furthermore, $(\Delta D)_{\max}$ is clearly higher than $|(\Delta D)_{\min}|$, regardless whether $h_1<h_2$ or $h_1>h_2$.

Based on the above discussion, it can be concluded that the quotient of $D_{inc}(x)$ and $D_{lev}(x)$ on the interval $(0,S)$ is not a constant in the case of the catenary. It is expressed by (3.33):

$$\frac{D_{inc}(x)}{D_{lev}(x)} \neq \text{const.} \qquad 0 < x < S \tag{3.33}$$

This way a special feature of the catenary has been revealed. Taking into consideration the adequate relation concerned to a parabola given by (4.34) in Section 4.3.2, a remarkable difference between the parabola and the catenary based calculations has been explored.

3.4.3 Practical Application of $\Delta D(x)$ and the Inclined Span Modelling

This section presents a practical application of $\Delta D(x)$ and also the *inclined span modelling* method through numerical examples, which use the main expressions derived in this chapter. The first one of the two examples deals with the sag computing at a mid–span point, while the other one at any other point of the span. Determining the equations for the conductor and the sag curves in the given level and the modelled inclined spans is shown in the first example of this section.

Example 3.3

Input data are given in Table 3.3. Using them, firstly $y_{lev}(x)$ and $D_{lev}(x)$ are obtained, then also $y_{inc}(x)$ and $D_{inc}(x)$ when the right–hand side support point is elevated with 200 metres (thus h_2=h_1+200m), but S and c data remain unchanged. In the next step, $D_{lev}(S/2)$ and $\Delta D(S/2)$ are calculated, and then added up to get $D_{inc}(S/2)$, i.e. the mid–span sag in a modelled inclined span. The result is checked by using (3.16) when x=S/2. Finally, the two conductor curves and their mid–span sags are drawn on the common diagram. The straight lines connecting the support points in two spans, $y_{line\ lev}(x)$ and $y_{line\ inc}(x)$, are also drawn on the same figure, to make the sag more visible.

Table 3.3: Data for level and inclined spans in Example 3.3

Span	S [m]	h_1 [m]	h_2 [m]	c [m]
Level	700	100	100	1000
Inclined	700	100	300	1000

Solution and calculations:

The use of (3.1), (3.3), (3.14) and (3.16) yields (3.34) – (3.37):

$$y_{lev}(x) = 1000 \cdot \cosh \frac{x-350}{1000} - 961.877819 \tag{3.34}$$

$$D_{lev}(x) = 1061.877819 - 1000 \cdot \cosh \frac{x-350}{1000} \tag{3.35}$$

$$y_{inc}(x) = 2000 \cdot \sinh \frac{x}{2000} \cdot \sinh \left(\frac{x}{2000} - 0.07357 \right) + 100 \tag{3.36}$$

$$D_{inc}(x) = 0.285714 \cdot x - 2000 \cdot \sinh\frac{x}{2000} \cdot \sinh\left(\frac{x}{2000} - 0.07357\right) \tag{3.37}$$

Inserting $x=S/2=350$ m into (3.35) and then the data from Table 3.3 into (3.24) give the following values:

$$D_{lev}(350\text{m}) = 1061.877819 - 1000 \cdot \cosh(0) = 61.878\,\text{m}$$

$$\Delta D(350\text{m}) = \frac{300-100}{700} \cdot 350 - 4 \cdot 1000 \cdot \sinh\frac{350}{2 \cdot 1000} \cdot \sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{300-100}{2 \cdot 1000 \cdot \sinh\frac{700}{2 \cdot 1000}}\right) \cdot$$

$$\cdot \cosh\left(\frac{700-350}{2 \cdot 1000} - \frac{1}{2}\operatorname{arcsinh}\frac{300-100}{2 \cdot 1000 \cdot \sinh\frac{700}{2 \cdot 1000}}\right) = 2.379\,\text{m}$$

The sum of the above two values yields the mid–span sag in a modelled inclined span, which has to be obtained:

$D_{inc}(350\,\text{m}) = 61.878\,\text{m} + 2.379\,\text{m} = 64.257\,\text{m}$

This result can be checked with the help of (3.37) obtained from (3.16). As it gives the same value, the correctness of the calculations has been proved.

$$D_{inc}(350\text{m}) = 0.285714 \cdot 350 - 2000 \cdot \sinh\frac{350}{2000} \cdot \sinh\left(\frac{350}{2000} - 0.07357\right) = 64.257\,\text{m}$$

The results are shown in Fig. 3.9.

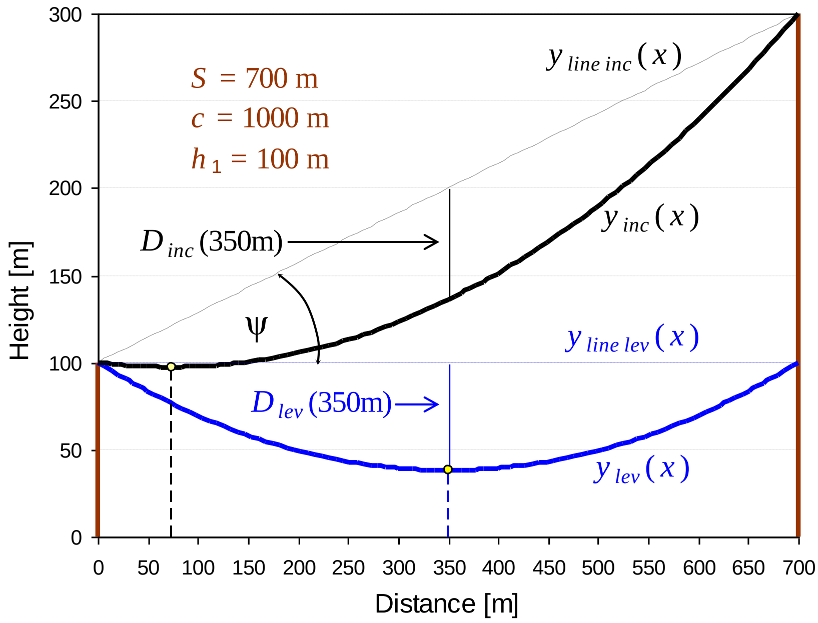


Fig. 3.9: Conductor curves and mid–span sags in level and inclined spans in Example 3.3

Example 3.4

The given data for the level span are the following: S, h, c and D_{lev}(535 m); the latter datum is a sag at x=535 metres, and h_1=h_2=h. The change of the sag is calculated when the right–hand side support point is elevated with 180 metres and data S, c remain unchanged. Then the sag at the actual point of a modelled inclined span is determined. Finally, the conductor curves and the sags at x=535 metres are drawn on the common diagram. The input data are separated in the following tables regarding to a level and an inclined span:

Table 3.4: Known data for a level span in Example 3.4

S [m]	h_1 [m]	h_2 [m]	c [m]	D_{lev}(535 m) [m]
800	100	100	1200	59.68448

Table 3.5: Known and unknown data for an inclined span in Example 3.4

S [m]	h_1 [m]	h_2 [m]	c [m]	D_{inc}(535 m) [m]
800	100	280	1200	?

Calculations:

Using (3.21), ΔD(535 m) can be calculated as follows:

$$\Delta D(535\text{m}) = \frac{280-100}{800}\cdot 535 - 4\cdot 1200\cdot \sinh\frac{535}{2\cdot 1200}\cdot \sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{280-100}{2\cdot 1200\cdot \sinh\dfrac{800}{2\cdot 1200}}\right)\cdot$$

$$\cdot \cosh\left(\frac{800-535}{2\cdot 1200} - \frac{1}{2}\operatorname{arcsinh}\frac{280-100}{2\cdot 1200\cdot \sinh\dfrac{800}{2\cdot 1200}}\right) = 1.93104\,\text{m}$$

So, D_{inc}(535 m) = D_{lev}(535 m) + ΔD(535 m) = 59.68448 m + 1.93104 m = 61.61552 m

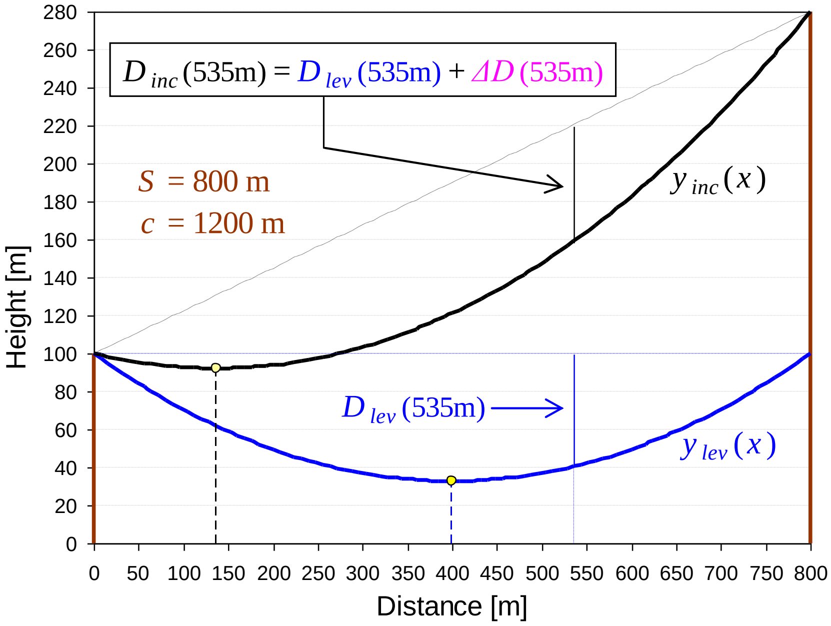


Fig. 3.10: Conductor curves and sags at $x = 535$ metres in Example 3.4

Thus, the sag in an inclined span is calculated from the given sag in a level span using expression for $\Delta D(x)$ applied at the actual point x=535 metres. The sag difference is 1.93104 metres. According to the graph of $\Delta D_3(x)$ curve in Fig. 3.6 (Example 3.1), this value is very close to the maximum of $\Delta D_3(x)$.

3.4.4 Examination of the Use of 1/cosψ for the Catenary Based Calculation

As it is mentioned in Section 3.2 some earlier literature proposes the use of 1/cosψ multiplier for computing the sag in an inclined span by a known sag in a level span. According to [41] the mentioned multiplier can be applied for the sag calculation at any point of the span. It is expressed mathematically by relation (3.38) containing ψ as the angle between the chord and the horizontal line:

$$D_{inc}(x) = \frac{1}{\cos\psi} \cdot D_{lev}(x) \quad \forall \quad 0 \le x \le S \tag{3.38}$$

Defining the span inclination, the angle ψ depends on the vertical distance between the support points, Δh. Taking into consideration (3.33), the quotient of $D_{inc}(x)$ and $D_{lev}(x)$ is not a constant, but according to (3.38) it is, because 1/cosψ=const. Since (3.33) is derived from the exact relation between $D_{inc}(x)$ and $D_{lev}(x)$, it means that (3.38) is an approximate relation and hence it produces errors in sag calculation. The absence of an adequate exact relation usable for a quick targeted sag calculation causes the need of the use of (3.38). In order to examine adequately the usability of (3.38) and thus also 1/cosψ multiplier, the mathematical background is presented with the application of the exact expressions for $D_{inc}(x)$ and $D_{lev}(x)$ from Section 3.3.

The multiplier $1/\cos\psi$ can be obtained by any of the two following expressions:

$$\frac{1}{\cos\psi} = \sqrt{1+\tan^2\psi} = \sqrt{1+\left(\frac{h_2-h_1}{S}\right)^2} \tag{3.39}$$

$$\frac{1}{\cos\psi} = \cosh(\operatorname{arcsinh}(\tan\psi)) = \cosh\left(\operatorname{arcsinh}\left(\frac{h_2-h_1}{S}\right)\right) \tag{3.40}$$

Using expressions (3.3) and (3.16), the equation for the sag error, denoted by $E(x)$, is defined by (3.41).

$$E(x) = D_{inc}(x) - \frac{1}{\cos\psi}\cdot D_{lev}(x) \tag{3.41}$$

Thus, the value of the actual sag error can be determined at any point within the span. Applying (3.39) the previous equation becomes (3.42):

$$E(x) = D_{inc}(x) - \sqrt{1+\left(\frac{h_2-h_1}{S}\right)^2}\cdot D_{lev}(x) \tag{3.42}$$

In order to analyse $E(x)$, five conductor curves are drawn in Fig. 3.11, one in a level span and four others in inclined spans. As it can be seen in Table 3.6, data S, c and h_1 are common ones, while h_2 differs in each case. This way $E(x)$ can be analysed when the span inclination (or Δh) changes. The lowest points of the curves are marked as *MIN* 1, *MIN* 2,…., *MIN* 5.

Table 3.6: Input data for curves in Fig. 3.11

Data	**1**	**2**	**3**	**4**	**5**
S [m]	700	700	700	700	700
h_1 [m]	100	100	100	100	100
h_2 [m]	100	150	200	250	300
c [m]	1000	1000	1000	1000	1000

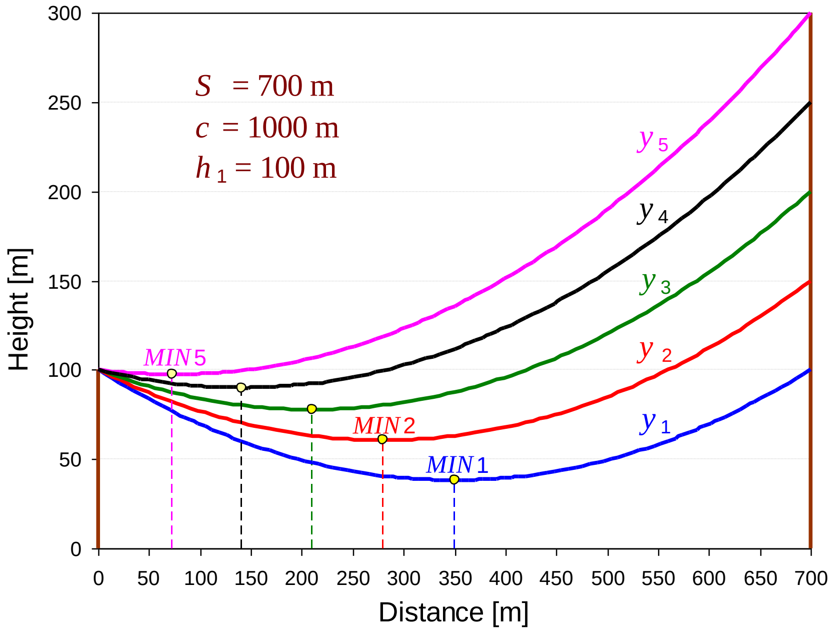


Fig. 3.11: One curve in a level span and four others in inclined spans

According to (3.43), the curve of $E(x)$ has been drawn in four different cases. These are denoted by $E_1(x)$, $E_2(x)$, $E_3(x)$, $E_4(x)$ and shown in Fig. 3.12. The superscripts in (3.43) refer to the adequate curves and data, but note that $h_1^{(i+1)}=h_1^{(1)}$ (i=1,2,3,4) here.

$$E_i(x) = D_{inc}^{(i+1)}(x) - \sqrt{1+\left(\frac{h_2^{(i+1)} - h_1^{(i+1)}}{S}\right)^2} \cdot D_{lev}^{(1)}(x) \qquad i = 1,2,3,4 \tag{3.43}$$

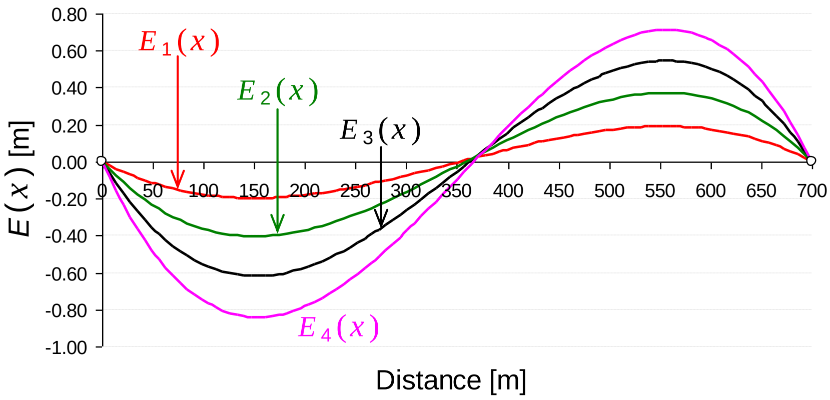


Fig. 3.12: *E*(*x*) curves concerned to different span inclinations

Considering all curves in Fig. 3.12, it can be concluded that the difference between $|E_{\min}|$ and $E_{\max}$ is not significant. The locations of $E_{\min}$ and $E_{\max}$ within the span are very close to $S/4$ and $3S/4$ respectively, when $h_1<h_2$. Furthermore, it is well seen that $E(x)$ has two fixed roots, x_1=0 and x_2=S, and between them also the third one. The latter slightly moves with the span inclination, toward the next nearer root, but it is always very close to the mid–span. If the mid–root is denoted by the symbol z, then:

$$E(0)=E(z)=E(S)=0 \tag{3.44}$$

Since $E(x)$ changes sign within the span, $D_{inc}(x)$ is lower than $(1/\cos\psi)\cdot D_{lev}(x)$ in one part of the span, but in the other one it is higher. It is mathematically described in the following way:

If $h_1<h_2$ then:

$$E(x)<0 \;\Rightarrow\; D_{inc}(x)<\frac{D_{lev}(x)}{\cos\psi} \quad \forall \quad x\in(0,z) \tag{3.45}$$

$$E(x)>0 \;\Rightarrow\; D_{inc}(x)>\frac{D_{lev}(x)}{\cos\psi} \quad \forall \quad x\in(z,S) \tag{3.46}$$

Similarly, it can be shown that the adequate relations concerned to the other type of an inclined span are given as:

If $h_1>h_2$ then:

$$E(x)>0 \;\Rightarrow\; D_{inc}(x)>\frac{D_{lev}(x)}{\cos\psi} \quad \forall \quad x\in(0,z) \tag{3.47}$$

$$E(x)<0 \;\Rightarrow\; D_{inc}(x)<\frac{D_{lev}(x)}{\cos\psi} \quad \forall \quad x\in(z,S) \tag{3.48}$$

Thus, sign of the sag error on the interval $(0,z)$ or (z,S) depends on the type of an inclined span, $h_1<h_2$ or $h_1>h_2$. It is worth mentioning that errors in sag calculation directly produce errors in clearance calculation. Hence, the use of the approximate relation given by (3.38) is unfavourable from more aspects. That is why the application of an exact equation for computing $D_{inc}(x)$ given by (3.16) should be recommended instead of (3.38).

3.5 Review of the Maximum Sag of the Catenary

Whether the conductor curve is considered as the parabola or the catenary, the maximum sag is the most relevant among all characteristic sags (mid–span sag, maximum sag, low point sag) of the conductor curve. This section analyses the existing expression for the maximum sag of the catenary, which is available in some earlier literature, then the new unique expression $D_{inc\,\max}=D_{inc\,\max}(D_{lev\,\max},S,c,\Delta h)$ has also been derived.

3.5.1 Analysis of the Existing Expression for the Maximum Sag

Expression (3.49), which is available in [41,45], regards to the maximum sag in an inclined span. Since its deduction has not been explained in details, it has to be appropriately analysed and discussed.

$$b = \frac{c}{a} \cdot \frac{2\sigma}{\gamma} \sinh^2 \frac{a\gamma}{4\sigma} \tag{3.49}$$

Here b is the maximum sag of the catenary in an inclined span, c presents the length of the straight line connecting the support points, a is the span length and finally the quotient σ/γ defines the catenary parameter. Expressing c by the *Pythagorean Theorem* and using labels from this chapter, (3.49) can be rewritten as:

$$D_{inc\max} = \frac{\sqrt{S^2 + (h_2 - h_1)^2}}{S} \cdot 2c \cdot \sinh^2 \frac{S}{4c} \tag{3.50}$$

The first fraction in the previous formula is in fact $1/\cos\psi$ multiplier from (3.38), while the remaining part of the formula presents the maximum sag of the catenary in a level span, due to (3.51).

$$D_{lev\max} = c \cdot \left(\cosh \frac{S}{2c} - 1 \right) = 2c \cdot \sinh^2 \frac{S}{4c} \tag{3.51}$$

Thus, (3.49) can be expressed in the following simplified form:

$$D_{inc\max} = \frac{1}{\cos\psi} \cdot D_{lev\max} \tag{3.52}$$

This formula can be considered only as an approximate one, since it evidently contains more mathematical inexactness. According to the conclusions from Section 3.4.4, the use of $1/\cos\psi$ for determining the catenary sag in an inclined span from the given sag in a level span does not give exact results. Moreover, since (3.52) is in fact (3.38) regarded to a mid–span, hence $D_{inc}(S/2)$ should stay in (3.52) instead of $D_{inc\ \max}$. It is because the maximum sag of the catenary in an inclined span is slightly moved from the mid–span, while in a level span it is not. Therefore, (3.49) cannot be an exact relation. The errors resulted by the use of (3.49) are not significant in spans with low inclination, but in steep spans they can be significant. However, the application of an exact formula given by (2.57) in Chapter 2 is recommended instead of (3.49) or (3.52).

3.5.2 Relation between $D_{inc\ \max}$ and $D_{lev\ \max}$

Returning to (3.52), in Section 3.5.1 it is considered as an approximate formula for computing the maximum sag of the catenary in an inclined span and then the adequate exact formula has been proposed. Taking into account the structure of (3.52) it can also be considered as relation between the maximum sags of the catenary in inclined and level spans. Due to the presence of $1/\cos\psi$ multiplier the actual relation can be only an approximate one, but not an exact. Since the adequate exact relation is presently not available in literature, it is worth determining it here using the above explained method for modelling an inclined span. This process needs three steps:

- Deriving $D_{inc}(x)$ so that it contains $D_{lev\,\max}$ in the final equation
- Finding the location of $D_{inc\,\max}$, i.e. x_C, by solving the equation $(D_{inc}(x))'=0$
- Inserting x_C into $D_{inc}(x)$ to get $D_{inc\,\max}$.

Based on Fig. 3.4 and the discussion in Section 3.3, $D_{inc}(x)$ can be derived as follows:

$$D_{inc}(x) = y_{lineinc}(x) - y_{inc}(x) = \frac{h_2 - h_1}{S} x + h_1 - y_{inc}(x) \tag{3.53}$$

$$D_{inc}(x) = \frac{h_2 - h_1}{S} x + h_1 - \left(y_{lev}(x+q) + h_1 - y_{lev}(q) \right) \tag{3.54}$$

Taking into consideration (3.55) and (3.56), the final expression for $D_{inc}(x)$ is given by (3.58).

$$y_{lev}(x+q) = c \cdot \cosh \frac{x - S/2 + q}{c} - c + h_1 - D_{lev\,\max} \tag{3.55}$$

$$y_{lev}(q) = c \cdot \cosh \frac{q - S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1 \tag{3.56}$$

$$\begin{aligned} D_{inc}(x) = {} & \frac{h_2 - h_1}{S} x + h_1 - \\ & - \left(\left(c \cdot \cosh \frac{x - S/2 + q}{c} - c + h_1 - D_{lev\,\max} \right) + h_1 - \left(c \cdot \cosh \frac{q - S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1 \right) \right) \end{aligned} \tag{3.57}$$

$$D_{inc}(x) = \frac{h_2 - h_1}{S} x - c \cdot \cosh \frac{x - S/2 + q}{c} + c \cdot \cosh \frac{q - S/2}{c} - c \cdot \cosh \frac{S}{2c} + c + D_{lev\,\max} \tag{3.58}$$

The rearrangement of the previous equation gives (3.59):

$$D_{inc}(x) = D_{lev\,\max} + \frac{h_2 - h_1}{S} x - 2c \cdot \sinh^2 \frac{S}{4c} + c \cdot \cosh \frac{q - S/2}{c} - c \cdot \cosh \frac{x - S/2 + q}{c} \tag{3.59}$$

Thus, the first derivative of $D_{inc}(x)$ is given by (3.60).

$$\frac{d}{dx} D_{inc}(x) = \frac{h_2 - h_1}{S} - \sinh \frac{x - S/2 + q}{c} \tag{3.60}$$

Solving (3.61) yields x_C:

$$\frac{d}{dx} D_{inc}(x) = 0 \quad \Rightarrow \quad x_C \tag{3.61}$$

$$\frac{h_2 - h_1}{S} - \sinh \frac{x_C - S/2 + q}{c} = 0 \tag{3.62}$$

$$\sinh \frac{x_C - S/2 + q}{c} = \frac{h_2 - h_1}{S} \tag{3.63}$$

$$\operatorname{arcsinh}\left(\sinh \frac{x_C - S/2 + q}{c}\right) = \operatorname{arcsinh} \frac{h_2 - h_1}{S} \tag{3.64}$$

$$\frac{x_C - S/2 + q}{c} = \operatorname{arcsinh} \frac{h_2 - h_1}{S} \tag{3.65}$$

$$x_C = \frac{S}{2} - q + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \tag{3.66}$$

Expressing q from (3.13) and substituting it into (3.66) results in (3.68).

$$q = c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} \tag{3.67}$$

$$x_C = \frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \tag{3.68}$$

The last step inserts x_C into $D_{inc}(x)$, according to (3.69) and (3.70), in order to get $D_{inc\,\max}$. Because of its significant length the deduction is detailed in Appendix 4. The final result is given by (3.71).

$$D_{inc\max} = D_{inc}(x_C) \tag{3.69}$$

$$D_{inc\max} = D_{lev\max} + \frac{h_2 - h_1}{S} x_C - 2c \cdot \sinh^2 \frac{S}{4c} + c \cdot \left(\cosh \frac{q - S/2}{c} - \cosh \frac{x_C - S/2 + q}{c}\right) \tag{3.70}$$

$$\begin{aligned} D_{inc\max} = D_{lev\max} &+ \frac{h_2 - h_1}{S} \cdot \left(\frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S}\right) - \\ &- 2c \cdot \sinh^2 \frac{S}{4c} + 2c \cdot \sinh\left(\frac{1}{2c}\left(\frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S}\right)\right) \cdot \\ &\cdot \sinh\left(\frac{1}{2c}\left(\frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S}\right)\right) \end{aligned} \tag{3.71}$$

This is an exact relation between the maximum sags of the catenary in inclined and level spans. Notice that $D_{inc\,\max}=D_{inc\,\max}(D_{lev\,\max},S,c,\Delta h)$. Using expression for x_{MIN} given by (2.29), the previous relation changes into (3.72):

$$D_{inc\max} = D_{lev\max} + \frac{h_2 - h_1}{S} \cdot \left(x_{MIN} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) - 2c \cdot \sinh^2 \frac{S}{4c} + $$
$$+ 2c \cdot \sinh\left(\frac{1}{2c}\left(x_{MIN} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right)\right) \cdot \sinh\left(\frac{1}{2c}\left(x_{MIN} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right)\right) \tag{3.72}$$

The simplified form of (3.72) is expression (3.73) where $\Delta D_{\max}=\Delta D_{\max}(S,c,\Delta h)$ presents the difference between the maximum sags of the catenary in inclined and level spans. If $D_{lev\max}$ is given, then $D_{inc\max}$ can be obtained by computing $\Delta D_{\max}$ and adding it up to $D_{lev\max}$.

$$D_{inc\max} = D_{lev\max} + \Delta D_{\max} \tag{3.73}$$

The applicability of (3.71) is presented below regarding to five catenaries drawn in Fig. 3.11 and using the input data from Table 3.6 (Section 3.4.4). Applying (3.51) and (3.71), the maximum sags for all catenaries drawn in Fig. 3.11 are computed and then listed in Table 3.7:

Table 3.7: Maximum sags of the catenaries drawn in Fig. 3.11

Curve	$D_{lev\max}$ [m]	$\Delta D_{\max}$ [m]	$D_{inc\max}$ [m]
y_1	61.87782	–	–
y_2	–	0.15241	62.03023
y_3	–	0.60740	62.48522
y_4	–	1.35844	63.23626
y_5	–	2.39515	64.27297

It is evident that the maximum sag of the catenary increases with the span inclination. Let us mention that each of five sags has different location within the span, even though S and c are common data. This is an important difference in comparison to a parabola, since its maximum sag is always located at a mid–span independently of the span inclination. The fact that the maximum sags in Tables 3.7 and 2.6 are equal, verifies the correctness of relation (3.71).

3.6 Summary of the Chapter

Chapter 3 shows in details how to model an inclined span by given basic data of a level one, when the span length and the catenary parameter are common data. The equations for the conductor curve and the sag are given in level and inclined spans as well.

Using the sag equations in both inclined and level spans, the exact relation between the conductor sags in two span types is derived. Since the relation is given as a function of x, it means that the sag at an arbitrary point of an inclined span can be directly calculated from the given sag at the same point of the appropriate level span. Also, the existing approximate relation between the sags in inclined and level spans that can be found in some earlier studies is adequately discussed. Having both the new and the earlier relations, the error produced by the use of the approximate relation can be exactly obtained at any point of the span.

Based on the function given as a quotient of $D_{inc}(x)$ and $D_{lev}(x)$ on the interval $(0,S)$, an important feature of the catenary is revealed and hence one of the differences between the parabola and the catenary based calculations is easily recognized.

The formula for the catenary's maximum sag in an inclined span, available in some earlier literature, has been appropriately analysed then the new exact one is proposed for use. Finally, the exact relation between the maximum sags in two span types is derived.

The new relations shown in this chapter are unique ones related to OHL. The practical applicability of the new relations and the developed *inclined span modelling* method is shown in Sections 4.4 and 4.5.

It is important to emphasize that the new method – due to its mathematical nature – can be also used in the case of the parabola. It is shown in Chapter 4.

4 APPLICATION OF THE PARABOLA MODEL

4.1 Introduction

The parabola is an algebraic function, differently from the catenary, which is a transcendental one. According to [70] a parabola is a two–dimensional, mirror–symmetrical curve, which is approximately U–shaped when oriented as shown in Fig. 4.1, but which can be in any orientation in its plane. *"Parabola fits any of several superficially different mathematical descriptions which can all be proved to define curves of exactly the same shape."* Two of them, explained in [70], are given below. *"One description of a parabola involves a point (the focus) and a line (the directrix). The focus does not lie on the directrix. The parabola is the locus of points in that plane that are equidistant from both the directrix and the focus."* It is shown in Figs. 4.1 and 4.2. The directrix is marked as *L* and the focus as *F* in the latter figure. *"The distance from any point on the parabola to the focus (P_nF) equals the perpendicular distance from the same point on the parabola to the directrix (P_nQ_n)."*

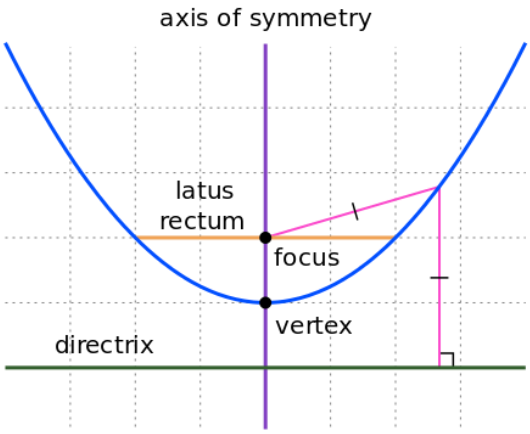


Fig. 4.1: Parabola with various features [70]

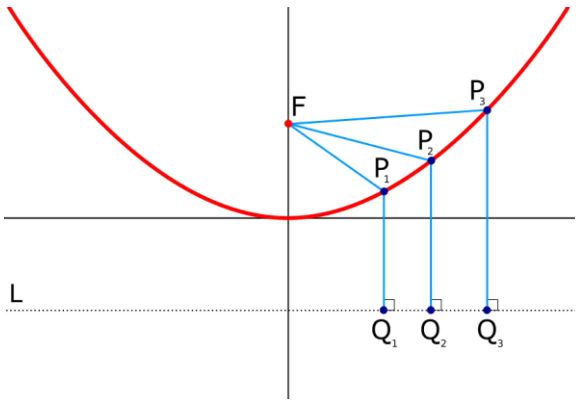


Fig. 4.2: Parabola, focus and directrix [70]

Another description of the parabola, which is much simpler and shorter than the previous one, is the following: *"a parabola is a graph of a quadratic function, such as $y=x^2$"*. This description is well suitable for the parabola based calculations in the field of OHL design. The way of the use of the coordinate system in this chapter is the same as it is detailed in Chapter 2, which is regarded to the catenary based calculation. Setting the origin on the line of the left–hand side support, instead of putting it at the vertex point, a wide mathematical background has been provided and several new expressions have been derived.

It is well known that the parabola based calculation is much simpler than the catenary based one, since the parabola is a quadratic function but the catenary is a hyperbolic one. However, their curves can be very similar, and that is why the catenary is frequently approximated by a parabola leading to a significant simplification of the calculation. The basic input data in this chapter are the following: span length, heights of the support points related to x–axis, and maximum sag. Thus, in comparison to the corresponding input data applied for the catenary based calculation, the maximum sag is used here instead of the catenary parameter.

There is a special relationship between the parabola and the catenary: the catenary is the locus of the focus of the parabola rolling along a straight line [71]. For instance, if the parabola $y_{par}=x^2$ is rolled along the x–axis, the locus of its focus is the catenary $y_{cat}=(1/4)\cdot\cosh(4x)$ [52]. The latter curve is drawn by a dashed line in Fig. 4.3, while the parabola is a solid curve.

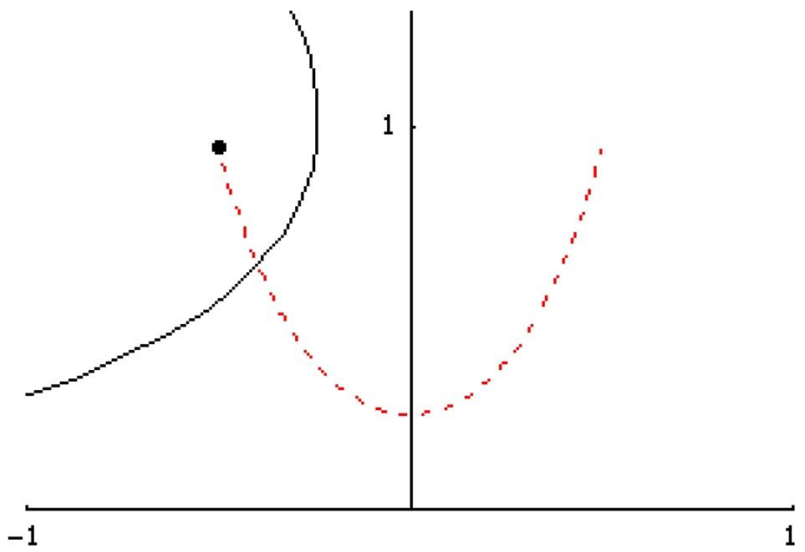


Fig. 4.3: Presentation of the relation between the parabola and the catenary [72]

This chapter is built as follows. After a short overview of the research field in Section 4.2, the parabolic sag equation is derived and discussed in details in Section 4.3. A universal equation for the conductor curve is defined in Section 4.4, while the determination of the vertex point by three different mathematical methods is presented in Section 4.5, which also deals with the low point sag. Section 4.6 contains a complex numerical example with 3 spans showing the usefulness of the main expressions obtained in the previous sections. Derivation of the special parabolic equations for the conductor curve in inclined spans is given in Section 4.7, as well as their practical application. Section 4.8 is a continuation of the previous one presenting formulas for the maximum sag and the low point sag in inclined spans. Section 4.9 introduces the parabolic approximation of the catenary in inclined spans and also provides a wide discussion of $1/\cos\psi$ multiplier's application. The last section in this chapter gives a short conclusion and summarizes the novel results.

4.2 Related Research

Lots of mathematical books (for instance [73,74]) discuss the parabola, but here such literature is listed which deals with concrete application of the parabola in OHL design, for drawing the conductor curve.

Similarly, as in the case of the catenary, for drawing the parabolic conductor curve, the related literature generally uses the coordinate system in a way that the vertex of the parabola is set at the origin of the coordinate system [34,39,75]. Therefore, the approximation of the catenary by a parabola is also shown in the same way. It is available in plenty of studies, for instance in [4,62,63].

A graphical method for drawing the parabolas by a given half–span distance and a mid–span sag is introduced in [76]. A very similar presentation of the mentioned method is shown in [41,44]. According to [76], if the sag exceeds about 5% of the span length, the sag correction is necessary. Using the Maclaurin's infinite series for hyperbolic functions, article [77] shows a *three–term* sag formula for a mid–span sag in a level span, where the first term is the parabola's sag, while the other two terms slightly increase it. This way, the difference between the calculated sag and the catenary's sag decreases and hence the sag calculation is more accurate. The brochure [7] also gives a *three–term* sag formula but in addition it provides practical numerical examples, showing that the use of two terms is customary for long spans, large sag transmission lines whilst the use of all three terms is necessary only for the most exacting problems in OHL design.

Some literature [75,78,79] gives the formula for calculating the maximum sag of the parabola, while some others [42,80] provide a sag equation which can be used to determine the sag at any point of the span. The latter equation is easily applicable for calculating the parabola's maximum sag knowing that it is always located at a mid–span.

Regarding to inclined spans, [81] introduces the conductor sagging in details by means of an inclined viewing line and also by means of a horizontal viewing line. Books [34,39,58,82,83] show the sag calculation in both level and inclined spans. The book [84] also discusses both span types and recommends the application of $1/\cos\psi$ multiplier for calculation of an inclined span sag from a given level span sag.

Besides explaining the parabola based calculation for OHL design, some literature also gives a basic recommendation whether the conductor curve can be considered as a parabola or it has to be considered as a catenary. According to [58], as long as the span is less than 300 metres and the sag is less than 5% of the span, the error between the catenary and the parabola approximates to 0.5%, and thus is not significant. The limit of 300 or 400 metres is a frequent criterion in literature for the maximum span length when the conductor curve can be considered as a parabola instead of the catenary. Obviously, this criterion does not distinguish level and inclined spans. One of the rare studies which propos the mentioned criterion separately for level and for inclined spans as well is an earlier Hungarian standard [68] for OHL design. It is worth mentioning that the following standard [85] does not contain any recommendation in connection with the use of the parabola or the catenary model. It means that the OHL design engineer has to select the appropriate model. The author of paper [86] provides practical numerical examples showing that the difference between the catenary and the parabola sags in inclined spans is bigger than in a level one. This way the author has shown that the span inclination plays an important role in selection of the parabola or the catenary model.

Several studies [47,76,87–89] show the use of the parabola sag templates which usually contain cold sag curve, hot sag curve, ground clearance curve and a tower footing curve, in order to make easier the OHL design. Nowadays this method is not frequent, due to the use of computers.

It has to be mentioned that numerous articles apply the parabolic approximation of the catenary for discussion of various topics in connection with the conductors of overhead lines, as mechanical state estimation of OHL [64], monitoring sag and tension of a transmission line [90], modified ruling span method [91] or factors which influence the accuracy of high temperature sag calculations [92] etc.

Paper [93] contains a section which deals with the conductor sag determination in an inclined span.

4.3 Parabolic Sag Equation

Calculating the conductor sag at a given point within the span is a frequent task in practice. To solve it, the sag equation is used, by which the conductor sag can be calculated at any point of the span. This section presents a new method for determination of the parabolic sag equation based on given maximum sag of the parabola. The maximum sag can be taken from the available sag–tension tables or obtained by a sag–tension calculation [94]. Applying mathematics, level and inclined spans are discussed separately in this section. Provided results show important features of the parabolic sag equation. It helps to shed light on some basic differences between the parabola and catenary functions, whose curves are used during planning and constructing electrical overhead lines. Despite the fact that their curves are often very similar, the equations of the parabola and the catenary are mathematically quite different. The presented method for determination of the parabolic sag equation is very practical for providing different algorithms useful for planning and designing overhead lines, especially in finding solutions for rare unconventional tasks in practice.

4.3.1 Sag Equation in a Level Span

For deriving the sag equation in a levelled span, Fig. 4.4 is used, which shows the conductor curve in the span with the support points on the same elevation. In this case the lowest point of the conductor (vertex of parabola) is at a mid–span, $S/2$. Data in Fig. 4.4 are the following: the span length, S, height of support points related to x–axis, h, and the maximum sag, D_{max}.

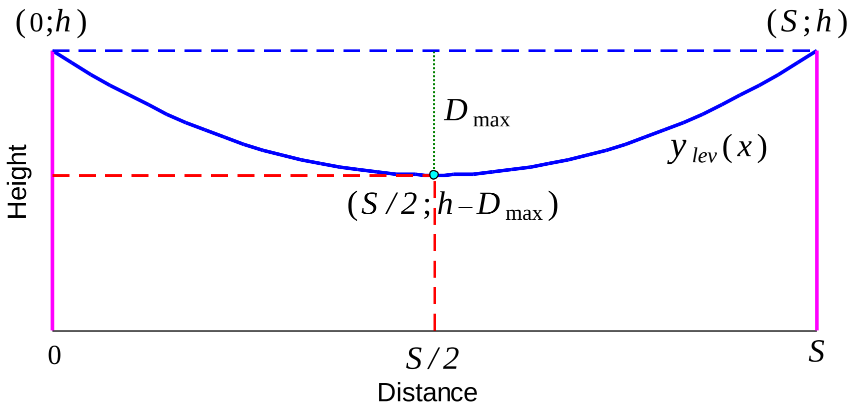


Fig. 4.4: Conductor curve in a level span

It is obvious that the sag at the support points is equal to zero. Also it is well seen that the sag has the largest value, D_{max}, at a mid–span. Taking into consideration these facts, the sag curve can be drawn as Fig. 4.5 shows.

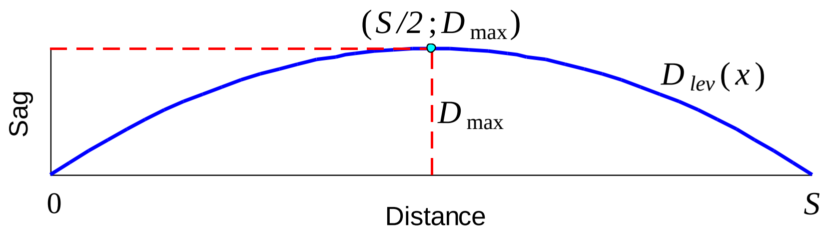


Fig. 4.5: Sag curve in a level span

Obtaining the equation of the curve shown in the previous figure, the conductor sag can be calculated at any point of the span. Since the actual curve is a parabola and has two roots (0 and S), its equation (4.4) can be derived by the use of the so called *factored form* of the parabola equation, which is given by (4.1):

$$D_{lev}(x) = a(x - x_1)(x - x_2) \qquad x \in [0, S] \tag{4.1}$$

$$x_1 = 0 \quad \wedge \quad x_2 = S \tag{4.2}$$

$$D_{lev}(x) = a(x - 0)(x - S) \qquad x \in [0, S] \tag{4.3}$$

$$D_{lev}(x) = a(x^2 - Sx) \qquad x \in [0, S] \tag{4.4}$$

The unknown a is the coefficient of the parabola and defines its shape. In the case of the sag curve, $a < 0$ (parabola opens downward and has a *global maximum*), while in case of the conductor curve, $a > 0$ (parabola opens upward and has a *global minimum*), but both parabolas open vertically. (Another type of the parabola opens horizontally, to the left or to the right.)

In order to determine the unknown coefficient a, the sag curve is replaced within the x–y coordinate system as it is shown in Fig. 4.6:

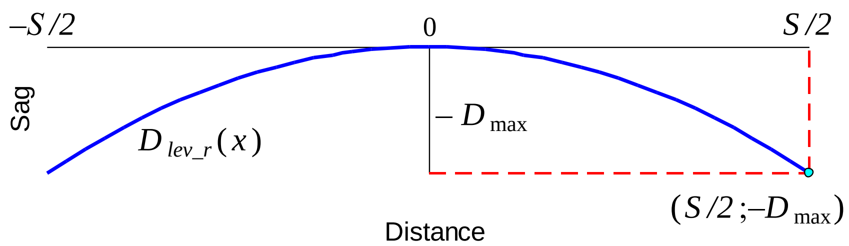


Fig. 4.6: Replaced sag curve with its top in the origin

Now the top of the curve is in the origin. In this case the equation of the curve D_{lev_r} is much simpler (4.5), but the coefficient a has not changed when the curve is replaced. In fact, that was the reason of the replacement. Inserting the coordinates of point ($S/2$; $-D_{max}$) in (4.5), coefficient a is obtained by (4.7) and can be substituted into (4.4) to complete the parabolic sag equation in a level span, i.e. to get (4.8):

$$D_{lev\_r}(x) = ax^2 \qquad x \in [-S/2, S/2] \tag{4.5}$$

$$x = S/2 \qquad D_{lev\_r}(S/2) = -D_{max} \tag{4.6}$$

$$-D_{max} = a\left(\frac{S}{2}\right)^2 \quad \Rightarrow \quad a = -\frac{4D_{max}}{S^2} \tag{4.7}$$

$$D_{lev}(x) = -\frac{4D_{max}}{S^2}(x^2 - Sx) \qquad x \in [0, S] \tag{4.8}$$

Equation (4.8) describes the sag curve from Fig. 4.5 and is usable for a sag calculation at an arbitrary point of the span. However, in Hungarian literature [45,84] another version of the sag equation is in use frequently. For this reason, transforming equation (4.8) into (4.13) is shown in the following lines. This way (4.8) is adequately verified.

$$D_{lev}(x) = -\frac{D_{\max}}{(S/2)^2}(x^2 - Sx) = -\frac{D_{\max}}{(S/2)^2}\left[x^2 - Sx + (S/2)^2 - (S/2)^2\right] \tag{4.9}$$

$$D_{lev}(x) = -D_{\max}\frac{(x - S/2)^2 - (S/2)^2}{(S/2)^2} = D_{\max}\left[1 - \frac{(x - S/2)^2}{(S/2)^2}\right] \tag{4.10}$$

$$D_{lev}(x) = D_{\max}\left[1 - \left(\frac{2x - S}{S}\right)^2\right] \qquad x \in [0, S] \tag{4.11}$$

Due to identity (4.12) [74], the previous equation can also be written by (4.13):

$$(x - y)^2 = (y - x)^2 \tag{4.12}$$

$$D_{lev}(x) = D_{\max}\left[1 - \left(\frac{S - 2x}{S}\right)^2\right] \qquad x \in [0, S] \tag{4.13}$$

Thus, (4.8) and (4.13) are equivalent equations. Being both parabola equations, they are transformable into a general form (4.14) or vertex form (4.15) as well.

$$D_{lev}(x) = -\frac{4D_{\max}}{S^2}x^2 + \frac{4D_{\max}}{S}x \qquad x \in [0, S] \tag{4.14}$$

$$D_{lev}(x) = -\frac{4D_{\max}}{S^2}\left(x - \frac{S}{2}\right)^2 + D_{\max} \qquad x \in [0, S] \tag{4.15}$$

Naturally, any equation from (4.8), (4.13), (4.14) and (4.15) can be used for sag calculation; the obtained results will be the same.

The basic equation of the parabola in general form is given by (4.16):

$$y(x) = ax^2 + bx + c \tag{4.16}$$

Comparing (4.14) to the previous equation, it can be seen that coefficient c is zero and coefficient a is negative (since S and $D_{\max}$ are always positive). As $a < 0$, the vertex point of the sag curve is the maximum point ($S/2$; $D_{\max}$). The coordinates of this point are easily recognizable in equation (4.15).

4.3.2 Sag Equation in an Inclined Span

The following method presented below is based on *inclined span modelling* from Chapter 3, but it is a bit modified here and adopted for the parabola. In order to obtain the sag equation in an inclined span, firstly the equation for the conductor curve in a level span has to be defined. Knowing coefficient a of the sag equation and the lowest point of the conductor curve in a level span, i.e. point ($S/2;h-D_{max}$), it is easy to obtain the vertex form of the equation for the conductor curve in a level span, by the application of (4.17).

$$y(x) = a(x - x_{MIN})^2 + y_{MIN} \tag{4.17}$$

The use of the coefficient a from either (4.14) or (4.15) but now with a "+" sign, yields equation (4.18), where D_{max} is the maximum sag in a level span. As shown in Fig. 4.7, the conductor curve in a level span is the curve between the points $A(0;h)$ and $B(S;h)$.

$$y_{lev}(x) = \frac{4D_{max}}{S^2}\left(x - \frac{S}{2}\right)^2 + h - D_{max} \qquad x \in [0, S] \tag{4.18}$$

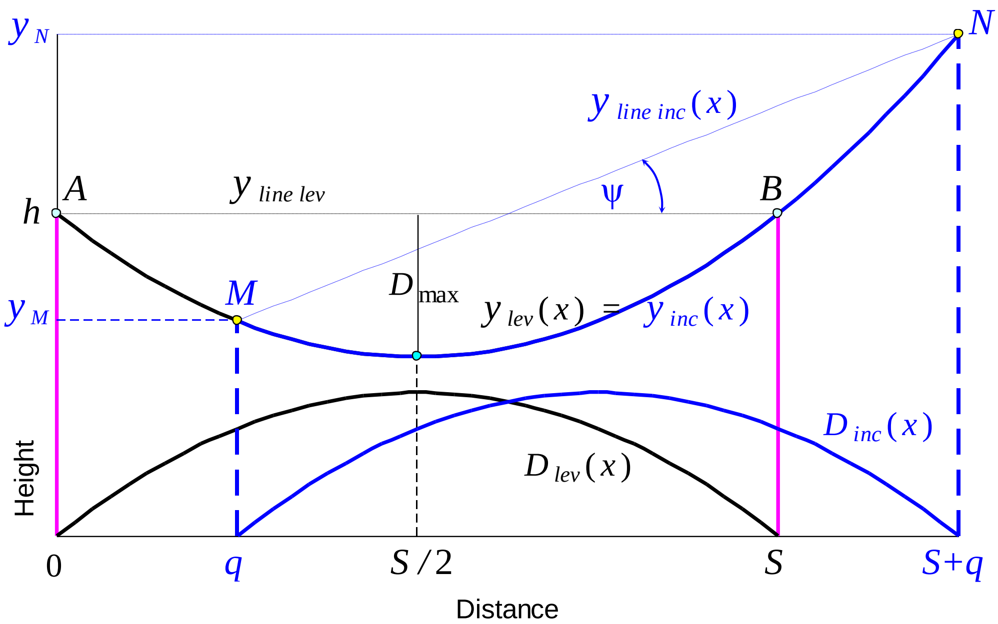


Figure 4.7: Conductor curves and sag curves in level and inclined spans with the same span length and coefficient *a* of the parabola

The conductor curve on the interval $[q,S+q]$ has the same equation (4.19) as the previous one, but it concerns now to an inclined span with the support points $M(q;y_M)$ and $N(S+q;y_N)$. Only the interval has changed (with keeping its size), but neither the span length nor coefficient a has changed.

$$y_{inc}(x) = \frac{4D_{max}}{S^2}\left(x - \frac{S}{2}\right)^2 + h - D_{max} \qquad x \in [q, S+q] \tag{4.19}$$

Equation (4.19) transformed into general form of the parabola is the following:

$$y_{inc}(x) = \frac{4D_{max}}{S^2}x^2 - \frac{4D_{max}}{S}x + h \qquad x \in [q, S+q] \tag{4.20}$$

Since the straight line connecting the support points of the formed inclined span on the interval [q,S+q] is given by (4.21), the sag equation (4.23) is classically obtained by (4.22).

$$y_{line\_inc}(x) = \frac{y_N - y_M}{S}(x-q) + y_M \qquad x \in [q, S+q] \tag{4.21}$$

$$D_{inc}(x) = y_{line\,inc}(x) - y_{inc}(x) \qquad x \in [q, S+q] \tag{4.22}$$

$$D_{inc}(x) = \frac{y_N - y_M}{S}(x-q) + y_M - \frac{4D_{\max}}{S^2}x^2 + \frac{4D_{\max}}{S}x - h \qquad x \in [q, S+q] \tag{4.23}$$

In the following lines the difference $y_N - y_M$ (4.26) is determined depending on q:

$$y_M = y(q) = \frac{4D_{\max}}{S^2}q^2 - \frac{4D_{\max}}{S}q + h \tag{4.24}$$

$$y_N = y(S+q) = \frac{4D_{\max}}{S^2}(S+q)^2 - \frac{4D_{\max}}{S}(S+q) + h \tag{4.25}$$

$$y_N - y_M = \frac{8D_{\max}}{S}q \tag{4.26}$$

Insertion of equations (4.24) and (4.26) into (4.23), yields (4.27). It can be transformed into (4.31).

$$D_{inc}(x) = \frac{8D_{\max}q}{S^2}x - \frac{8D_{\max}q^2}{S^2} + \frac{4D_{\max}q^2}{S^2} - \frac{4D_{\max}q}{S} + h - \frac{4D_{\max}}{S^2}x^2 + \frac{4D_{\max}}{S}x - h \qquad x \in [q, S+q] \tag{4.27}$$

$$D_{inc}(x) = -\frac{4D_{\max}}{S^2}x^2 + \frac{4D_{\max}}{S}x - D_{\max} + \frac{8D_{\max}q}{S^2}x - \frac{4D_{\max}q}{S} - \frac{4D_{\max}q^2}{S^2} + D_{\max} \qquad x \in [q, S+q] \tag{4.28}$$

$$D_{inc}(x) = -\frac{4D_{\max}}{S^2}\left[x^2 - Sx + \left(\frac{S}{2}\right)^2 - 2qx + Sq + q^2\right] + D_{\max} \qquad x \in [q, S+q] \tag{4.29}$$

$$D_{inc}(x) = -\frac{4D_{\max}}{S^2}\left[\left(x - \frac{S}{2}\right)^2 - 2\left(x - \frac{S}{2}\right)q + q^2\right] + D_{\max} \qquad x \in [q, S+q] \tag{4.30}$$

$$D_{inc}(x) = -\frac{4D_{\max}}{S^2}\left(x - \frac{S}{2} - q\right)^2 + D_{\max} \qquad x \in [q, S+q] \tag{4.31}$$

$$D_{inc}(x) = D_{lev}(x-q) \qquad x \in [q, S+q] \tag{4.32}$$

Thus, the parabola’s sag in an inclined span does not differ from the sag in a level span. This conclusion can be expressed in the following mathematical relation:

$$\text{If} \quad a_{inc} = a_{lev} \ \wedge \ S_{inc} = S_{lev} \ \Rightarrow \ D_{inc}(x) \equiv D_{lev}(x) \tag{4.33}$$

Practically, the above relation with its conditions concerns the case when the conductor curves in both level and inclined spans are parts of the same parabola curve but on different intervals with the same size.

All the following features of the parabola steam from (4.33):

- Sag curve does not depend on the span inclination
- The maximum sags of the parabola in level and inclined spans are equal
- The maximum sag of the parabola is always located at a mid–span, i.e. in both level and inclined spans.

The latter parabola's feature is generally known or accepted in literature, but the detailed mathematical background is not available. Here it is appropriately presented from OHL designer's view point and given as a partial conclusion of relation (4.33). In other words, the already known feature of the parabola is proved analytically in this section. It is worth mentioning that – relating to the catenary – none of the above listed features is valid. It is a direct consequence of the fact that in the case of the catenary the sag curves in inclined and level spans differ from each other.

Returning to (3.33) in Chapter 3, due to (4.33) the corresponding relation for the parabola is the following:

$$\frac{D_{inc}^{(\mathrm{par})}(x)}{D_{lev}^{(\mathrm{par})}(x)} = \left(\frac{D_{inc}^{(\mathrm{par})}}{D_{lev}^{(\mathrm{par})}}\right)(x) = 1 \qquad 0 < x < S \tag{4.34}$$

Thus, the quotient of the sag functions in inclined and level spans on the interval $(0,S)$ is equal to 1, and thus is a constant.

Furthermore, in the case of the parabola identity $D(x) = D(S-x)$ when $0 \le x \le S/2$ is valid in all span types, independently of the span inclination. This way the following feature of the parabola is identified: the sag function of its curve replaced from interval $[0, S]$ to $[-S/2, S/2]$ is an *even* function in the case of level and inclined spans as well. Considering (4.34) and denoting $D_{lev}(x)$ and $D_{inc}(x)$ with the same symbol, $D^{(\mathrm{par})}(x)$, the mentioned feature of the parabola can be expressed mathematically as follows:

$$D^{(\mathrm{par})}\left(-x+\frac{S}{2}\right) = D^{(\mathrm{par})}\left(x+\frac{S}{2}\right) \quad \Rightarrow \quad D^{(\mathrm{par})}\left(x+\frac{S}{2}\right) \ \underline{\text{is an } \textit{even} \text{ function}} \tag{4.35}$$

4.4 Universal Equation for the Parabolic Conductor Curve

Equation for the conductor curve is usable for drawing the conductor curve and also for determining the vertex point (low point) or the conductor height at any point of the span.

Derivation of this equation is shown below with the help of Fig. 4.8 which presents the conductor curve in the span with the support points on different elevations.

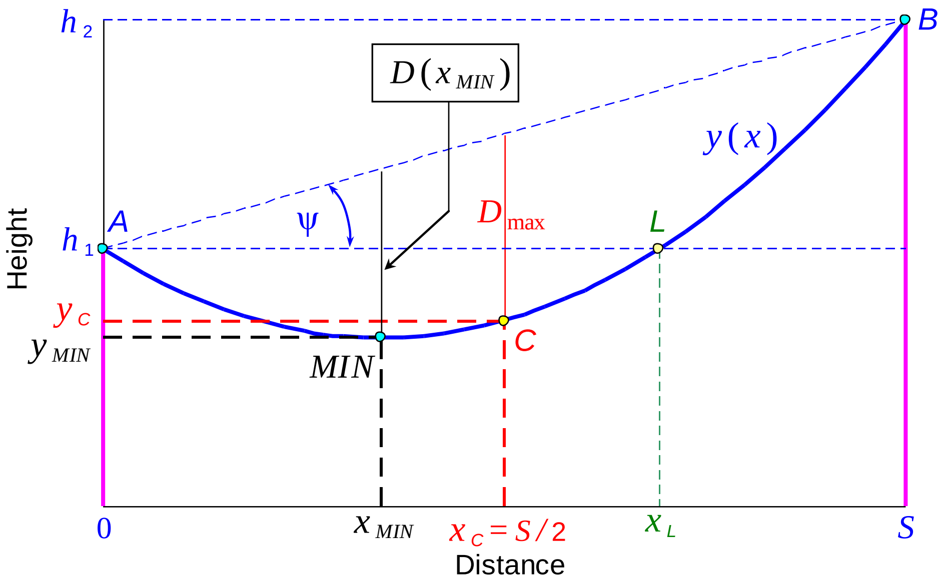


Figure 4.8: Conductor curve in inclined span with $h_1 < h_2$

The fact that the parabola is completely defined when any three points of its curve are known, is practically used here for defining the parabolic conductor curve. Two support points of the span, $A(0;h_1)$ and $B(S;h_2)$ are known points, while the third necessary point, C, is defined by the known maximum sag. The feature of the parabola that its maximum sag is always located at a mid–span is applied. (In the case of the catenary it is different.) Thus, the x–coordinate of point C is known ($x_C=S/2$), while its y–coordinate is obtainable by (4.36), and hence C is given by (4.37).

$$y_C = \frac{h_1 + h_2}{2} - D_{\max} \tag{4.36}$$

$$C\left(\frac{S}{2};\frac{h_1 + h_2}{2} - D_{\max}\right) \tag{4.37}$$

Based on three points A, B, C of the parabolic conductor curve, the system of three algebraic equations, (4.38) – (4.40), in three unknowns (a, b, c) is written by utilizing the basic parabola equation in general form (4.16).

$$h_1 = c \tag{4.38}$$

$$h_2 = aS^2 + bS + c \tag{4.39}$$

$$\frac{h_1 + h_2}{2} - D_{\max} = a\left(\frac{S}{2}\right)^2 + b\left(\frac{S}{2}\right) + c \tag{4.40}$$

The system of the three previous equations, expressed in matrix form, is the following:

$$\begin{bmatrix} 0 & 0 & 1 \\ S^2 & S & 1 \\ (S/2)^2 & S/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ (h_1 + h_2)/2 - D_{max} \end{bmatrix} \tag{4.41}$$

The solution of this system is given by (4.42) and it presents coefficients *a*, *b*, *c* of the parabola equation (4.16).

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4D_{max}/S^2 \\ (h_2 - h_1 - 4D_{max})/S \\ h_1 \end{bmatrix} \tag{4.42}$$

After substituting *a*, *b*, *c* into (4.16), the equation for the conductor curve in general form is derived:

$$y(x) = \frac{4D_{max}}{S^2} x^2 + \frac{h_2 - h_1 - 4D_{max}}{S} x + h_1 \qquad x \in [0, S] \tag{4.43}$$

The previous equation is universal, since it is usable in both types of an inclined span ($h_1 < h_2$ and $h_1 > h_2$) and in level spans ($h_1 = h_2$) as well.

Having the minimum turning point, the conductor curve is a cup–shaped parabola. Differently from the sag equation, coefficient *a* is in this case positive, because the span length and the maximum sag are both positive. Equation (4.43) can be checked by using a conclusion from Section 4.3 that the parabolic sag equation is the same in level and inclined spans. Considering it, subtraction of the sag equation, $D(x)$, from the equation for the straight line connecting the support points, $y_{line}(x)$, should also provide (4.43). According to (4.44) it obviously does and this way (4.43) is verified.

$$y(x) = y_{line}(x) - D(x) = \frac{h_2 - h_1}{S} x + h_1 - \left(-\frac{4D_{max}}{S^2} x^2 + \frac{4D_{max}}{S} x \right) \qquad x \in [0, S] \tag{4.44}$$

In a level span $h_1 = h_2 = h$ and hence equation (4.43) changes into (4.45).

$$y_{lev}(x) = \frac{4D_{max}}{S^2} x^2 - \frac{4D_{max}}{S} x + h \qquad x \in [0, S] \tag{4.45}$$

4.5 Vertex Point of the Parabolic Conductor Curve, Low Point Sag

Before deriving the vertex point of the parabola it is worth mentioning the obvious difference between the catenary and the parabola in connection with the determination of their equations. While in the case of the catenary the coordinates of the vertex point are necessary data for defining its equation, the parabola's equation can be obtained even without the vertex point. It is shown in Section 4.3 where the equation of the parabola is obtained by its three known points, but none of them is the vertex. In level spans only, one of the three points is the vertex, because it is located at a mid–span then. Taking into consideration that the vertex point is very important for clearance calculation, this section shows its determination. Once the equation for the conductor curve is derived, different mathematical techniques are applicable to define the vertex point (low point) of the conductor, on the basis of a given maximum sag. The following three methods are detailed below:

- Derivative of the conductor curve
- Finding the longest level subspan within an inclined span
- Transforming parabola equation from general form into vertex form.

The validity of the listed methods is proved by their identical results.

It is worth noting that the sag at the vertex point is defined if the vertex point is also the lowest point of the conductor, otherwise it is not.

4.5.1 Derivative of the Conductor Curve

The basic way to find the *x*–coordinate of the extreme point (minimum or maximum) of the curve $y(x)$ is to find the first derivative, dy/dx, and to solve the equation $dy/dx=0$. Then by substituting the obtained result into the equation for the curve, the *y*–coordinate of the extreme point is also defined. The application of this method on the conductor curve, shown by (4.46) – (4.60), yields the vertex point of the conductor curve given by (4.61).

$$\frac{dy}{dx}=\frac{8D_{\max}}{S^2}x+\frac{h_2-h_1-4D_{\max}}{S} \tag{4.46}$$

$$\frac{dy}{dx}=0 \quad \Rightarrow \quad x_{MIN} \tag{4.47}$$

$$\frac{8D_{\max}}{S^2}x_{MIN}+\frac{h_2-h_1-4D_{\max}}{S}=0 \tag{4.48}$$

$$x_{MIN}=\frac{S}{8D_{\max}}\left[4D_{\max}-\left(h_2-h_1\right)\right] \tag{4.49}$$

$$x_{MIN}=\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right) \tag{4.50}$$

This way the *x*–coordinate of the vertex point is obtained. Expression (4.50) can be considered as the horizontal distance from the vertex point to the left–hand side support. On the other hand, expression (4.51) presents the horizontal distance from the vertex point to the right–hand side support.

$$S - x_{MIN} = \frac{S}{2}\left(1 + \frac{h_2 - h_1}{4D_{\max}}\right) \tag{4.51}$$

Based on (4.50) and (4.51), it can be concluded that the vertex point of the parabola in an inclined span is located on the distance of $S(h_2–h_1)/8D_{\max}$ units from the mid–span, measured horizontally toward the lower support point. Obviously, the mentioned distance increases with the span inclination (or $h_2–h_1$), but in level spans it is equal to zero.

Once the x–coordinate of the vertex point is obtained, the y–coordinate can be defined by substituting x_{MIN} into the equation for the conductor curve according to (4.52). The deduction is shown in the following lines:

$$y_{MIN} = y(x_{MIN}) \tag{4.52}$$

$$y_{MIN} = \frac{4D_{\max}}{S^2} x_{MIN}{}^2 + \frac{h_2 - h_1 - 4D_{\max}}{S} x_{MIN} + h_1 \tag{4.53}$$

$$y_{MIN} = \frac{4D_{\max}}{S^2}\left[\frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right]^2 + \frac{h_2 - h_1 - 4D_{\max}}{S} \cdot \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) + h_1 \tag{4.54}$$

$$y_{MIN} = D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)^2 + \frac{h_2 - h_1 - 4D_{\max}}{2} \cdot \left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) + h_1 \tag{4.55}$$

$$y_{MIN} = h_1 + \left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\left[D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) + \frac{h_2 - h_1 - 4D_{\max}}{2}\right] \tag{4.56}$$

$$y_{MIN} = h_1 + \left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\left(D_{\max} - \frac{h_2 - h_1}{4} + \frac{h_2 - h_1}{2} - 2D_{\max}\right) \tag{4.57}$$

$$y_{MIN} = h_1 + \left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\left(-D_{\max} + \frac{h_2 - h_1}{4}\right) \tag{4.58}$$

$$y_{MIN} = h_1 - D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) \tag{4.59}$$

$$y_{MIN} = h_1 - D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)^2 \tag{4.60}$$

According to (4.50) and (4.60), the vertex point $MIN(x_{MIN};y_{MIN})$ is given by (4.61).

$$MIN\left(\frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right);\ h_1 - D_{\max}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)^2\right) \tag{4.61}$$

4.5.2 Finding the Longest Level Subspan within an Inclined Span

For presentation of this method Fig. 4.8 is used which contains all necessary points and symbols. It is well seen that there is one point (denoted by *L*) of the conductor curve which lies on the same elevation, h_1, as point *A* does (4.62). By determination of the *x*–coordinate of the point *L*, the *x*–coordinate of the point *MIN* can be easily defined, since it is exactly half of the distance between the points *A* and *L*. Thus $x_{MIN}=x_L/2$. The x_L is in fact the length of the longest level subspan within the given inclined span. Finding x_{MIN} (4.68) is shown in the following lines:

$$y_A = y(0) = y_L = y(x_L) = h_1 \tag{4.62}$$

$$h_1 = \frac{4D_{\max}}{S^2} x_L^2 + \frac{h_2 - h_1 - 4D_{\max}}{S} x_L + h_1 \tag{4.63}$$

$$\frac{4D_{\max}}{S^2} x_L^2 + \frac{h_2 - h_1 - 4D_{\max}}{S} x_L = 0 \tag{4.64}$$

$$x_L\left[4D_{\max} x_L + S(h_2 - h_1) - 4SD_{\max}\right] = 0 \tag{4.65}$$

It is clear that $x_L=0$ is not an appropriate solution, therefore it is necessary to solve equation (4.66) in order to get x_L (4.67), and then also (4.68):

$$4D_{\max} x_L + S(h_2 - h_1) - 4SD_{\max} = 0 \tag{4.66}$$

$$x_L = \frac{4SD_{\max} - S(h_2 - h_1)}{4D_{\max}} = S\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) \tag{4.67}$$

$$x_{MIN} = \frac{x_L}{2} = \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right) \tag{4.68}$$

Since the *x*–coordinate of the point *MIN* is obtained, its *y*–coordinate can be defined in the same way shown in the previous method, i.e. by (4.52). Both presented methods, the actual and the previous one, are provided for the case $h_1<h_2$, but the case $h_1>h_2$ also produces the same result.

4.5.3 Transforming Parabola Equation from General into Vertex Form

The fact that each parabola equation in general form (4.16) can be also written in vertex form (4.17) can be practically used to find the coordinates of the vertex point, as they are readable from the parabolic equation given in vertex form. Transforming the above derived equation for the conductor curve, from its general form (4.43) into vertex form (4.73), is shown through expressions (4.69) – (4.72), by the use of *the completing the square method* as follows:

$$y(x)=\frac{4D_{\max}}{S^2}\left(x^2+\frac{S^2}{4D_{\max}}\cdot\frac{h_2-h_1-4D_{\max}}{S}x\right)+h_1 \tag{4.69}$$

$$y(x)=\frac{4D_{\max}}{S^2}\left[x^2+S\left(\frac{h_2-h_1}{4D_{\max}}-1\right)x\right]+h_1 \tag{4.70}$$

$$\begin{aligned}y(x)=\frac{4D_{\max}}{S^2}\left[x^2-S\left(1-\frac{h_2-h_1}{4D_{\max}}\right)x\right]+h_1+\frac{4D_{\max}}{S^2}\left[\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2-\\-\frac{4D_{\max}}{S^2}\left[\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2\end{aligned} \tag{4.71}$$

$$\begin{aligned}y(x)=\frac{4D_{\max}}{S^2}\left\{x^2-S\left(1-\frac{h_2-h_1}{4D_{\max}}\right)x+\left[\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2\right\}+h_1-\\-\frac{4D_{\max}}{S^2}\left[\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2\end{aligned} \tag{4.72}$$

$$y(x)=\frac{4D_{\max}}{S^2}\left[x-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2+h_1-D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2 \qquad x\in[0,S] \tag{4.73}$$

$$y(x)=\frac{4D_{\max}}{S^2}(x-x_{MIN})^2+y_{MIN} \tag{4.74}$$

According to (4.74), the coordinates of the vertex point, x_{MIN} and y_{MIN}, are easily recognizable in (4.73). These are the same ones as (4.50) and (4.60) which are previously obtained in Section 4.5.1.

4.5.4 Low Point Sag

Since the lowest point of the conductor is also called shortly as the *low point*, hence the sag at the lowest point of the conductor is also called as the *low point sag*. Note that the conductor sag is defined only within the span, the low point can be only within the span, and that the vertex point is in most cases within the span, but in sharply steep spans it can be out of the span. The latter case is a rare one when the low point and the vertex point differ in their location, while in all other cases they do not, since they are the same point then. It means that in most spans the low point sag can be defined by obtaining the sag at the vertex point, since the vertex point is the low point as well. Then the expression for the low point sag can be defined analytically by substituting the *x*–coordinate of the vertex point into the sag equation, as it is shown below by (4.75) – (4.80).

$$D(x_{MIN}) = -\frac{4D_{\max}}{S^2}x_{MIN}^2 + \frac{4D_{\max}}{S}x_{MIN} \tag{4.75}$$

$$D(x_{MIN}) = -\frac{4D_{\max}}{S^2}\left[\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2 + \frac{4D_{\max}}{S}\cdot\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right) \tag{4.76}$$

$$D(x_{MIN}) = -D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2 + 2D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right) \tag{4.77}$$

$$D(x_{MIN}) = D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\left[-\left(1-\frac{h_2-h_1}{4D_{\max}}\right)+2\right] \tag{4.78}$$

$$D(x_{MIN}) = D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\left(1+\frac{h_2-h_1}{4D_{\max}}\right) \tag{4.79}$$

$$D(x_{MIN}) = D_{\max}\left[1-\left(\frac{h_2-h_1}{4D_{\max}}\right)^2\right] \quad \forall \quad x_{MIN} \in [0,S] \tag{4.80}$$

The previous expression can be used for the computation of the sag at the lowest point of the conductor. The ordered condition given by $0 \le x_{MIN} \le S$ prevents the use of (4.80) when the vertex is out of the span. In that case, i.e. when $x_{MIN} < 0$ or $x_{MIN} > S$, the low point sag is in fact the sag at the lower support point, and hence it is zero. Thus, the basic discussion about the special cases of an inclined span given in Chapter 2, relating to the catenary, refers also to the parabola in the same way.

4.6 Practical Usage of Equations for Conductor and Sag Curves

The practical usefulness of the above shown equations is presented below by a numerical example with two inclined spans and one level span. The maximum sag is a common datum in each case, as well as the span length, but the heights of the support points differ. The input data are given in Table 4.1.

Example 4.1

Table 4.1: Input data in Example 4.1

Data	Case 1 $h_1 < h_2$	Case 2 $h_1 = h_2$	Case 3 $h_1 > h_2$
S [m]	400	400	400
h_1 [m]	25	45	65
h_2 [m]	65	45	25
D_{max} [m]	20	20	20

Based on (4.43), the equations for the conductor curves in three cases are the following:

$$y_1(x) = 5\cdot 10^{-4}\cdot x^2 - 0.1\cdot x + 25 \qquad x\in[0,400] \tag{4.81}$$

$$y_2(x) = 5\cdot 10^{-4}\cdot x^2 - 0.2\cdot x + 45 \qquad x\in[0,400] \tag{4.82}$$

$$y_3(x) = 5\cdot 10^{-4}\cdot x^2 - 0.3\cdot x + 65 \qquad x\in[0,400] \tag{4.83}$$

Equations (4.81) – (4.83) are given in general form. According to the sag definition, the sag equation can be obtained by (4.84):

$$D(x) = y_{line}(x) - y(x) \qquad x\in[0,S] \tag{4.84}$$

Thus, firstly the equations for the straight lines connecting the support points have to be defined in all three actual cases:

$$y_{line1}(x) = 0.1\cdot x + 25 \qquad x\in[0,400] \tag{4.85}$$

$$y_{line2} = 45\,\mathrm{m} \tag{4.86}$$

$$y_{line3}(x) = -0.1\cdot x + 65 \qquad x\in[0,400] \tag{4.87}$$

The application of (4.84) provides a parabolic sag equation which is the same in all three given cases.

$$D(x) = D_1(x) = D_2(x) = D_3(x) = -5\cdot 10^{-4}\cdot x^2 + 0.2\cdot x \qquad x\in[0,400] \tag{4.88}$$

This way the conclusion (4.33) provided analytically in Section 4.3.2 is confirmed here numerically. Thus, the parabolic sag equation does not depend on the span inclination (or the height difference between the support points). Note that the coefficient a of the parabola in equation (4.88) is negative, that is why the sag curve is a hat–shaped parabola.

Using expressions (4.80), (4.50) and (4.60), the low point sag and the coordinates of the vertex point are calculated in each case then the results are listed in Table 4.2. Due to the fact that case 3 is a *mirror image* of case 1, and vice versa, that is why y_{MIN} is equal in these two cases, as well as $D(x_{MIN})$ is. In case 2 the low point sag is in fact the maximum sag, because this case presents a level span.

Table 4.2: Results of Example 4.1

Results	Case 1 $h_1 < h_2$	Case 2 $h_1 = h_2$	Case 3 $h_1 > h_2$
x_{MIN} [m]	100	200	300
y_{MIN} [m]	20	25	20
$D(x_{MIN})$ [m]	15	20	15

According to (4.74), the equations for the three conductor curves in vertex form are the following:

$$y_1(x) = 5\cdot 10^{-4}\cdot (x-100)^2 + 20 \qquad x\in[0,400] \tag{4.89}$$

$$y_2(x) = 5\cdot 10^{-4}\cdot (x-200)^2 + 25 \qquad x\in[0,400] \tag{4.90}$$

$$y_3(x) = 5\cdot 10^{-4}\cdot (x-300)^2 + 20 \qquad x\in[0,400] \tag{4.91}$$

Now all the results from the given example can be appropriately presented on the common diagram (see Fig. 4.9).

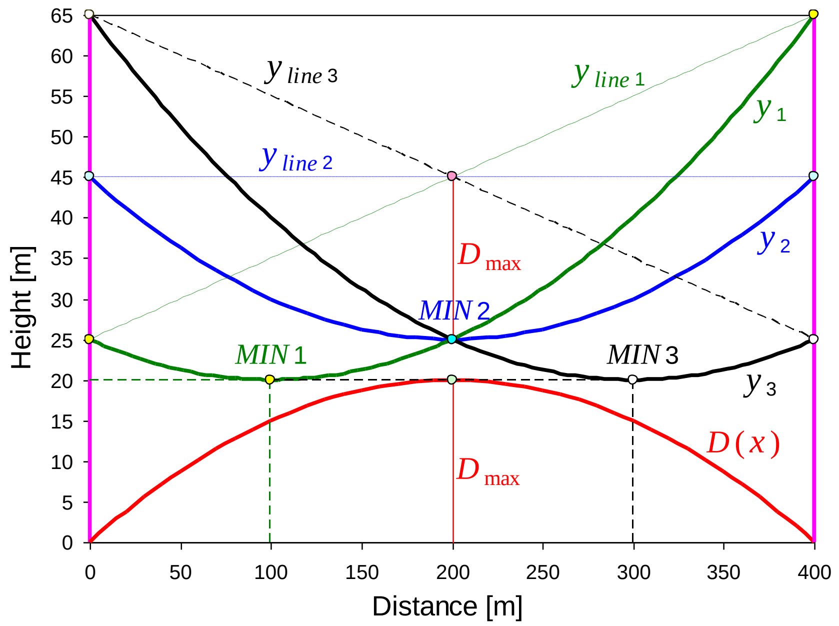


Fig. 4.9: Sag and conductor curves from Example 4.1

4.7 Special Equation for Conductor Curve in Inclined Spans

This section shows a method for deriving a parabolic equation for the conductor curve using x, y coordinates of two support points and only one coordinate (either x or y) of the vertex point. Thus, regarding to the three points, only five out of six coordinates are needed for defining the new equation. Differently from equation obtained by 3 points (see Section 4.4), which is universal and for a frequent use, the new one is for a very special use, in inclined spans strictly, i.e. when $h_1 \neq h_2$.

In order to derive the new equation, firstly $y=y(S, D_{max}, h_1, h_2, x)$ is transformed into $y=y(h_1, x_{MIN}, y_{MIN}, x)$. It is shown in the following lines, where (4.92) is obtained by rearranging (4.43):

$$y(x) = \frac{4D_{max}}{S^2}x^2 - \frac{4D_{max}}{S}\left(1-\frac{h_2-h_1}{4D_{max}}\right)x + h_1 \qquad x\in[0,S] \tag{4.92}$$

$$y(x)=\frac{4D_{\max}}{S^2}\cdot\frac{\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2}{\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2}x^2-\frac{4D_{\max}}{S}\cdot\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\cdot\frac{\left(1-\frac{h_2-h_1}{4D_{\max}}\right)}{\left(1-\frac{h_2-h_1}{4D_{\max}}\right)}x+h_1 \tag{4.93}$$

$$y(x)=\frac{D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2}{\left(\frac{S}{2}\right)^2\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2}x^2-2\cdot\frac{D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2}{\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)}x+h_1 \tag{4.94}$$

$$y(x)=\frac{h_1-\left[h_1-D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2\right]}{\left[\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2}x^2-2\cdot\frac{h_1-\left[h_1-D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2\right]}{\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)}x+h_1 \tag{4.95}$$

Considering expressions for x_{MIN} and y_{MIN}, given by (4.50) and (4.60) retrospectively, (4.95) gets a simplified form:

$$y(x)=\frac{h_1-y_{MIN}}{x_{MIN}^2}x^2-2\cdot\frac{h_1-y_{MIN}}{x_{MIN}}x+h_1 \tag{4.96}$$

The previous equation, given as $y=y(h_1, x_{MIN}, y_{MIN}, x)$, can be used to obtain $y=y(S, h_1, h_2, x_{MIN}, x)$ and also $y=y(S, h_1, h_2, y_{MIN}, x)$. To achieve it, the following expressions are necessary which are defined by equating coefficients b (according to $y=ax^2+bx+c$) from equations (4.92) and (4.96) and then also coefficients a from the same two equations:

$$b=-\frac{4D_{\max}}{S}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)=-2\cdot\frac{h_1-y_{MIN}}{x_{MIN}} \quad\Rightarrow\quad x_{MIN}=\frac{-2S(h_1-y_{MIN})}{h_2-h_1-4D_{\max}} \tag{4.97}$$

$$a=\frac{4D_{\max}}{S^2}=\frac{h_1-y_{MIN}}{x_{MIN}^2} \quad\Rightarrow\quad y_{MIN}=h_1-D_{\max}\left(\frac{2x_{MIN}}{S}\right)^2 \tag{4.98}$$

Combining (4.96), (4.97) and (4.98) is shown in the two following sections in order to define the new parabolic equations.

4.7.1 Deriving Equation for the Conductor Curve by Given *S*, h_1, h_2, x_{MIN}

Expressing $D_{\max}=D_{\max}(S, h_1, h_2, x_{MIN}, y_{MIN})$ from (4.97) by (4.99), and substituting it into (4.98) yields (4.100), which can be applied to define coefficients a and b (according to $y=ax^2+bx+c$) in (4.103), both expressed without y_{MIN}. These are given by (4.101) and (4.102). (The complete deduction is shown in Appendix 5.)

$$D_{\max} = \frac{1}{4x_{MIN}} \cdot \left[2S(h_1 - y_{MIN}) + (h_2 - h_1)x_{MIN}\right] \quad \forall \quad h_1 \neq h_2 \tag{4.99}$$

$$y_{MIN} = h_1 - \frac{(h_2 - h_1)x_{MIN}^2}{S(S - 2x_{MIN})} \quad \forall \quad h_1 \neq h_2 \tag{4.100}$$

$$a = \frac{h_1 - y_{MIN}}{x_{MIN}^2} = \frac{h_2 - h_1}{S(S - 2x_{MIN})} \tag{4.101}$$

$$b = -2 \cdot \frac{h_1 - y_{MIN}}{x_{MIN}} = -\frac{2(h_2 - h_1)x_{MIN}}{S(S - 2x_{MIN})} \tag{4.102}$$

Thus, the parabolic equation for the conductor curve, $y=y(S, h_1, h_2, x_{MIN}, x)$, in general form is the following:

$$y(x) = \frac{h_2 - h_1}{S(S - 2x_{MIN})} x^2 - \frac{2(h_2 - h_1)x_{MIN}}{S(S - 2x_{MIN})} x + h_1 \quad \forall \quad h_1 \neq h_2 \quad \wedge \quad x \in [0, S] \tag{4.103}$$

It is obvious that x, y coordinates of the two support points (these are defined by given data S, h_1 and h_2) and only x–coordinate of the vertex point are sufficient data for defining the new parabolic equation for the conductor curve which is usable for the determination of the conductor height, related to x–axis, at any point of the span. Another important application of this equation is for drawing a conductor curve. The correctness of (4.103) is verified below by the use of the input data from Example 4.1 (case 1), but x_{MIN} is applied instead of $D_{\max}$. The obtained equation is the same as $y_1(x)$ in general form, defined in the mentioned example.

$$y(x) = \frac{65 - 25}{400 \cdot (400 - 2 \cdot 100)} x^2 - \frac{2 \cdot (65 - 25) \cdot 100}{400 \cdot (400 - 2 \cdot 100)} x + 25 = \frac{40}{400 \cdot 200} x^2 - \frac{80 \cdot 100}{80\,000} x + 25 =$$
$$= \frac{x^2}{2000} - \frac{x}{10} + 25 = 5 \cdot 10^{-4} \cdot x^2 - 0.1 \cdot x + 25$$

The following equation is vertex form of (4.103). The correctness of it is also verified numerically using the same data that are used in (4.103). The result is vertex form of $y_1(x)$ from Example 4.1 (case 1).

$$y(x) = \frac{h_2 - h_1}{S(S - 2x_{MIN})}(x - x_{MIN})^2 + h_1 - \frac{(h_2 - h_1)x_{MIN}^2}{S(S - 2x_{MIN})} \quad \forall \quad h_1 \neq h_2 \quad \wedge \quad x \in [0, S] \tag{4.104}$$

$$y(x) = \frac{65 - 25}{400 \cdot (400 - 2 \cdot 100)}(x - 100)^2 + 25 - \frac{(65 - 25) \cdot 100^2}{400 \cdot (400 - 2 \cdot 100)} =$$
$$= 5 \cdot 10^{-4} \cdot (x - 100)^2 + 25 - 5 = 5 \cdot 10^{-4} \cdot (x - 100)^2 + 20$$

4.7.2 Deriving Equation for the Conductor Curve by Given *S*, *h*₁, *h*₂, y_{MIN}

Expressing $D_{\max}=D_{\max}(S, h_1, x_{MIN}, y_{MIN})$ from (4.97) by (4.105), and substituting it into (4.98) yields (4.106), which can be applied to define coefficients a and b (according to $y=ax^2+bx+c$) in (4.109), both expressed without x_{MIN}. These are given by (4.107) and (4.108). (The complete deduction is presented in Appendix 6.)

$$D_{\max} = \frac{S^2(h_1 - y_{MIN})}{4x_{MIN}^2} \tag{4.105}$$

$$x_{MIN} = \frac{S(h_1 - y_{MIN})}{h_2 - h_1}\left(\sqrt{\frac{h_2 - y_{MIN}}{h_1 - y_{MIN}}} - 1\right) \qquad \forall \quad h_1 \neq h_2 \tag{4.106}$$

$$a = \frac{h_1 - y_{MIN}}{x_{MIN}^2} = \left[\frac{h_2 - h_1}{S\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)}\right]^2 \tag{4.107}$$

$$b = -2 \cdot \frac{h_1 - y_{MIN}}{x_{MIN}} = \frac{-2(h_2 - h_1)}{S\left(\sqrt{\frac{h_2 - y_{MIN}}{h_1 - y_{MIN}}} - 1\right)} \tag{4.108}$$

Taking into consideration (4.16), (4.96) (4.107) and (4.108), the parabolic equation for the conductor curve, $y=y(S, h_1, h_2, y_{MIN}, x)$, in general form is given by (4.109):

$$y(x) = \left[\frac{h_2 - h_1}{S\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)}\right]^2 \cdot x^2 - \frac{2(h_2 - h_1)}{S\left(\sqrt{\frac{h_2 - y_{MIN}}{h_1 - y_{MIN}}} - 1\right)} x + h_1 \qquad \forall \quad h_1 \neq h_2 \quad \wedge \quad x \in [0, S] \tag{4.109}$$

In this case x, y coordinates of the two support points (defined by given data S, h_1 and h_2) and only y–coordinate of the vertex point are sufficient data for defining the new parabolic equation for the conductor curve, which is usable for the determination of the conductor height, related to x–axis, at any point of the span. The equation is also applicable for drawing the conductor curve. Correctness of (4.109) is verified below by the use of the input data from Example 4.1 (case 3), but y_{MIN} is applied here instead of $D_{\max}$. The obtained equation is the same as $y_3(x)$ in standard form (4.83), defined in the mentioned example.

$$y(x) = \left[\frac{25 - 65}{400 \cdot \left(\sqrt{25 - 20} - \sqrt{65 - 20}\right)}\right]^2 \cdot x^2 - \frac{2 \cdot (25 - 65)}{400 \cdot \left(\sqrt{\frac{25 - 20}{65 - 20}} - 1\right)} x + 65$$

$$y(x)=\left[\frac{-40}{400\cdot\left(\sqrt{5}-\sqrt{45}\right)}\right]^2\cdot x^2-\frac{2\cdot(25-65)}{400\cdot\left(\sqrt{\frac{5}{45}}-1\right)}x+65=$$

$$=\left[\frac{-1}{10\cdot\left(\sqrt{5}-3\sqrt{5}\right)}\right]^2\cdot x^2-\frac{2\cdot(-40)}{400\cdot\left(\frac{\sqrt{5}}{3\sqrt{5}}-1\right)}x+65=$$

$$=\left[\frac{-1}{10\cdot\left(-2\sqrt{5}\right)}\right]^2\cdot x^2-\frac{-80}{400\cdot\left(\frac{1}{3}-1\right)}x+65=0.0005\cdot x^2-0.3\cdot x+65$$

Vertex form of (4.109) is given by (4.110). The correctness of it is also verified numerically using the same data which are used in (4.109). The result is vertex form of $y_3(x)$ from Example 4.1 (case 3).

$$y(x)=\left[\frac{h_2-h_1}{S\left(\sqrt{h_2-y_{MIN}}-\sqrt{h_1-y_{MIN}}\right)}\right]^2\cdot\left[x-\frac{S(h_1-y_{MIN})}{h_2-h_1}\left(\sqrt{\frac{h_2-y_{MIN}}{h_1-y_{MIN}}}-1\right)\right]^2+y_{MIN} \tag{4.110}$$

$$\forall \quad h_1\neq h_2 \quad \wedge \quad x\in[0,S]$$

$$y(x)=\left[\frac{25-65}{400\cdot\left(\sqrt{25-20}-\sqrt{65-20}\right)}\right]^2\cdot\left[x-\frac{400\cdot(65-20)}{25-65}\left(\sqrt{\frac{25-20}{65-20}}-1\right)\right]^2+20=$$

$$=0{,}0005\cdot\left[x-\frac{400\cdot 45}{-40}\left(\sqrt{\frac{5}{45}}-1\right)\right]^2+20=0.0005\cdot\left[x+450\cdot\left(\frac{\sqrt{5}}{3\sqrt{5}}-1\right)\right]^2+20=$$

$$=0{,}0005\cdot\left[x+450\cdot\left(\frac{-2}{3}\right)\right]^2+20=0.0005\cdot(x-300)^2+20$$

4.8 Maximum Sag and Low Point Sag in Inclined Spans

Similarly to the previous section, this one is also based on the use of 5 coordinates of three points and the maximum sag is unknown datum. Taking into consideration the importance of the maximum sag, it is worth deriving its formula for an easy and quick computation. This section shows new formulas for the maximum sag and also the low point sag in the two cases, when *x* or *y* coordinate of the vertex point is given, besides the given both coordinates of the two support points. All new formulas are obtained by combining the expressions from Section 4.7 and are usable in inclined spans strictly.

4.8.1 Case 1 (Given Data are *S*, *h*1, *h*2, x_{MIN})

In this case $D_{\max}$ can be defined by equating coefficients *a*, from equations (4.92) and (4.103), according to (4.111):

$$a = \frac{4D_{\max}}{S^2} = \frac{h_2 - h_1}{S(S - 2x_{MIN})} \tag{4.111}$$

$$D_{\max} = \frac{S(h_2 - h_1)}{4(S - 2x_{MIN})} \quad \forall \quad h_1 \neq h_2 \tag{4.112}$$

The obtained formula for the maximum sag is applicable in all inclined spans. Its correctness can be easily checked by using data (S, h_1, h_2, x_{MIN}) from Example 4.1 (case 1), where the value of $D_{\max}$ is 20 metres and $h_1 < h_2$.

$$D_{\max} = \frac{S(h_2 - h_1)}{4(S - 2x_{MIN})} = \frac{400 \cdot (65 - 25)}{4 \cdot (400 - 2 \cdot 100)} = \frac{400 \cdot 40}{4 \cdot 200} = 20\,\text{m}$$

This way the correctness of (4.112) is confirmed.

The formula for the low point sag, $D(x_{MIN})$, is defined below by substituting (4.112) into (4.80), as follows:

$$D(x_{MIN}) = \frac{S(h_2 - h_1)}{4(S - 2x_{MIN})} \left\{ 1 - \left[\frac{h_2 - h_1}{4\dfrac{S(h_2 - h_1)}{4(S - 2x_{MIN})}} \right]^2 \right\} = \frac{S(h_2 - h_1)}{4(S - 2x_{MIN})} \cdot \frac{S^2 - (S - 2x_{MIN})^2}{S^2} \tag{4.113}$$

$$D(x_{MIN}) = \frac{(h_2 - h_1) \cdot \left[S^2 - \left(S^2 - 4Sx_{MIN} + 4x_{MIN}^2 \right) \right]}{4S(S - 2x_{MIN})} = \frac{4(h_2 - h_1)(S - x_{MIN})x_{MIN}}{4S(S - 2x_{MIN})} \tag{4.114}$$

$$D(x_{MIN}) = \frac{(h_2 - h_1)(S - x_{MIN})x_{MIN}}{S(S - 2x_{MIN})} \quad \forall \quad h_1 \neq h_2 \quad \wedge \quad x_{MIN} \in [0, S] \tag{4.115}$$

The previous formula can be examined by the use of the same data as in the case of checking (4.103). According to Example 4.1 (case 1), the result should be 15 metres.

$$D(x_{MIN}) = \frac{(65 - 25) \cdot (400 - 100) \cdot 100}{400 \cdot (400 - 2 \cdot 100)} = \frac{40 \cdot 300 \cdot 100}{400 \cdot 200} = \frac{1\,200\,000}{80\,000} = 15\,\text{m}$$

4.8.2 Case 2 (Given Data are *S*, *h*1, *h*2, y_{MIN})

When y_{MIN} is given instead of x_{MIN}, the formula for the maximum sag (4.117) is obtainable by equating coefficients *a*, from equations (4.92) and (4.109), according to (4.116):

$$a = \frac{4D_{\max}}{S^2} = \left[\frac{h_2 - h_1}{S\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}} \right)} \right]^2 \tag{4.116}$$

$$D_{\max} = \left[\frac{h_2 - h_1}{2\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)} \right]^2 \qquad \forall \quad h_1 \neq h_2 \tag{4.117}$$

The correctness of (4.117) is verified below, by using input data (S, h_1, h_2, y_{MIN}) from Example 4.1 (case 3), where $D_{\max}$=20 metres and h_1>h_2.

$$D_{\max} = \left[\frac{25-65}{2\left(\sqrt{25-20} - \sqrt{65-20}\right)} \right]^2 = \left[\frac{-40}{2\left(\sqrt{5} - \sqrt{45}\right)} \right]^2 = \left[\frac{-40}{2\left(\sqrt{5} - 3\sqrt{5}\right)} \right]^2 = \left[\frac{-40}{2 \cdot (-2\sqrt{5})} \right]^2 = 20\,\mathrm{m}$$

The formula for the low point sag given by (4.120) can be defined by substituting (4.117) into (4.80) and rearranging the obtained expression, as it is shown in the following lines:

$$D(x_{MIN}) = \left[\frac{h_2 - h_1}{2\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)} \right]^2 \cdot \left\{ 1 - \left[\frac{h_2 - h_1}{4\left(\frac{1}{2} \cdot \frac{h_2 - h_1}{\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}} \right)^2} \right]^2 \right\} \tag{4.118}$$

$$D(x_{MIN}) = \left[\frac{h_2 - h_1}{2\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)} \right]^2 \cdot \left[1 - \frac{\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)^4}{\left(h_2 - h_1\right)^2} \right] \tag{4.119}$$

$$D(x_{MIN}) = \frac{\left(h_2 - h_1\right)^2 - \left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)^4}{\left[2\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)\right]^2} \qquad \forall \quad h_1 \neq h_2 \quad \wedge \quad x_{MIN} \in [0, S] \tag{4.120}$$

The previous formula for the low point sag is examined by the use of the same data as in the case of checking (4.109). According to Example 4.1 (case 3), the correct result is 15 metres.

$$D(x_{MIN}) = \frac{(25-65)^2 - \left(\sqrt{25-20} - \sqrt{65-20}\right)^4}{\left[2\left(\sqrt{25-20} - \sqrt{65-20}\right)\right]^2} = \frac{(-40)^2 - \left(\sqrt{5} - \sqrt{45}\right)^4}{\left[2\left(\sqrt{5} - \sqrt{45}\right)\right]^2} = \frac{1600 - \left(\sqrt{5} - 3\sqrt{5}\right)^4}{\left[2\left(\sqrt{5} - 3\sqrt{5}\right)\right]^2} =$$

$$= \frac{1600 - \left(-2\sqrt{5}\right)^4}{\left[2 \cdot \left(-2\sqrt{5}\right)\right]^2} = \frac{1600 - 2^4 \cdot 5^2}{4^2 \cdot 5} = \frac{1600 - 400}{80} = 15\,\mathrm{m}$$

Notice that case 3 presents the inclined span where the right–hand side support point is lower than the left–hand side one, while case 1 shows an opposite example, i.e. the right–hand side support point is higher than the other one.

4.9 Use of 1/cosψ Multiplier in Parabola Based Calculation

Several new equations for the conductor and the sag curves have been derived in the previous sections, as well as the coordinates of the parabola's vertex point and a formula for the low point sag. However, the parabola based calculation for OHL design cannot be considered as entirely completed without the appropriate discussion of 1/cosψ multiplier's use, which has an importance in inclined spans. Literature generally recommends the application of the mentioned multiplier, but wide mathematical background is not available. This section gives an adequate explanation of how 1/cosψ affects the parabolic curve and the sag giving an opportunity for the readers to get a clear resolution in connection with the use of 1/cosψ. Since the parabola is the approximation of the catenary, it is necessary to compare the parabola with the catenary, without and then also with the use of 1/cosψ for a parabola modification. Taking into consideration that the multiplier has an effect in inclined spans, the comparison of the mentioned curves has to be related to inclined spans. It produces an additional problem namely that literature discusses the parabolic approximation of the catenary in level spans only. For this reason the currently absent mathematical background for the parabolic approximation of the catenary in inclined spans is created below in the separate section, in order to make possible the adequate discussion of 1/cosψ multiplier's use.

4.9.1 Parabolic Approximation of the Catenary in Inclined Spans

As a final result in this section, the equation of the parabola has been defined, which presents the approximation of the catenary in an inclined span, but in case of $h_1=h_2=h$ changes into equation which presents the approximation of the catenary in a level span. Note: 1/cosψ multiplier is not used here.

Initial conditions are the following: S, h_1 and h_2 are common data for both the catenary and the parabola, in inclined and level spans as well. The catenary parameter is the same in both spans, as well as the parabola's coefficient a_p. Due to this condition, the maximum sag of the parabola can be expressed by the given catenary parameter, c, as it is detailed below, in order to determine the y–coordinate of the point $C(x_C;y_C)$ in Fig. 4.8. Since the support points are fixed and known, the parabolic equation can be derived by the 3–point method presented in Section 4.4. Note: in order to make the symbols of the catenary parameter and the parabola's third coefficient differ from each other, parabola's coefficients a, b, c get here subscript p.

Due to (4.121) [72], the parabolic approximation of the catenary in a level span is given by (4.122).

$$y_{\text{cat}}(x) = c\cdot\cosh\left(\frac{x}{c}\right) = c\sum_{n=0}^{\infty}\frac{1}{(2n)!c^{2n}}x^{2n} = c\cdot\left(1+\frac{1}{2!c^2}x^2+\frac{1}{4!c^4}x^4+\frac{1}{6!c^6}x^6+....\right)= \\ = c+\frac{1}{2!c}x^2+\frac{1}{4!c^3}x^4+\frac{1}{6!c^5}x^6+....\approx c+\frac{1}{2!c}x^2 = c+\frac{1}{2c}x^2 \tag{4.121}$$

$$y_{\text{par}}(x) = c+\frac{1}{2c}x^2 \tag{4.122}$$

The parabola curve of (4.122) has a vertex at point (0;c). Replacing the parabola in the coordinate system in a way that the vertex point is set in the origin, the equation of the parabola changes into (4.123), where 1/2c is the coefficient of the parabola and is denoted by a_p.

$$y(x) = \frac{1}{2c} x^2 = a_p \cdot x^2 \tag{4.123}$$

Equating a_p from (4.123) with a from (4.92), the parabola's maximum sag can be expressed by the catenary parameter, c, as follows:

$$\frac{1}{2c} = \frac{4D_{max}}{S^2} \quad \Rightarrow \quad D_{max} = \frac{S^2}{8c} \tag{4.124}$$

Since c is the same in level and inclined spans, as well as the span length and also coefficient a_p, it means that the maximum sag of the parabola is the same in both spans. Knowing the parabola's feature, that its maximum sag is always located at a mid–span and using the coordinate system according to Fig. 4.8, the third point, C, of the parabola – due to (4.125) – is here given by (4.126), while the points A and B are as in Fig. 4.8.

$$y_C = \frac{h_1 + h_2}{2} - D_{max} = \frac{h_1 + h_2}{2} - \frac{S^2}{8c} \tag{4.125}$$

$$C\left(\frac{S}{2}; \frac{h_1 + h_2}{2} - \frac{S^2}{8c}\right) \tag{4.126}$$

Based on three points A, B, C of the parabolic curve, the system of three algebraic equations, (4.127) – (4.128), in three unknowns (a_p, b_p, c_p) can be written by using the basic parabola equation in general form (4.16).

$$h_1 = c_p \tag{4.127}$$

$$h_2 = a_p S^2 + b_p S + c_p \tag{4.128}$$

$$\frac{h_1 + h_2}{2} - \frac{S^2}{8c} = a_p \left(\frac{S}{2}\right)^2 + b_p \left(\frac{S}{2}\right) + c_p \tag{4.129}$$

The system of the three previous equations, expressed in matrix form, is the following:

$$\begin{bmatrix} 0 & 0 & 1 \\ S^2 & S & 1 \\ (S/2)^2 & S/2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_p \\ b_p \\ c_p \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ (h_1 + h_2)/2 - S^2/8c \end{bmatrix} \tag{4.130}$$

The solution of this system is given by (4.131) and it presents coefficients of the parabola equation (4.16).

$$\begin{bmatrix} a_p \\ b_p \\ c_p \end{bmatrix} = \begin{bmatrix} 1/2c \\ (h_2 - h_1)/S - S/2c \\ h_1 \end{bmatrix} \tag{4.131}$$

After substituting a_p, b_p, c_p into (4.16), the equation which presents the parabolic approximation of the catenary in an inclined span, given in general form is (4.132):

$$y(x) = \frac{1}{2c}x^2 + \left(\frac{h_2 - h_1}{S} - \frac{S}{2c}\right)x + h_1 \qquad x \in [0, S] \tag{4.132}$$

The previous equation transformed into vertex form is (4.133) and hence the coordinates of the vertex point are expressed by (4.134) and (4.135).

$$y(x) = \frac{1}{2c}\left\{x - \frac{S}{2}\left[1 - \frac{2c}{S^2}(h_2 - h_1)\right]\right\}^2 + h_1 - \frac{1}{2c}\left\{\frac{S}{2}\left[1 - \frac{2c}{S^2}(h_2 - h_1)\right]\right\}^2 \qquad x \in [0, S] \tag{4.133}$$

$$x_{MIN} = \frac{S}{2}\left[1 - \frac{2c}{S^2}(h_2 - h_1)\right] \tag{4.134}$$

$$y_{MIN} = h_1 - \frac{1}{2c}\left\{\frac{S}{2}\left[1 - \frac{2c}{S^2}(h_2 - h_1)\right]\right\}^2 \tag{4.135}$$

Using (4.124) and the equations (4.14) and (4.15), the sag equations in general and vertex form are given as follows:

$$D(x) = -\frac{1}{2c}x^2 + \frac{S}{2c}x \qquad x \in [0, S] \tag{4.136}$$

$$D(x) = -\frac{1}{2c}\left(x - \frac{S}{2}\right)^2 + \frac{S^2}{8c} \qquad x \in [0, S] \tag{4.137}$$

Derived equations (4.132), (4.133), (4.136) and (4.137) are all universal, since each of them is usable in both types of an inclined span ($h_1 < h_2$ and $h_1 > h_2$) and in level spans ($h_1 = h_2$) as well.

4.9.2 Parabolic Approximation of the Catenary in Level Spans

According to the above mentioned, in case of $h_1 = h_2 = h$ the equation which presents the parabolic approximation of the catenary in an inclined span changes into an appropriate equation concerned to a level span. This way equations (4.132) and (4.133) change into the two following ones, while the vertex point is then given by (4.140).

$$y(x) = \frac{1}{2c}x^2 - \frac{S}{2c}x + h \qquad x \in [0, S] \tag{4.138}$$

$$y(x) = \frac{1}{2c}\left(x - \frac{S}{2}\right)^2 + h - \frac{S^2}{8c} \qquad x \in [0, S] \tag{4.139}$$

$$MIN\left(\frac{S}{2};h-\frac{S^2}{8c}\right) \tag{4.140}$$

Due to (4.33), the sag equation in a level span is the same as in an inclined span, i.e. is given by (4.136) or (4.137).

4.9.3 Mathematical Background of 1/cosψ Multiplier's Use

Based on (4.137) the sag of the parabolic curve does not depend on Δh (and thus neither on the span inclination), because there is no $\Delta h = h_2 - h_1$ in the sag equation. Furthermore, coefficient a is also independent of the span inclination (or Δh).

Since $D_{inc}(x) \equiv D_{lev}(x)$, it means that mathematically there is no difference between the sags of the parabola in inclined and level spans and it is valid at each point of the span. On the other hand, the sag of the catenary is dominantly characterised by the relation $D_{inc}(x) > D_{lev}(x)$ on the interval $(0,S)$. Thus, there is an evident contradiction between the sags of the catenary and the parabola, which can have a negative impact on the approximation of the catenary by the parabola. In practice it is partly compensated by using 1/cosψ multiplier [41] which increases the sag of the parabola in an inclined span in comparison to the sag in a level span. The following formula is used:

$$D_{inc\psi}(x) = \frac{1}{\cos\psi} \cdot D_{lev}(x) \quad \forall \quad 0 \le x \le S \tag{4.141}$$

It is obvious that the previous expression concerns not only to the mid–span sag, but to a sag at any point within a span. The angle ψ is discussed in Section 3.4.4. Using 1/cosψ multiplier the parabola gets an important feature given by the following relation between the parabola sags in inclined and level spans:

$$D_{inc\psi}(x) > D_{lev}(x) \quad \forall \quad 0 < x < S \tag{4.142}$$

The caused effect is illustrated in Fig. 4.10.

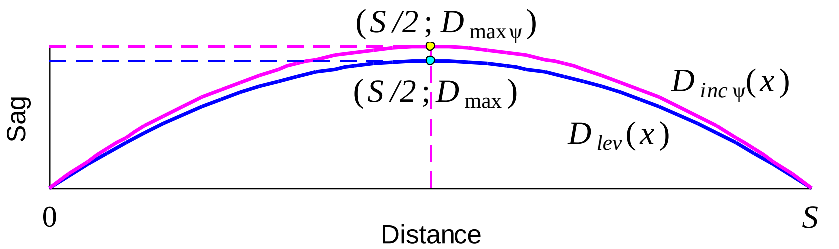


Fig. 4.10: Sag curves in level and inclined spans with the application of 1/cosψ for the latter

Naturally, a larger inclination (i.e. larger angle ψ) of the span causes a bigger increase of 1/cosψ, and hence also the sag in an inclined span. After application of 1/cosψ the maximum sag in an inclined span is denoted by $D_{\max\,\psi}$ and is given by (4.143), where $D_{\max}$ is the maximum sag of the parabola in a level span.

$$D_{\max\,\psi} = \frac{D_{\max}}{\cos\psi} \tag{4.143}$$

The previous expression is the relation between the maximum sags of the parabola in inclined and level spans, when 1/cosψ is used. The sag equation in inclined spans then gets the following form (4.144):

$$D_{inc\psi}(x) = \frac{-4D_{\max\,\psi}}{S^2}\left(x - \frac{S}{2}\right)^2 + D_{\max\,\psi} \qquad x \in [0, S] \tag{4.144}$$

Analysing (4.144), it can be concluded that the maximum sag is still located at a mid–span, thus 1/cosψ does not produce the movement of the maximum sag from the mid–span. It means that the above mentioned feature of the parabola that its maximum sag is always located at a mid–span, can be complemented in a way that it is even independent of the application of 1/cosψ.

Taking into consideration (4.143), the previous equation can be written by (4.145) and is usable for the sag calculation at any point within the inclined span by the given maximum sag in a level span, $D_{\max}$.

$$D_{inc\psi}(x) = \frac{-4D_{\max}}{S^2\cos\psi}\left(x - \frac{S}{2}\right)^2 + \frac{D_{\max}}{\cos\psi} \qquad x \in [0, S] \tag{4.145}$$

Note that the sign of angle ψ does not have an effect on the results obtained by (4.145), since the cosine is an *even* function:

$$\cos(-\psi) = \cos(\psi) \tag{4.146}$$

Furthermore, (4.145) is a universal sag equation, because it can also be used for a sag calculation at any point within a level span. In that case ψ=0 and then (4.145) becomes (4.15), since cos(0)=1. In the case of the mid–span (x=S/2), equation (4.145) changes into (4.143).

The equation for the modified parabolic conductor curve in inclined spans can be obtained by the use of (4.147) or simply substituting (4.143) into (4.73) instead of $D_{\max}$. The actual equation in vertex form is given by (4.148), and is based on a given maximum sag in a level span, $D_{\max}$.

$$y_{inc\psi}(x) = y_{line}(x) - D_{inc\psi}(x) \qquad x \in [0, S] \tag{4.147}$$

$$y_{inc\psi}(x) = \frac{4D_{\max}}{S^2\cos\psi}\left[x - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\cos\psi\right)\right]^2 + h_1 - \frac{D_{\max}}{\cos\psi}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\cos\psi\right)^2 \quad x \in [0, S] \tag{4.148}$$

The previous equation is valid for both cases of inclined spans, $h_1<h_2$ and $h_1>h_2$. In a level span ($\psi=0$ and $h_1=h_2=h$) an actual equation becomes (4.18). The simplified form of (4.148) is the following:

$$y_{inc\psi}(x) = a_\psi (x - x_{MIN\,\psi})^2 + y_{MIN\,\psi} \qquad x \in [0, S] \tag{4.149}$$

According to (4.149), coefficient a_ψ and the coordinates of the vertex point $MIN_\psi(x_{MIN\,\psi};y_{MIN\,\psi})$ of the modified parabola are easily readable from (4.148) and are given by (4.150) – (4.152):

$$a_\psi = \frac{4D_{max}}{S^2 \cos\psi} \tag{4.150}$$

$$x_{MIN\,\psi} = \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{max}}\cos\psi\right) \tag{4.151}$$

$$y_{MIN\,\psi} = h_1 - \frac{D_{max}}{\cos\psi}\left(1 - \frac{h_2 - h_1}{4D_{max}}\cos\psi\right)^2 \tag{4.152}$$

Since $1/\cos\psi > 1$, and also $D_{max} > 0$ and $S > 0$, therefore $a_\psi > a$. Knowing the relation between the parabola's parameter, p, and the coefficient a of the parabola, given by (4.153) [95], and considering expressions (4.154) and (4.155), it can be concluded that $p_\psi < p$. (The parabola's parameter is considered as the distance from the focus point to the directrix, see in Fig. 4.2.)

$$p = \frac{1}{2a} \tag{4.153}$$

$$p = \frac{S^2}{8D_{max}} \tag{4.154}$$

$$p_\psi = \frac{S^2 \cos\psi}{8D_{max}} \tag{4.155}$$

Thus, $1/\cos\psi$ increases the parabola's sag in an inclined span by increasing the parabola's coefficient, a, i.e. by reducing the parabola's parameter, p, in dependence of the span inclination (or Δh). Summarizing the above discussion, the basic and the modified parabolas can be described as two parabolas with a different parameter and common start and end points which are located on different elevation. As a consequence, the vertex points of the two curves differ in their position.

Finally, it is worth mentioning that the equation for the modified parabolic conductor curve and the corresponding sag equation in general form are given by (4.156) and (4.157):

$$y_{inc\psi}(x) = \frac{4D_{max}}{S^2\cos\psi}x^2 + \frac{1}{S}\left(h_2 - h_1 - \frac{4D_{max}}{\cos\psi}\right)x + h_1 \qquad x \in [0, S] \tag{4.156}$$

$$D_{inc\psi}(x) = \frac{-4D_{\max}}{S^2\cos\psi}x^2 + \frac{4D_{\max}}{S\cos\psi}x \qquad x \in [0,S] \tag{4.157}$$

Returning to (4.34), due to (4.141) the corresponding relation for the modified parabola is the following:

$$\frac{D_{inc\psi}^{(\mathrm{par}\,\psi)}(x)}{D_{lev}^{(\mathrm{par})}(x)} = \left(\frac{D_{inc\psi}^{(\mathrm{par}\,\psi)}}{D_{lev}^{(\mathrm{par})}}\right)(x) = \frac{1}{\cos\psi} \qquad 0 < x < S \tag{4.158}$$

Thus, when $1/\cos\psi$ multiplier is used, the quotient of the sag functions in inclined and level spans on the interval $(0,S)$ is a constant. Another important conclusion in connection with the application of $1/\cos\psi$, expressed mathematically, is the following one:

$$D_{inc\psi}\left(-x+\frac{S}{2}\right) = D_{inc\psi}\left(x+\frac{S}{2}\right) \quad \Rightarrow \quad D_{inc\psi}\left(x+\frac{S}{2}\right) \ \underline{\text{is an } \textit{even} \text{ function}} \tag{4.159}$$

4.9.3.1 Equation for the Modified Parabolic Conductor Curve Based on Catenary Parameter

Considering the mathematical background of $1/\cos\psi$ multiplier's use and (4.73), the vertex form of the equation for the modified parabolic conductor curve in inclined spans, which is based on the given catenary parameter, can be directly obtained by substituting (4.160) into (4.73) instead of $D_{\max}$. Another mode is complementing (4.133) in a way as (4.73) is complemented into (4.148). The actual equation is (4.161).

$$D_{\max\psi} = D_{\max}\cdot\frac{1}{\cos\psi} = \frac{S^2}{8c}\cdot\frac{1}{\cos\psi} \tag{4.160}$$

$$y_{inc\psi}(x) = \frac{1}{2c\cdot\cos\psi}\left\{x-\frac{S}{2}\left[1-\frac{2c(h_2-h_1)}{S^2}\cos\psi\right]\right\}^2 + h_1 - \frac{1}{2c\cdot\cos\psi}\left\{\frac{S}{2}\left[1-\frac{2c(h_2-h_1)}{S^2}\cos\psi\right]\right\}^2$$
$$x \in [0,S] \tag{4.161}$$

4.9.4 Practical Example without and with the Use of 1/cosψ Multiplier

Having the equations for the parabolic approximation of the catenary for both cases, without and with the application of $1/\cos\psi$, they can be used in a practical example in order to analyse the effect of the mentioned multiplier. The input data are given in Table 4.3 concerning the catenary in three cases (one level and two inclined spans). Data S, h_1 and c are common in each case. Using the above obtained expressions, firstly the vertex's coordinates of each curve and the parabola's coefficient a are determined. After that the equation of the catenary, and the equations of its approximations by basic and modified parabolas are defined in all three cases separately. The numerical example helps to draw the concrete conclusion in connection with the application of 1/cosψ, which is analytically not possible.

Example 4.2

Table 4.3: Input data in Example 4.2

Data	Case 1	Case 2	Case 3
S [m]	400	400	400
h_1 [m]	30	30	30
h_2 [m]	30	70	110
c [m]	1000	1000	1000

Based on data from Table 4.3, three actual catenary curves are drawn in Fig. 4.11, by the use of (2.34). The vertex points of the curves are denoted by *MIN* 1, *MIN* 2 and *MIN* 3. Both inclined spans are a classic type, thus the vertex point is also the low point in each case.

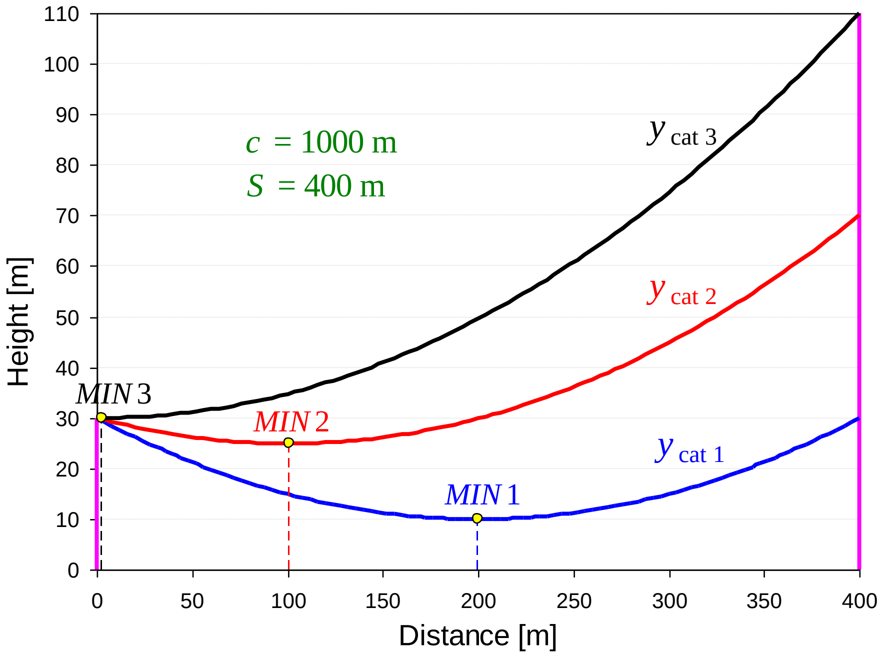


Figure 4.11: Catenary curves in Example 4.2

Table 4.4: Results of case 1 in Example 4.2

Results	Catenary	Basic parabola	Modified parabola
a [m^{-1}]	–	$5{\cdot}10^{-4}$	$5{\cdot}10^{-4}$
x_{MIN} [m]	200	200	200
y_{MIN} [m]	9.933244	10	10

Thus, in a level span the x–coordinate of the vertex point in each case is $S/2$. Furthermore, the modified parabola does not differ from the basic one, or, in other words, $1/\cos\psi$ has no effect in a level span. Equations of the catenary and its approximation are given as follows:

$$y_{\text{cat}\,1}(x) = 2\cdot 10^3 \cdot \sinh^2\left(\frac{x-200}{2\cdot 10^3}\right) + 9.933244 \qquad x \in [0,400] \tag{4.162}$$

$$y_{\text{par}\,1}(x) = y_{\text{par}\,\psi\,1}(x) = 5\cdot 10^{-4}\cdot (x-200)^2 + 10 \qquad x \in [0,400] \tag{4.163}$$

Table 4.5: Results of case 2 in Example 4.2

Results	Catenary	Basic parabola	Modified parabola
a [m^{-1}]	–	$5{\cdot}10^{-4}$	$50249{\cdot}10^{-8}$
x_{MIN} [m]	100.826218	100	100.496281
y_{MIN} [m]	24.912729	25	24.925063

$$y_{\text{cat}\,2}(x) = 2\cdot 10^3 \cdot \sinh^2\left(\frac{x-100.826218}{2\cdot 10^3}\right) + 24.912729 \qquad x \in [0,400] \tag{4.164}$$

$$y_{\text{par}\,2}(x) = 5\cdot 10^{-4}\cdot (x-100)^2 + 25 \qquad x \in [0,400] \tag{4.165}$$

$$y_{\text{par}\,\psi\,2}(x) = 50249\cdot 10^{-8}\cdot (x-100.496281)^2 + 24.925063 \qquad x \in [0,400] \tag{4.166}$$

Table 4.6: Results of case 3 in Example 4.2

Results	Catenary	Parabola	Modified parabola
a [m^{-1}]	–	$5{\cdot}10^{-4}$	$5099{\cdot}10^{-7}$
x_{MIN} [m]	2.611421	0	3.883865
y_{MIN} [m]	29.996590	30	29.992308

$$y_{\text{cat}\,3}(x) = 2\cdot 10^3 \cdot \sinh^2\left(\frac{x-2.611421}{2\cdot 10^3}\right) + 29.996590 \qquad x \in [0,400] \tag{4.167}$$

$$y_{\text{par}\,3}(x) = 5\cdot 10^{-4}\cdot x^2 + 30 \qquad x \in [0,400] \tag{4.168}$$

$$y_{\text{par}\,\psi\,3}(x) = 5099\cdot 10^{-7}\cdot (x-3.883865)^2 + 29.992308 \qquad x \in [0,400] \tag{4.169}$$

Despite the fact that the three equations are mathematically different, their curves are so similar that the difference between them is hardly visible on the small diagram, as it is in Fig. 4.11. For this reason the curves of $\Delta y(x) = y_{\text{par}}(x) - y_{\text{cat}}(x)$ and $\Delta y_\psi(x) = y_{\text{par}\,\psi}(x) - y_{\text{cat}}(x)$ are drawn in Fig. 4.12, on the common diagram. This way the effect of $1/\cos\psi$, when the catenary is mathematically approximated by a parabola, is made visible and hence can be evaluated. Note that due to (4.163), $\Delta y_1(x) \equiv \Delta y_{\psi 1}(x)$.

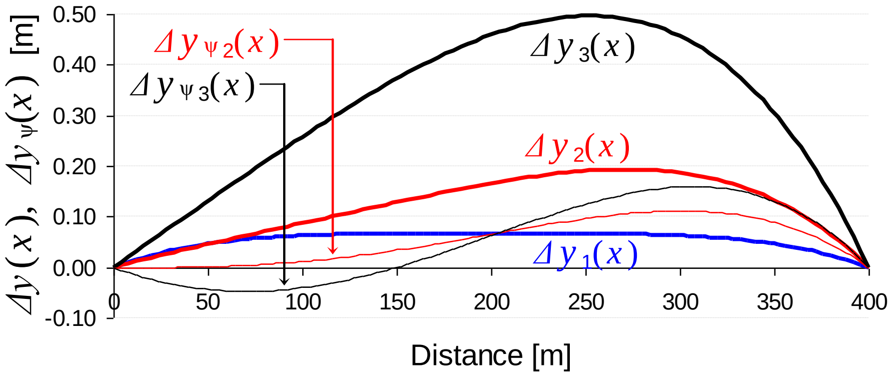


Figure 4.12: Curves $\Delta y(x) = y_{par}(x) - y_{cat}(x)$ and $\Delta y_{\psi}(x) = y_{par\ \psi}(x) - y_{cat}(x)$ in Example 4.2

The analysis of all curves in the previous figure brings a clear conclusion that the use of 1/cosψ appropriately reduces the deviation of the parabola from the catenary. It practically means that the modified parabola, $y_{par\ \psi}(x)$, resembles the catenary better than the basic parabola, $y_{par}(x)$. It is expressed mathematically by the following inequality with the catenary and its basic and modified approximations by the parabola, valid for inclined spans:

$$\left| y_{par\,\psi}(x) - y_{cat}(x) \right| < y_{par}(x) - y_{cat}(x) \quad \forall \quad 0 < x < S \quad \wedge \quad h_1 \neq h_2 \qquad (4.170)$$

Thus, the use of 1/cosψ multiplier is advisable. However, the phenomena from case 3 has to be mentioned, where $\Delta y_{\psi}(x)$ changes sign within the span and the y–coordinate of the modified parabola's vertex point is positioned lower than the y–coordinate of the catenary's vertex point (see results in Table 4.6). From the aspect of a clearance calculation between the live overhead conductors and the ground or objects, it is a disadvantage. This is one of the reasons because the use of a parabola in spans with significant inclination (or h_2–h_1) is generally avoided, i.e. it is recommended to consider the conductor curve as a catenary.

4.10 Summary of the Chapter

Chapter 4 deals with a parabola based calculation for OHL design using the coordinate system for drawing the conductor curve in the same way as in the case of the catenary, introduced in Chapter 2. Due to it, a universal parabolic equation is derived, which is applicable for the conductor height calculation in all spans, independently of the span inclination. The input data are the same as in the case of the catenary, but instead of the catenary parameter the maximum sag of the parabola is used. Besides the universal parabolic equation for a frequent use, the special equation for the conductor curve in inclined spans is also derived by two support points and only one coordinate of the vertex point. In this case the maximum sag is not a needed datum, but can be calculated as its formula is also defined. Deductions in this chapter are generally based on the feature of the parabola that its maximum sag is always located at a mid–span. The applicability of the new equations in practice is presented in numerical examples containing different span types.

A separated section details the effect of $1/\cos\psi$ multiplier in the case of the parabola and also evaluates its use in practice. It is shown that $1/\cos\psi$ modifies the parabola in a way that the sag in inclined span is increased in comparison to the sag in a level span. This is valid at each point of the span except the start and end points. In fact $1/\cos\psi$ modifies both the conductor curve and the sag curve, but the point is that both modified curves are still parabolas. In order to make possible the comparison of the basic parabola and also the modified parabola with the catenary, the equation for the parabolic approximation of the catenary in an inclined span is derived previously. This unique equation, which contains the catenary parameter instead of the parabola's maximum sag, is a universal one as it is usable in all span types. Differently from the case of the catenary where the use of $1/\cos\psi$ multiplier should be avoided, since it produces errors in sag calculation, in case of the parabola it is advisable. Its use is recommended, because it results a better parabolic approximation of the catenary. According to a related numerical example, this positive effect of $1/\cos\psi$ multiplier is more significant in spans with a high inclination.

5 UNIVERSAL FORMULAS FOR THE CONDUCTOR LENGTH

5.1 Introduction and Related Research

Because of the sag of overhead lines, the conductor within the span is always longer than the span itself. Thus, the conductor length calculation also has an importance when constructing overhead lines. In the case of the parabola based calculation the maximum sag is a necessary datum for the length calculation, whereas in the case of the catenary based calculation it is the parameter of the catenary, besides the span length and the heights of the two support points as necessary data in each case.

Studies for OHL design generally give a solution for the conductor length calculation in level spans, but very rarely in inclined ones. Hence, the length formula for level spans is frequently in use in inclined spans as well, despite the fact that it produces errors in calculations. Furthermore, the available length formulas are defined for determining the conductor length in a full span, i.e. for frequent conventional tasks, but not for rare unconventional tasks, for instance, the conductor length calculation in an arbitrary part of the span, either a level or inclined one. These are the reasons for deriving the algorithm for calculation of the conductor length, which ensures adequate calculation in each case, i.e. in level and inclined spans as well, and also in a full span and in its part. Such a complex task can be effectively solved by the application of the integral calculus. Naturally, the calculation has to be shown separately when the conductor curve is considered as a parabola or a catenary.

Regarding to related studies, lots of them (e.g. [4,78,83,96]) show the length formula for the catenary conductor curve in a full level span, and then the expansion of the formula into a series is also given, according to (5.1) [97]. The first two terms are commonly used for computing the length of the parabola which is the approximation of the catenary. The other terms in a series are negligible. Hence, due to (5.1) and considering that $c=S^2/8D_{\max}$ (from (4.124)), we can write expression (5.2).

$$\sinh\,x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+..... \tag{5.1}$$

$$L_{\mathrm{cat}}^{(lev)}=2c\cdot\sinh\,\frac{S}{2c}\approx 2c\left(\frac{S}{2c}+\left(\frac{S}{2c}\right)^3\cdot\frac{1}{3!}\right)=S+\frac{S^3}{24c^2}=S+\frac{S^3}{24}\cdot\left(\frac{8D_{\max}}{S^2}\right)^2=S+\frac{8D_{\max}^2}{3S}=L_{\mathrm{par}}^{(lev)} \tag{5.2}$$

Thus, the length formula for the catenary in a level span is shown far left [35,40,44], whereas the length formula for the parabolic approximation of the catenary can be seen far right

[2,60,98,99] in expression (5.2), where datum D_{max} presents the maximum sag of the parabola.

The book [81] is a rare one, which deals with the length calculation of the parabolic conductor curve in inclined spans. On the other hand, [47] is one of the rare studies dealing with the length calculation of the catenary conductor curve in inclined spans. Similarly, publication [100] derives the expression, which is also applicable for the latter length calculation, expressing L. Using labels from this work, the mentioned expression can be rewritten as follows:

$$L^2 - (h_2 - h_1)^2 = 4c^2 \cdot \sinh^2\left(\frac{S}{2c}\right) \tag{5.3}$$

It is evident that if $h_1=h_2$, then (5.3) becomes an expression which can be modified for calculating the length of the catenary in level spans (see (5.2) far left).

The author of the book [41] presents a unique relation between the catenary lengths in inclined and level spans, when the catenary parameter is the same in both spans, as well as the span length. It can be expressed by (5.4), where ψ is the angle of the span inclination.

$$L_{\mathrm{cat}}^{(inc)} = \frac{1}{\cos\psi} \cdot L_{\mathrm{cat}}^{(lev)} \tag{5.4}$$

It is shown in Chapter 3 that the application of $1/\cos\psi$ in the case of the catenary is not a mathematically exact calculation, but an approximate one. Thus, the previous formula can also be classified there. However, the use of (5.4) in the case of low inclinations produces very small errors, which can be neglected. On the other hand, errors in steeply inclined spans can be significant, since the error increases with the span inclination.

Referring to the conductor length in part of the span, the publication [63] has to be mentioned, where the total length of the parabolic conductor curve in an inclined span is given as a sum of the conductor lengths in two subspans. According to the shown method, the x–coordinate of the lowest point of the conductor is the one which divides the given span into two subspans. Thus, these depend on the location of the lowest point, but are not selected arbitrarily.

Taking into consideration the above mentioned, it is obvious that all length formulas for quick targeted calculations of the conductor length are not available. Furthermore, some of the proposed formulas are suited for approximate calculations. In order to complete the collection of the length formulas, the absent ones have to be defined, and the approximate ones have to be replaced by mathematically exact formulas. For this reason, the universal length formula is derived (separately for the parabola and the catenary) in this chapter, which is suitable for obtaining three more formulas and hence to cover together the four characteristic cases of the length calculation, i.e. in level and inclined spans, in a full span and in any span–part. Thus, four different length formulas are defined for the parabola and four others for the catenary. Each of them is an exact formula. The coordinate system is used in the same way as in the three previous chapters with the aim of keeping the uniformity of the entire work. It is well known that the horizontal or/and vertical translation of the curve in the coordinate system does not cause a change in its length.

The structure of this chapter is as follows. After a short overview of related research, which is given in this section, the parabola length calculation is presented in Section 5.2, firstly in inclined spans, then also in level ones. In both span types the calculation is shown separately in part of a span and in a full span. Following the same order as in the case of the parabola, Section 5.3 deals with calculation of the catenary length. Section 5.4 shows a practical example by using the length formulas derived in the two previous sections and the data of the three conductor curves (catenary, basic parabola and modified parabola) from Chapter 4. Section 5.5 gives a short conclusion and a summary of the novel results.

5.2 Parabolic Conductor Curve

The length of the parabola in an inclined span differs from its length in a level span, even if the span length is the same in both cases, as well as the parabola's coefficient a. That is why the separate length formulas are needed regarding to level and inclined spans. In the following, firstly the deduction concerned to an inclined span is shown then the defined final formula is appropriately modified for the use in a level span, considering the latter as a special case of an inclined span where there is no difference in height of the support points.

All new length formulas, presented in this section, are concerned to the basic parabola, but these can be easily transformed to the corresponding length formulas for the modified parabola (see Chapter 4), substituting $D_{max}/\cos\psi$ instead of D_{max}. The lengths of the basic and modified parabola are compared in the scope of Section 5.4.

5.2.1 Calculations in Inclined Spans

This section deals with the conductor length calculation in an inclined span, separately in two possible cases, i.e. in the part of the span and in the full span. For this purpose Fig. 5.1 is used, which is based on Fig. 4.8, but contains two additional points, E and F, needed for the calculation of the conductor length in the part of the span. Thus, the used symbols are all as in Fig. 4.8, except for the here unnecessary L and $D(x_{MIN})$, with additional ones, listed below:

x_1 – start point of part of a span $[x_1, x_2] \subseteq [0, S]$

x_2 – end point of part of a span $[x_1, x_2] \subseteq [0, S]$

$E\ (x_1; y(x_1))$ – start point of the conductor in part of a span

$F\ (x_2; y(x_2))$ – end point of the conductor in part of a span.

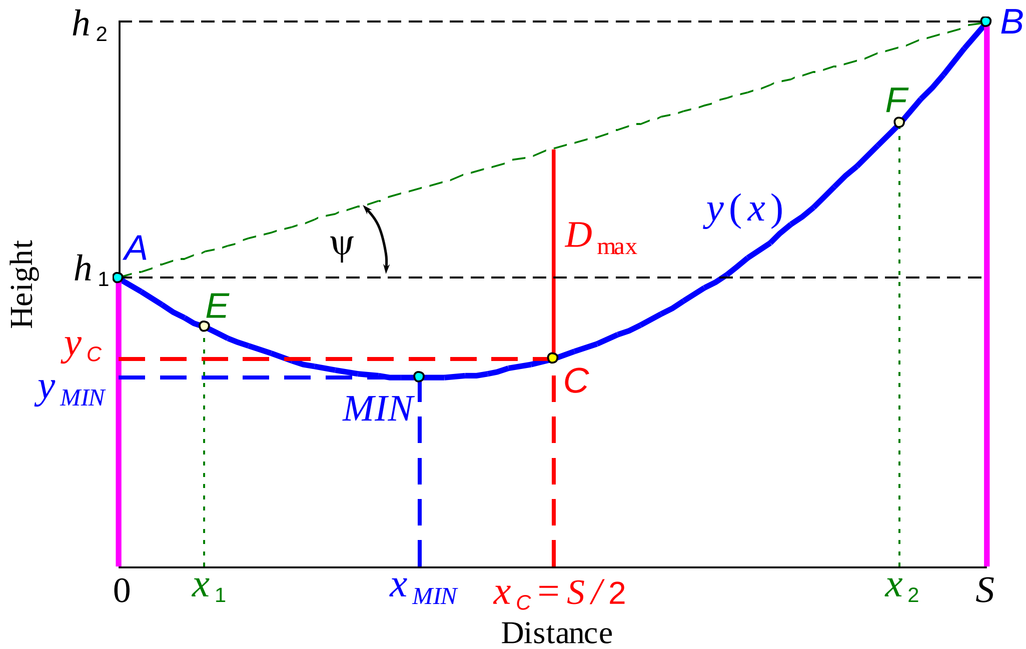


Fig. 5.1: Parabolic conductor curve in an inclined span with $h_1 < h_2$

The equation for the conductor curve (4.73) is an essential for the calculation of the conductor length. The deduction has been simplified by the application of (4.17), expressions for x_{MIN} and a have been used from (4.73) at the end of the deduction.

5.2.1.1 Conductor (Parabola) Length in Part of an Inclined Span

The length of the parabola on the interval $[x_1, x_2]$, shown in Fig. 5.1, can be determined by the following well known mathematical formula for the arc length [74,101,102]:

$$L_{x_1 x_2} = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (5.5)$$

The first derivative of (4.17) is (5.6). Squaring it results in (5.7):

$$\frac{dy}{dx} = 2a(x - x_{MIN}) \tag{5.6}$$

$$\left(\frac{dy}{dx}\right)^2 = [2a(x - x_{MIN})]^2 \tag{5.7}$$

Inserting (5.7) into (5.5) and evaluating the integral [103] by the application of the substitution method are shown below, step by step.

$$L_{x_1 x_2} = \int_{x_1}^{x_2} \sqrt{1 + [2a(x - x_{MIN})]^2}\, dx \tag{5.8}$$

$$2a(x - x_{MIN}) = \sinh t \quad \Rightarrow \quad t = \operatorname{arcsinh}(2a(x - x_{MIN})) \tag{5.9}$$

$$2a dx = \cosh t \cdot dt \quad \Rightarrow \quad dx = \frac{1}{2a}\cosh t \cdot dt \tag{5.10}$$

$$\left| \begin{array}{lll} x = x_1 & \Rightarrow & t_1 = \operatorname{arcsinh}(2a(x_1 - x_{MIN})) \\ x = x_2 & \Rightarrow & t_2 = \operatorname{arcsinh}(2a(x_2 - x_{MIN})) \end{array} \right| \tag{5.11}$$

$$L_{t_1 t_2} = \int_{t_1}^{t_2} \sqrt{1 + \sinh^2 t} \cdot \frac{1}{2a}\cosh t \cdot dt = \frac{1}{2a}\int_{t_1}^{t_2} \cosh^2 t \cdot dt \tag{5.12}$$

$$L_{t_1 t_2} = \frac{1}{2a}\int_{t_1}^{t_2} \frac{1 + \cosh 2t}{2} \cdot dt = \frac{1}{4a}\int_{t_1}^{t_2} (1 + \cosh 2t) \cdot dt \tag{5.13}$$

$$L_{t_1 t_2} = \frac{1}{4a}\left(t + \frac{\sinh\ 2t}{2}\right)_{t_1}^{t_2} = \frac{1}{4a}(t + \sinh\ t \cdot \cosh t)_{t_1}^{t_2} \tag{5.14}$$

$$\begin{aligned} &L_{x_1 x_2} = \\ &\frac{1}{4a}\left[\operatorname{arcsinh}(2a(x - x_{MIN})) + \sinh(\operatorname{arcsinh}(2a(x - x_{MIN}))) \cdot \cosh(\operatorname{arcsinh}(2a(x - x_{MIN})))\right]_{x_1}^{x_2} \end{aligned} \tag{5.15}$$

$$L_{x_1 x_2} = \frac{1}{4a}\left[\operatorname{arcsinh}(2a(x - x_{MIN})) + 2a(x - x_{MIN}) \cdot \sqrt{1 + (2a(x - x_{MIN}))^2}\right]_{x_1}^{x_2} \tag{5.16}$$

$$L_{x_1 x_2} = \left[\frac{1}{4a}\operatorname{arcsinh}(2a(x - x_{MIN})) + \frac{1}{2}(x - x_{MIN}) \cdot \sqrt{1 + (2a(x - x_{MIN}))^2}\right]_{x_1}^{x_2} \tag{5.17}$$

$$\begin{aligned} L_{x_1 x_2} = &\frac{1}{4a}\operatorname{arcsinh}(2a(x_2 - x_{MIN})) - \frac{1}{4a}\operatorname{arcsinh}(2a(x_1 - x_{MIN})) + \\ &+ \frac{1}{2}(x_2 - x_{MIN}) \cdot \sqrt{1 + (2a(x_2 - x_{MIN}))^2} - \frac{1}{2}(x_1 - x_{MIN}) \cdot \sqrt{1 + (2a(x_1 - x_{MIN}))^2} \end{aligned} \tag{5.18}$$

Substituting (4.50) and a=4$D_{\max}$/S^2 (see (4.42)) into previous expression yields (5.19):

$$
\begin{aligned}
L_{x_1x_2} &= \frac{S^2}{16D_{\max}}\operatorname{arcsinh}\left(\frac{8D_{\max}}{S^2}\left(x_2-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right)\right)- \\
&-\frac{S^2}{16D_{\max}}\operatorname{arcsinh}\left(\frac{8D_{\max}}{S^2}\left(x_1-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right)\right)+ \\
&+\frac{1}{2}\left(x_2-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right)\cdot\sqrt{1+\left(\frac{8D_{\max}}{S^2}\left(x_2-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right)\right)^2}- \\
&-\frac{1}{2}\left(x_1-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right)\cdot\sqrt{1+\left(\frac{8D_{\max}}{S^2}\left(x_1-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right)\right)^2}
\end{aligned}
\tag{5.19}
$$

Formula (5.19) is a universal one for the conductor length calculation based on the parabola model, since it can be directly used for deriving the final formulas for calculations in a full inclined span, but also in a level span (a full or its part).

5.2.1.2 Conductor (Parabola) Length in a Full Inclined Span

In order to obtain the formula for the conductor length calculation in a full inclined span, expression (5.8) can be used, but the integral limits have to be changed into: $x_1=0$ and $x_2=S$. In fact, this is a special case of the span–part when the integral limits present the x–coordinates of the two support points of the conductor in a given span. The main steps for the determination of the final formula are the following:

$$
L=\int_0^S\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{5.20}
$$

$$
L=\left[\frac{1}{4a}\operatorname{arcsinh}\left(2a\left(x-x_{MIN}\right)\right)+\frac{1}{2}\left(x-x_{MIN}\right)\cdot\sqrt{1+\left(2a\left(x-x_{MIN}\right)\right)^2}\right]_0^S \tag{5.21}
$$

$$
\begin{aligned}
L &= \frac{1}{4a}\operatorname{arcsinh}\left(2a\left(S-x_{MIN}\right)\right)-\frac{1}{4a}\operatorname{arcsinh}\left(2a\left(0-x_{MIN}\right)\right)+ \\
&+\frac{1}{2}\left(S-x_{MIN}\right)\cdot\sqrt{1+\left(2a\left(S-x_{MIN}\right)\right)^2}-\frac{1}{2}\left(0-x_{MIN}\right)\cdot\sqrt{1+\left(2a\left(0-x_{MIN}\right)\right)^2}
\end{aligned}
\tag{5.22}
$$

$$
\begin{aligned}
L &= \frac{1}{4a}\operatorname{arcsinh}\left(2a\left(S-x_{MIN}\right)\right)+\frac{1}{4a}\operatorname{arcsinh}\left(2ax_{MIN}\right)+ \\
&+\frac{1}{2}\left(S-x_{MIN}\right)\cdot\sqrt{1+\left(2a\left(S-x_{MIN}\right)\right)^2}+\frac{1}{2}x_{MIN}\cdot\sqrt{1+\left(2ax_{MIN}\right)^2}
\end{aligned}
\tag{5.23}
$$

After the substitution of expressions for x_{MIN} and a into (5.23), it becomes (5.24):

$$L=\frac{S^2}{16D_{max}}\operatorname{arcsinh}\left(\frac{4D_{max}}{S}\left(1+\frac{h_2-h_1}{4D_{max}}\right)\right)+\frac{S^2}{16D_{max}}\operatorname{arcsinh}\left(\frac{4D_{max}}{S}\left(1-\frac{h_2-h_1}{4D_{max}}\right)\right)+$$
$$+\frac{S}{4}\left(1+\frac{h_2-h_1}{4D_{max}}\right)\cdot\sqrt{1+\left(\frac{4D_{max}}{S}\left(1+\frac{h_2-h_1}{4D_{max}}\right)\right)^2}+ \tag{5.24}$$
$$+\frac{S}{4}\left(1-\frac{h_2-h_1}{4D_{max}}\right)\cdot\sqrt{1+\left(\frac{4D_{max}}{S}\left(1-\frac{h_2-h_1}{4D_{max}}\right)\right)^2}$$

Equation (5.24) is the final formula for the length calculation of the conductor within the whole span. It is needed more frequently than the corresponding formula for the conductor length in the part of the span, given by (5.19). Let us mention that both (5.19) and (5.24) are obtained by the application of the same algorithm, with the use of the appropriate integral limits in two different cases.

5.2.2 Calculations in Level Spans

The level span is a special case when the support points of the conductor are on the same elevation and the vertex point of the conductor curve is located at a mid–span. Actually, it is a simplification of an inclined span. Fig. 5.2 presents a level span with equal supports in a flat terrain.

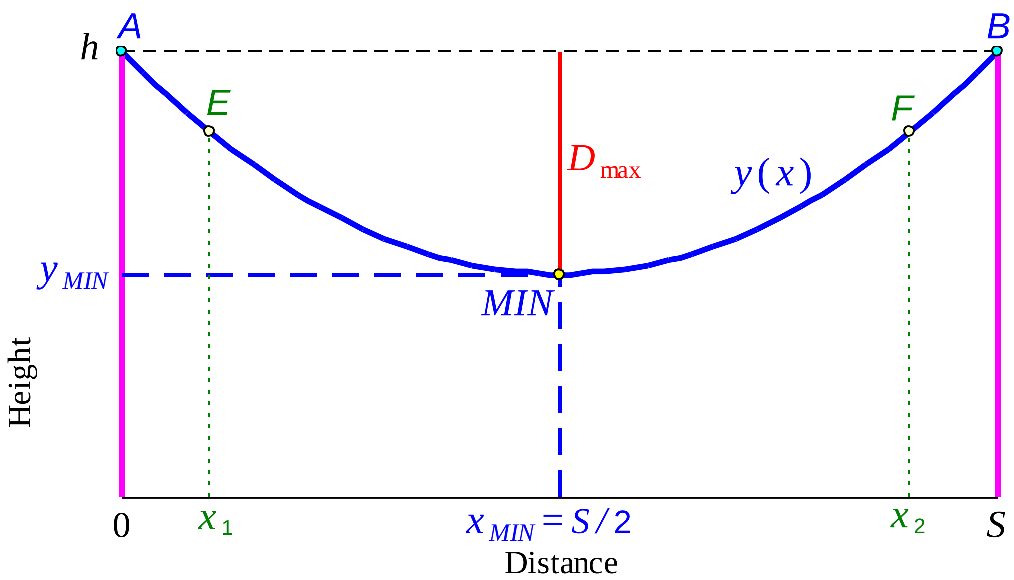


Fig. 5.2: Parabolic conductor curve in a level span

5.2.2.1 Conductor (Parabola) Length in Part of a Level Span

Using (5.19) and taking into consideration that $h_1=h_2=h$, it is easy to get the special formula for the conductor length calculation in part $[x_1, x_2]$ of a level span. The length of the parabola on the interval $[x_1, x_2]$, shown in Fig. 5.2, can be determined by the following formula:

$$L_{x_1x_2} = \frac{S^2}{16D_{max}}\operatorname{arcsinh}\left(\frac{8D_{max}}{S^2}\left(x_2-\frac{S}{2}\right)\right) - \frac{S^2}{16D_{max}}\operatorname{arcsinh}\left(\frac{8D_{max}}{S^2}\left(x_1-\frac{S}{2}\right)\right) + \\ + \frac{1}{2}\left(x_2-\frac{S}{2}\right)\cdot\sqrt{1+\left(\frac{8D_{max}}{S^2}\left(x_2-\frac{S}{2}\right)\right)^2} - \frac{1}{2}\left(x_1-\frac{S}{2}\right)\cdot\sqrt{1+\left(\frac{8D_{max}}{S^2}\left(x_1-\frac{S}{2}\right)\right)^2} \tag{5.25}$$

Instead of the full algorithm shown in Section 5.2.1, its final formula is directly applied here to obtain (5.25). The size and location of the span–part [x_1, x_2] is arbitrarily selectable on the interval $[0, S]$.

5.2.2.2 Conductor (Parabola) Length in a Full Level Span

Since the parabola is an even function, the length of its curve in a full level span can be determined as a double length of the curve in one half of the span, according to (5.26). The main steps for deriving the final formula (5.29) are given by (5.27) and (5.28).

$$L = \int_0^S \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 2\int_0^{S/2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{5.26}$$

$$L = \frac{1}{2a}\left[\operatorname{arcsinh}\left(2a\left(x-\frac{S}{2}\right)\right) + 2a\left(x-\frac{S}{2}\right)\cdot\sqrt{1+\left(2a\left(x-\frac{S}{2}\right)\right)^2}\right]_0^{S/2} \tag{5.27}$$

$$L = \frac{1}{2a}\operatorname{arcsinh}(aS) + \frac{S}{2}\cdot\sqrt{1+(aS)^2} \tag{5.28}$$

$$L = \frac{S^2}{8D_{max}}\operatorname{arcsinh}\left(\frac{4D_{max}}{S}\right) + \frac{S}{2}\cdot\sqrt{1+\left(\frac{4D_{max}}{S}\right)^2} \tag{5.29}$$

Expression (5.29) is a formula for the conductor length calculation in a full level span. Naturally, the appropriate application of (5.19) or (5.25) would give the same formula, but here the aim was to show a special use of the algorithm from Section 5.2.1. Formula (5.29) also has another form (5.31) obtained by using identity (5.30) [74,103].

$$\operatorname{arcsinh}(x) = \ln\left(x+\sqrt{1+x^2}\right) \tag{5.30}$$

$$L = \frac{S^2}{8D_{max}}\ln\left(\frac{4D_{max}}{S} + \sqrt{1+\left(\frac{4D_{max}}{S}\right)^2}\right) + \frac{S}{2}\cdot\sqrt{1+\left(\frac{4D_{max}}{S}\right)^2} \tag{5.31}$$

5.3 Catenary Conductor Curve

Similarly as in the case of the parabola, the length formulas are derived in four cases, but the conductor curve is here considered as a catenary:

- Catenary length in part of an inclined span
- Catenary length in a full inclined span
- Catenary length in part of a level span
- Catenary length in a full level span.

5.3.1 Calculations in Inclined Spans

In this section Fig. 5.1 from Section 5.2.1 is used with the same symbols, except for the maximum sag of the parabola. Instead of it the catenary parameter is applied. The deduction has been done regarding to inclined and level spans separately. The equation for the catenary conductor curve, which is obtained in Chapter 2, is used here.

5.3.1.1 Conductor (Catenary) Length in Part of an Inclined Span

The length formula in this section is derived by using (2.18). The square of its first derivative is given by (5.32):

$$\left(\frac{dy}{dx}\right)^2 = \sinh^2 \frac{x - x_{MIN}}{c} \tag{5.32}$$

Substituting (5.32) into (5.5) and evaluating the integral result in (5.34).

$$L_{x_1x_2} = \int_{x_1}^{x_2} \sqrt{1 + \sinh^2 \frac{x - x_{MIN}}{c}} \cdot dx = \int_{x_1}^{x_2} \cosh \frac{x - x_{MIN}}{c} \cdot dx = c \cdot \sinh \left. \frac{x - x_{MIN}}{c} \right|_{x_1}^{x_2} \tag{5.33}$$

$$L_{x_1x_2} = c \cdot \left(\sinh \frac{x_2 - x_{MIN}}{c} - \sinh \frac{x_1 - x_{MIN}}{c} \right) \tag{5.34}$$

Applying identity (5.35), the previous expression can be transformed into (5.36), then into (5.37) after substitution of (2.29).

$$\sinh x - \sinh y = 2 \cdot \cosh \frac{x + y}{2} \cdot \sinh \frac{x - y}{2} \tag{5.35}$$

$$L_{x_1x_2} = 2c \cdot \cosh \frac{x_2 + x_1 - 2x_{MIN}}{2c} \cdot \sinh \frac{x_2 - x_1}{2c} = 2c \cdot \sinh \frac{x_2 - x_1}{2c} \cdot \cosh \left(\frac{x_1 + x_2}{2c} - \frac{x_{MIN}}{c} \right) \tag{5.36}$$

$$L_{x_1x_2} = 2c \cdot \sinh \frac{x_2 - x_1}{2c} \cdot \cosh \left(\frac{x_1 + x_2 - S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh (S/2c)} \right) \tag{5.37}$$

Formula (5.37) is a universal one for the conductor length calculation based on the catenary model, since it can be directly used for deriving the final formulas for calculations in a full inclined span, but also in a level span (a full or its part).

5.3.1.2 Conductor (Catenary) Length in a Full Inclined Span

Having obtained the formula for the length of the catenary in part of an inclined span, given by (5.37), it can be simply transformed into an adequate formula for the catenary length in a full inclined span, setting $x_1=0$ and $x_2=S$. (The reverse order is not possible.) It is detailed below, where at first (5.37) has been changed into (5.38).

$$L = 2c \cdot \sinh \frac{S}{2c} \cdot \cosh\left(\operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right) \tag{5.38}$$

Taking into consideration (5.39), the previous length formula can be transformed into (5.42), as follows:

$$\cosh u = \sqrt{1 + \sinh^2 u} \tag{5.39}$$

$$\cosh\left(\operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right) = \sqrt{1 + \sinh^2\left(\operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right)} = \sqrt{1 + \frac{\left(h_2 - h_1\right)^2}{4c^2 \sinh^2\left(S/2c\right)}} \tag{5.40}$$

$$L = 2c \cdot \sinh \frac{S}{2c} \cdot \sqrt{1 + \frac{\left(h_2 - h_1\right)^2}{4c^2 \sinh^2\left(S/2c\right)}} \tag{5.41}$$

$$L = \sqrt{4c^2 \sinh^2\left(S/2c\right) + \left(h_2 - h_1\right)^2} \tag{5.42}$$

This formula is corresponding to the one which is presented in [47].

5.3.2 Calculations in Level Spans

This section does not need a new figure either, that one from Section 5.2.2 can be used here as well. Two formulas for the length of the catenary in a level span are defined, one in part of the span and one in a full span. As it can be seen below, obtaining both formulas is significantly simpler than obtaining adequate formulas for the parabola length. Furthermore, the formulas themselves are also much simpler.

5.3.2.1 Conductor (Catenary) Length in Part of a Level Span

When $h_1=h_2$, then $x_{MIN}=S/2$ and hence (5.34) changes into (5.43). Using identity (5.35) yields (5.44).

$$L_{x_1 x_2} = c \cdot \left(\sinh \frac{x_2 - S/2}{c} - \sinh \frac{x_1 - S/2}{c} \right) \tag{5.43}$$

$$L_{x_1 x_2} = 2c \cdot \sinh \frac{x_2 - x_1}{2c} \cdot \cosh \frac{x_1 + x_2 - S}{2c} \tag{5.44}$$

The previous formula is also obtainable by considering that $h_1=h_2$ in (5.37).

5.3.2.2 Conductor (Catenary) Length in a Full Level Span

The easiest way to define the formula for the catenary length in a full level span is to use the adequate formula for the catenary length in a full inclined span, given by (5.42), and to set $h_1 = h_2$. It results in (5.45).

$$L = \sqrt{4c^2 \sinh^2 \frac{S}{2c}} = 2c \cdot \sinh \frac{S}{2c} \tag{5.45}$$

Notice that the obtained formula is the same as (5.2) far left, which is available in literature. This way the correctness of the new method is verified.

5.4 Practical Example and Analysis of the Results

Having obtained the length formulas for the catenary and the parabola (basic and modified), all necessary conditions have been established for comparison of the lengths of the three mentioned curves when the latter two ones are approximations of the first one. In order to make a comprehensive comparison, it is recommended to include the following cases:

- Level span, $h_1 = h_2$,
- Incline span with low inclination, i.e. with small $|h_2 - h_1|$,
- Incline span with high inclination, i.e. with big $|h_2 - h_1|$.

Such a suitable and practical example has been created below (Example 5.1) by the use of the five catenaries, which are drawn in Fig. 3.11, according to data from Table 3.6. Having data for the catenaries, the additional necessary data (D_{max} and $D_{max\,\psi}$) for computing the length of the catenary approximation by a parabola (basic and modified) are obtained in all five cases separately and listed in Table 5.1. After that the lengths of the three curves have been calculated and then also compared in each case, in order to draw conclusions.

Example 5.1

According to data from Table 3.6, the maximum sag of the parabola is computed as $D_{max} = S^2/8c$, and then also the maximum sag of the modified parabola as $D_{max\,\psi} = D_{max}/\cos\psi$. These data – together with h_1, h_2, S from Table 3.6 – are sufficient for all length computations.

Table 5.1: Catenary parameter and maximum sags of the parabolas

According to Table 3.6	Catenary parameter c [m]	Maximum sag of modified parabola $D_{\max\,\psi}$ [m]	Maximum sag of basic parabola $D_{\max}$ [m]
Case 1	10^3	61.25	61.25
Case 2	10^3	61.40605	61.25
Case 3	10^3	61.87184	61.25
Case 4	10^3	62.64047	61.25
Case 5	10^3	63.70096	61.25

Table 5.2: Lengths of catenaries (from Fig. 3.11) and their parabolic approximations

According to Table 3.6	Length of catenary L_{cat} [m]	Length of modified parabola $L_{\text{par}\,\psi}$ [m]	Length of basic parabola L_{par} [m]
Case 1	714.37946	714.03991	714.03991
Case 2	716.12709	715.79359	715.72366
Case 3	721.34459	721.02879	720.75464
Case 4	729.95754	729.66983	729.07293
Case 5	741.84770	741.59654	740.58204

Based on the previous table, the following one is prepared for analysing the length differences of the three curves in the five given cases, according to Fig. 3.11 and Table 3.6.

Table 5.3: Differences of lengths from Table 5.2

According to Table 3.6	$L_{\text{cat}} - L_{\text{par}\,\psi}$ [m]	$L_{\text{cat}} - L_{\text{par}}$ [m]	$L_{\text{par}\,\psi} - L_{\text{par}}$ [m]
Case 1	0.33955	0.33955	0
Case 2	0.33350	0.40343	0.06993
Case 3	0.31580	0.58995	0.27415
Case 4	0.28771	0.88461	0.59690
Case 5	0.25116	1.26566	1.01450

Thus, we can write relation (5.46) regarding to the level span (case 1), and relation (5.47) regarding to the inclined spans (cases 2 – 5):

$$L_{\text{cat}}^{(lev)} > L_{\text{par}\,\psi}^{(lev)} \equiv L_{\text{par}}^{(lev)} \tag{5.46}$$

$$L_{\text{cat}}^{(inc)} > L_{\text{par}\,\psi}^{(inc)} > L_{\text{par}}^{(inc)} \tag{5.47}$$

The following relation results from the previous one:

$$L_{\text{cat}}^{(inc)} - L_{\text{par}\,\psi}^{(inc)} < L_{\text{cat}}^{(inc)} - L_{\text{par}}^{(inc)} \tag{5.48}$$

So, the use of the multiplier 1/cosψ (see Chapter 4) for the parabola in inclined spans is recommended from the aspect of the length computation, since it ensures results closer to the catenary length than when 1/cosψ is not used. Let us mention that due to cos(0)=1, the multiplier does not have influence when computing the length of the parabola in level spans.

According to Fig. 3.11 in Section 3.4.4, relations (5.46) and (5.47) have been derived for inclined spans with $h_1 < h_2$. It is worth mentioning that the same relations also concern to another type of inclined spans, i.e. when $h_1 > h_2$. It confirms the universality of the developed method, which is achieved due to this work's strictly mathematical approach.

Another important conclusion, drawn by analysing the results from Table 5.3, is that when the span inclination (or $|h_2 - h_1|$) increases, then the difference in lengths of the catenary and its approximation by the modified parabola (see Chapter 4) decreases, whereas the difference in lengths of the catenary and its approximation by the basic parabola increases. Expressing it mathematically by the use of $|h_2 - h_1|$ instead of the angle of the span inclination ψ, regarding to both inclined span types, it results in relations (5.49) and (5.50), as follows:

$$L_{\text{cat}\,2}^{(inc)} - L_{\text{par}\,\psi\,2}^{(inc)} < L_{\text{cat}\,1}^{(inc)} - L_{\text{par}\,\psi\,1}^{(inc)} \quad \forall \quad \left|h_2^{(2)} - h_1^{(2)}\right| > \left|h_2^{(1)} - h_1^{(1)}\right| \tag{5.49}$$

$$L_{\text{cat}\,2}^{(inc)} - L_{\text{par}\,2}^{(inc)} > L_{\text{cat}\,1}^{(inc)} - L_{\text{par}\,1}^{(inc)} \quad \forall \quad \left|h_2^{(2)} - h_1^{(2)}\right| > \left|h_2^{(1)} - h_1^{(1)}\right| \tag{5.50}$$

Note: The four previous relations are valid for inclined spans which occur in OHL practice. Some of these relations can be invalid for very extreme span inclinations only, which never occur in OHL practice and hence are not a target of this work.

5.5 Summary of the Chapter

Taking into consideration that currently there is not any publication which deals widely with the calculation of the conductor length in a span, and also no publication gives the length formulas for all characteristic cases or, if there is, it gives only approximate ones; this chapter shows the derivations of the following formulas, covering both very frequent and very rare tasks in practice:

1. Formula for the parabola length in part of an inclined span
2. Formula for the parabola length in a full inclined span
3. Formula for the parabola length in part of a level span

4. Formula for the parabola length in a full level span
5. Formula for the catenary length in part of an inclined span
6. Formula for the catenary length in a full inclined span
7. Formula for the catenary length in part of a level span
8. Formula for the catenary length in a full level span.

Formula 1 is a universal one for computing the length of the parabola, since formulas 2, 3, 4 can be directly obtained from it, taking into account the following self–evident facts:

- Full span is a special case of the span–part, when the start and end points of the latter are the *x*–coordinates of the two support points in a given span.
- Level span is a special case of an inclined span, when the support points are on the same elevation.

Similarly, formula 5 is a universal one for computing the length of the catenary, as formulas 6, 7, 8 can be directly defined from it.

Comparing the length formulas, and also their derivations in the case of the parabola and the catenary, all presented in this chapter, it is evident that these are significantly more complicated in the case of the parabola. Considering Chapters 2–4, all other derivations are obviously more complicated in the case of the catenary. The conductor length calculation is the only one which is simpler within the catenary based calculation than within the parabola based calculation.

Using exact formulas obtained in this chapter, the lengths of the catenary and its approximation by the parabola (basic and modified) have been compared separately in level and inclined spans. This way some important conclusions have been drawn and also the application of 1/cosψ multiplier has been evaluated from the aspect of the length computation in case of the parabola. It is formulated within the thesis.

6 EXTENSION OF THE NEW METHODS

6.1 Introduction

The new algorithms and equations presented in Chapters 2–5 were originally obtained for the application in single dead–end spans but these are also usable in all spans with the conductors fixed at the rigid support insulators (pin insulator/post insulator). However, the high voltage overhead lines are mostly built with hanging suspension insulators mounted on the support towers (poles) between two dead–end (tension) towers. *At a typical suspension structure, the conductor is supported vertically by a suspension insulator assembly, but allowed to move freely in the direction of the conductor axis. This conductor movement is possible due to insulator swing along the conductor axis. Changes in conductor tension between spans, caused by changes in temperature and load are normally equalized by insulator swing, eliminating horizontal tension differences across suspension structures* [63]. In the following it will be shown that taking into consideration the *Ruling Span Theory* [104,105] and the movement of the support points, the use of the most methods shown in Chapters 2–5 can be extended in any span and also in the entire section of OHL with several support spans between the two dead–end structures. It is worth mentioning that the aim of this dissertation is not to replace the existing mechanical calculation, but to adequately upgrade it (using their results) by the exact and up to now missing mathematical calculations, instead of the existing deficient and inexact ones. The mechanical calculation (in literature so cold the *Sag–tension calculation*) already has ready methods for the determination of the horizontal movement (Δx) of the suspension points [41]. Knowing the insulator length also the vertical movement (Δy) of the suspension point can be easily obtained. Thus, the new x,y coordinates of the latter and so the modified span length can be considered as the known input data. Besides the maximum sag of the parabola (or the catenary parameter) these data are sufficient for writing the equations for the conductor curve and sag, as well as many other formulas shown in Chapters 2–5.

6.2 Ruling Span Theory

The publication [104] explains well the *Ruling Span Theory* as follows. *If all spans in a section of line are of the same length then the tension on individual span will be equal. Keeping the span lengths the same is possible on lines constructed on open terrains. However, for construction along highways and residential areas, the span lengths can never be equal. The owner of the property wants the poles be installed on the boundary of his/her* *lot. This causes a diverse length of spans that will affect the sag and conductor tension of the individual spans. A ruling span, also known as equivalent span or mean effective span (MES), is an assumed uniform design span which approximately portray the mechanical performance of a section of line between its dead–end supports. The ruling span is used in the design and construction of a line to provide a uniform span length which is a function of the various lengths of spans between dead–ends. This uniform span length allows sags and clearance to be readily calculated for structure spotting and conductor stringing.* Due to written in [106] *the ruling span may be defined as that span length in which the tension in the conductor, under changes in temperature and loading, will most nearly agree with the average tension in a series of spans of varying lengths between dead ends. A more common definition is that the ruling span is the span length used as a basis for calculating the conductor sags and tensions, constructing the sag template, and preparing the stringing tables.*

According to [85], the ruling span length can be determined by (6.1).

$$S_R = \frac{\sum_{i=1}^{n} \frac{k_i^3}{S_i^2}}{\sum_{i=1}^{n} \frac{k_i^2}{S_i}} \cdot \sqrt{\frac{\sum_{i=1}^{n} S_i^3}{\sum_{i=1}^{n} S_i}} \tag{6.1}$$

where:

S_R – ruling span length

k_i – distance between the suspension points in i^{th} span

(Note: in inclined spans $k_i > S_i$, while in level spans $k_i = S_i$.)

$S_1, S_2, \ldots, S_n$ – the 1st, 2nd, …, n^{th} span length respectively.

Since the tension in all of the suspension spans is equal (or nearly so), the maximum sag in any of the corresponding suspension spans can be calculated using the following formula [105]:

$$D_{\max i} = D_R \cdot \left(\frac{S_i}{S_R} \right)^2 \tag{6.2}$$

where:

D_R – maximum sag obtained using the ruling span length

$D_{\max i}$ – maximum sag of the i^{th} span

S_i – length of the i^{th} span.

Fig. 6.1 shows the i^{th} support span. Because of the conductor's temperature change the tension changes and the suspension points I and J move to I' and J'. This way the span length S_i changes into S'_i.

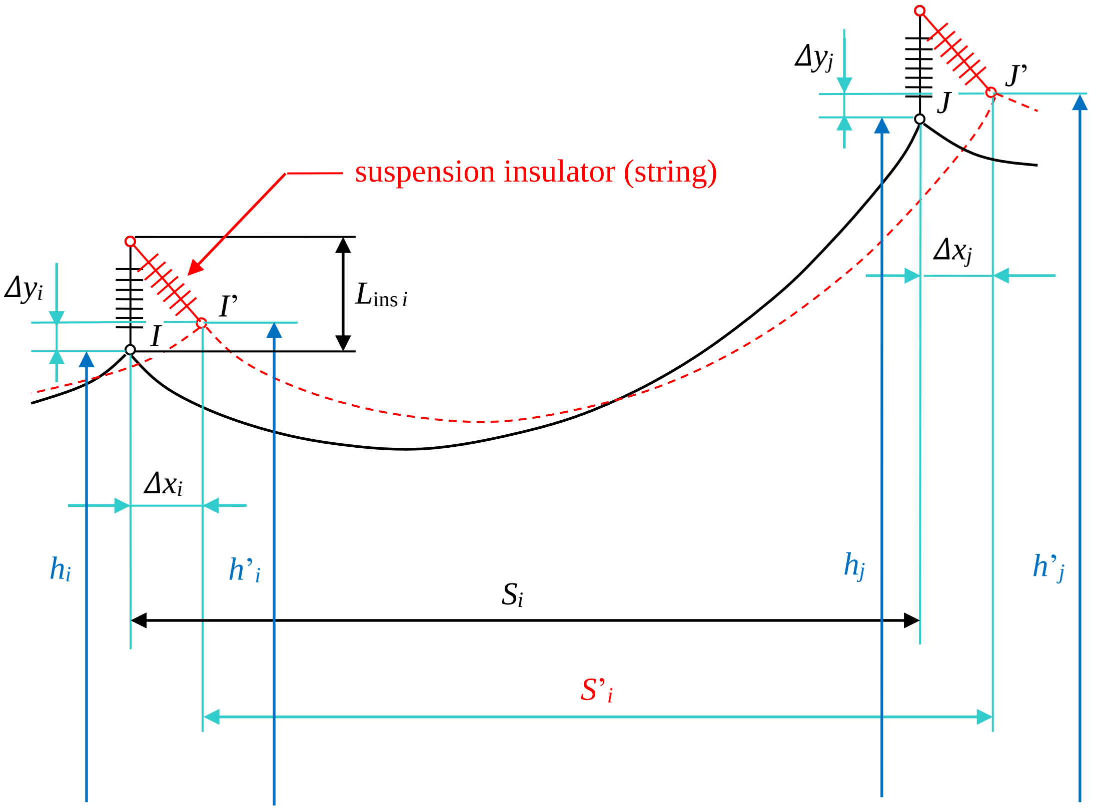


Fig. 6.1: The *i*th suspension span between the towers *I* and *J*

Each change in temperature causes different degrees of movement of the suspension points. Accordingly, each mathematical calculation considers to one selected temperature of the conductor by the application of the adequate results of the sag–tension calculation. The latter is not the target of this dissertation because it is easily available and explained well in the existing literature, for instance [41]. Actually, the mathematical calculations shown in this work builds on existing sag–tension calculation using their main results. Expression (6.3) is the equation for the parabolic conductor curve in the i^{th} span, taking into account the insulator swing. The x–axis is considered to go through the left–hand side tension tower of the OHL section between the two tension towers.

$$y' = y(x, h'_i, h'_j, D'_{\max i}, S'_i) = \frac{4D'_{\max i}}{S'^2_i}\left[x - \frac{S'_i}{2}\left(1 - \frac{h'_j - h'_i}{4D'_{\max i}}\right)\right]^2 + h'_i - D'_{\max i}\left(1 - \frac{h'_j - h'_i}{4D'_{\max i}}\right)^2 \qquad (6.3)$$

$$x \in \left[S'_1 + \ldots + S'_{i-1}, S'_1 + \ldots + S'_i\right]$$

If the OHL section consists of n spans and thus n+1 towers, where all towers are support ones, except for the first and the last one, which are dead–end towers, then the parabolic equation for the conductor curve considered to all spans is given by the following expression:

$$y' = \begin{cases} y'(x, h_1, h'_2, D'_{\max 1}, S'_1) \quad \forall \quad x \in [0, S'_1] \\ y'(x, h'_2, h'_3, D'_{\max 2}, S'_2) \quad \forall \quad x \in [S'_1, S'_2] \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y'(x, h'_i, h'_j, D'_{\max i}, S'_i) \quad \forall \quad x \in [S'_1 + S'_2 + \dots + S'_{i-1}, S'_1 + S'_2 + \dots + S'_{i-1} + S'_i] \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y'(x, h'_n, h_{n+1}, D'_{\max n}, S'_n) \quad \forall \quad x \in [S'_1 + S'_2 + \dots + S'_{n-1}, S'_1 + S'_2 + \dots + S'_{n-1} + S'_n] \end{cases} \tag{6.4}$$

where:
$\Delta x_1=0$, $\Delta x_{n+1}=0$, $\Delta y_1=0$, $\Delta y_{n+1}=0$, $h'_1=h_1$, $h'_{n+1}=h_{n+1}$ since the 1st and the (n+1)th towers are dead–end ones. (The tension insulators on dead–end towers are considered as the continuation of the conductor.)
$\Delta x_2,\dots, \Delta x_i, \Delta x_j,\dots, \Delta x_n$ are data taken from the sag–tension calculation [41].

$\Delta x_i>0 \quad \vee \quad \Delta x_i=0 \quad \vee \quad \Delta x_i<0$

$S'_1=S_1+\Delta x_2$, $S'_2=S_2-\Delta x_2+\Delta x_3,\dots$, $S'_i=S_i-\Delta x_i+\Delta x_j$, $S'_j=S_j-\Delta x_j+\Delta x_{j+1},\dots$, $S'_n=S_n-\Delta x_n$ are modified span lengths.

$$\sum_{i=1}^{n} S_i = \sum_{i=1}^{n} S'_i \tag{6.5}$$

$\Delta y_2,\dots, \Delta y_i, \Delta y_j,\dots, \Delta y_n$ are data obtained by (6.6), where $L_{\text{ins}\,i}$ is the length of the suspension insulator in the ith span.

$$\Delta y_i = L_{\text{ins}i} - \sqrt{L_{\text{ins}i}^2 - (\Delta x_i)^2} \tag{6.6}$$

$\Delta y_i>0 \quad \vee \quad \Delta y_i=0$

$h'_2=h_2+\Delta y_2$, $h'_3=h_3+\Delta y_3,\dots$, $h'_i=h_i+\Delta y_i$, $h'_j=h_j+\Delta y_j,\dots$, $h'_n=h_n+\Delta y_n$ are the modified heights of the suspension points of the conductors on the support towers.
Summarizing this section it can be said that taking into consideration the insulator swing and *Ruling Span Theory*, the input data for the mathematical calculations are S'_i, h'_i, h'_j, $D'_{\max i}$ instead of S_i, h_i, h_j, $D_{\max i}$.

6.3 When not to apply the Ruling Span Theory

In actual construction shown in Fig. 6.2, the stringing section is not a single dead–end span but it consists of series of unequal spans between the rigid dead–end supports. During stringing, the conductors can freely move between spans because it is temporarily supported by free–wheeling rollers. Under these conditions, the conductor behaves according to the Ruling Span Theory. [105]

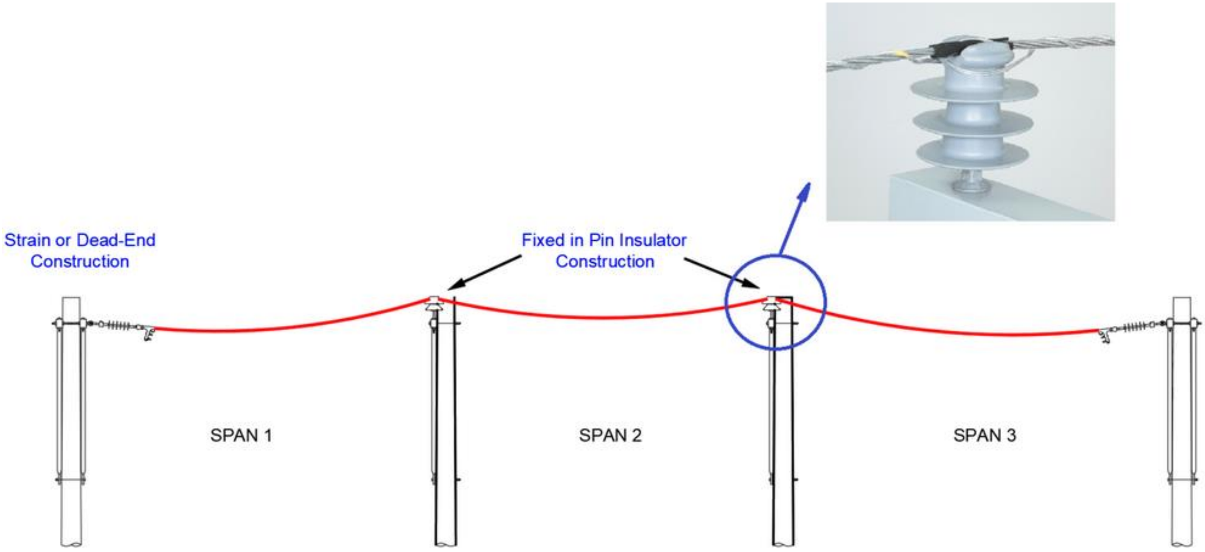


Fig. 6.2: Mechanically independent spans [**105**]

However, when the conductor are tied or fixed at the rigid support insulators (pin insulator/post insulator) the conductor can no longer move freely between spans. The spans, in a sense, become dead–end spans or mechanically independent spans. Then, the future behaviour of the conductor under various loading conditions will not follow the Ruling Span Theory. Its behaviour can be determined based on calculation procedures used for single dead–end spans. The difference in horizontal tension between span will then cause longitudinal movement or flexing in the supporting structures or insulators.

On the other hand, when the conductor is fixed at suspension insulator or strings, the difference in horizontal tension between spans will be compensated by the longitudinal or transverse movement or swing of the strings. Hence, it is safe to assume that the conductor will behave according to Ruling Span Theory. [105]

NEW SCIENTIFIC RESULTS

Executive Summary

Electrical network can be divided into two basic groups, underground cables and overhead lines (OHL). It is well known that the construction of OHL is less expensive but its design is more complex. One of the reasons of the latter disadvantageous fact is the conductor sag, which directly affects clearance calculations. The OHL have to be designed and operated so that they would not cause injuries to people, therefore maintaining adequate distance between energized conductors and ground or other objects is a particularly important task of OHL design. When designing electrical network a special attention should be paid to the safety of its environment. My dissertation has been written in this spirit.

Focusing on the conductor sag in a span, this dissertation introduces novel methods, algorithms and equations, which are creatable or obtainable by the use of the given major result of the sag–tension calculation (catenary parameter or parabola's maximum sag), besides the span length and the heights of the support points. Both the catenary and the parabola based calculations have been discussed, as well as the special link between them, providing a wide mathematical background, which can help to solve not only standard and frequent tasks in OHL practice, but also some rare unconventional ones. This work is a complex mathematical module, which practically connects the results of the sag–tension calculation with clearance calculation, and in this way it contributes to safe electrical network planning.

Introduction deals with the objectives and the structure of the dissertation, and also gives a brief introduction of the bases of overhead lines and their design. The novel results are reported in Chapters 2–5 and are grouped in four theses.

Chapter 1 introduces the drone as an unmanned aerial vehicle and its wide applicability for the inspection of overhead lines giving priority to autonomous drones in comparison to remote controlled ones. It is highlighted that the future usage of drones will be mounting the sensors (as temperature and vibration sensors, etc.) on energized conductors, i.e. without disconnection of the power supply. For planning the trajectory of an autonomous drone used for inspection of overhead lines, it is necessary to know the equation for the conductor curve. The correspondent mathematical algorithms for determining the equations for the conductor curve, either the catenary or parabola, and also its length are presented in details in Chapters 2–5.

Chapter 2 widens the current catenary based calculation developing universal equations for the conductor and the sag curves, usable in all span types. It is achieved by the new way of applying the coordinate system, which differs from the way that is generally used in literature. Besides the determination of the maximum sag and its location in an inclined span, the formulas for other characteristic sags (maximum sag, mid–span sag and low point sag) of the catenary have been obtained and also special cases of the inclined spans have been appropriately explained.

Chapter 3 introduces a mathematical solution for inclined span modelling by the known data of a given level span, with the possibility of the arbitrary selection of the span inclination or the difference in elevation of the support points. The developed method is presented in the case of the catenary, but it is applicable in the case of the parabola as well. Using it, unique relations between the catenary sags in inclined and level spans have been derived when the span length is a common datum in both spans, as well as the catenary parameter.

Chapter 4 describes the determination of the universal parabolic equation for the conductor curve taking into consideration the fact that the maximum sag of the parabola is always located at a mid–span, i.e. in both level and inclined spans. Three methods for defining the vertex point of the parabola have also been shown and explained. In addition, the special parabolic equations for the conductor curve, usable strictly in inclined spans, have been obtained by the known $(x; y)$ coordinates of two support points and only one coordinate of the parabola's vertex point. The parabolic approximation of the catenary in an inclined span has been created mathematically.

Chapter 5 deals with the conductor length calculation for the cases of the catenary and the parabola using the equations for the conductor curve from Chapters 2 and 4. Applying the integral calculus, universal formulas have been derived for the conductor length calculation in level and inclined spans, and also in a span–part and in a full span as well. Moreover, the length of the catenary is compared to the length of its approximation by the parabola.

Chapter 6 explains the extension of the new methods, shown in Chapters 2–5, in the entire section of OHL consisting of several support spans between the two dead–end structures.

Összefoglalás

A villamos hálózat két csoportba sorolható, földkábeles és szabadvezeték hálózatra. Közismert, hogy az utóbbi megépítése olcsóbb, viszont a tervezése bonyolultabb. A bonyolultabb tervezés egyik oka a vezeték belógása, ami a vezetékektől való távolság számításával kapcsolatos. A szabadvezeték hálózatot úgy kell tervezni és működtetni, hogy személyi sérülést ne okozzon, ezért a megfelelő távolság (villamos szigetelőképesség) fenntartása a feszültség alatt álló vezetékek és a föld vagy más tárgyak között különösen fontos feladat. A villamos hálózat tervezésekor annak a környezetével szembeni biztonságára kiemelt figyelmet kell fordítani. Ennek szellemében készült a disszertációm.

A disszertáció azokat az új módszereket, algoritmusokat és egyenleteket mutatja be a vezeték oszlopközbeni belógásával kapcsolatban, melyek a szabadvezetékek mechanikai méretezésének adott főeredményén (láncgörbe paraméterén vagy a parabola legnagyobb belógásán) alapulnak, az oszlopköz hosszára és a felfüggesztési pontok magasságaira vonatkozó adatok mellett. Mind a láncgörbe, mind pedig a parabola alapú számítás megtárgyalásra kerül, valamint azok speciális matematikai kapcsolata is, amely segítséget nyújthat nemcsak a szokásos gyakorlati feladatok megoldásában, hanem egyes ritka, speciális feladatok esetében is. A munka eredménye egy összetett matematikai modul, amely praktikusan összeköti a szabadvezetékek mechanikai méretezésének eredményeit a vezetékektől való távolság számítással, és így hozzájárul a biztonságos hálózat tervezéséhez.

A *Bevezetés* című fejezet a disszertáció céljait és szerkezetét mutatja be, valamint rövid ismertetést ad a szabadvezeték hálózatról és annak tervezéséről. Az új eredményeket négy tézisbe csoportosítottam, ezek a 2–5. fejezetben vannak ismertetve.

Az 1. fejezetben bemutatásra kerül a drón, mint pilóta nélküli légi jármű és annak széleskörű alkalmazhatósága a szabadvezeték hálózat ellenőrzése területén prioritást adva az autonóm drónnak a távirányított drónnal szemben. Hangsúlyozva van a drónok jövőbeli használata a különböző szenzorok (mint hőmérséklet és rezgés szenzorok, stb.) felszereléséhez a feszültség alatt álló vezetékekre, a villamos hálózat kikapcsolása nélkül. A szabadvezeték hálózat ellenőrzésére alkalmazott autonóm drónok trajektóriájának a tervezéséhez szükség van a vezetékgörbe egyenletére. A releváns matematikai algoritmusok a vezetékgörbe egyenletének meghatározásához, mind a láncgörbe, mint pedig a parabola esetén, valamint azok hosszának a számításához a 2–5. fejezetek mutatják be részletesen.

A 2. fejezetben kibővítésre kerül a jelenleg szokásos láncgörbe alapú számítás. Itt a vezetékgörbe és a belógási görbe univerzális egyenletei vannak bemutatva, melyek minden típusú felfüggesztési közben érvényesek. Ezt a koordináta rendszer alkalmazásának egy új módja tette lehetővé, amely eltér a szakirodalomban használttól. A legnagyobb belógásnak és annak elhelyezésének a meghatározása mellett a láncgörbe többi jellegzetes belógásainak formulái (belógás az oszlopköz felénél, belógás a vezeték légmélyebb pontjában) is meghatározásra kerültek ferde felfüggesztésre vonatkozóan. A ferde felfüggesztés különleges eseteit is tárgyaltam.

A 3. fejezetben egy matematikai megoldás van bemutatva a ferde felfüggesztési köz modellezésére az adott vízszintes felfüggesztési köz adatai alapján, amelynél az oszlopköz ferdesége vagy a felfüggesztési pontok közötti függőleges távolság tetszőlegesen választható. A kidolgozott módszer láncgörbére van bemutatva, de a módszer a parabolánál is alkalmazható. Ennek használatával a láncgörbe ferde és vízszintes felfüggesztésre vonatkozó belógásai között egyedi összefüggéseket dolgoztam ki arra az esetre, amikor az oszlopköz hossza és a láncgörbe paramétere is közös adat mindkét féle felfüggesztésnél.

A 4. fejezetben a vezetékgörbe univerzális parabolikus egyenlete van megadva, felhasználva azt a tényt, hogy a parabola legnagyobb belógása mindig az oszlopköz felénél helyezkedik el mind vízszintes, mind pedig ferde felfüggesztés esetén. A parabola legmélyebb pontjának a meghatározásához három módszert mutattam be és fejtettem ki. Ezen túlmenően a ferde felfüggesztésre vonatkozóan a vezetékgörbe speciális parabolikus egyenletei kerültek kidolgozásra, a két felfüggesztési pont (x; y) koordinátái és a parabola legmélyebb pontjának egy koordinátája alapján. A láncgörbe parabolával való közelítésére ferde felfüggesztés esetén egy matematikai átalakítást is ismertettem.

Az 5. fejezetben a vezetékhossz számítása van kidolgozva mind a láncgörbe, mind pedig a parabola esetére felhasználva a vezetékgörbe egyenleteit a második és negyedik fejezetekből. Integrálszámítás alkalmazásával univerzális formulák kerültek levezetésre a vezetékhossz számításához vízszintes és ferde felfüggesztés esetén, valamint a teljes oszlopközben és annak tetszőleges részében egyaránt. A láncgörbe hosszát összehasonlítottam a parabolagörbe hosszával.

A 6. fejezet a 2–5. fejezetekben bemutatott új módszerek kiterjesztését tárgyalja a két feszítő oszlop közötti szabadvezeték hálózat teljes szakaszába, amely több oszlopközből áll.

Thesis 1

Relating to the drawing of the conductor curve considered as a catenary, I have derived universal equations for the conductor and the sag curves which are applicable for determining the conductor height and the sag at any point of the span, in all possible span types with any span inclination. New equations also cover the special cases of inclined spans where the catenary's vertex point and the conductor's low point differ in their location.

Universal equation for the conductor curve:

$$y(x) = 2c \cdot \left\{ \sinh^2 \left[\frac{S}{4c} - \frac{x}{2c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right] - \sinh^2 \left[\frac{S}{4c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right] \right\} + h_1$$

$$x \in [0, S]$$

Universal equation for the sag curve:

$$D(x) = \frac{h_2 - h_1}{S} x - 2c \cdot \left\{ \sinh^2 \left[\frac{S}{4c} - \frac{x}{2c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right] - \right.$$
$$\left. - \sinh^2 \left[\frac{S}{4c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right] \right\} \qquad x \in [0, S]$$

I have shown that the new sag equation can be used for determining the location of the maximum sag in a span and also for deriving the special formulas for the characteristic sags: the maximum sag, the mid–span sag and the low point sag.

Maximum sag formula:

$$D_{\max} = 2c \cdot \left\{ \frac{h_2 - h_1}{2S} \left[\frac{S}{2c} - \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} + \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right] - \right.$$
$$\left. - \sinh^2 \left(\frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) + \sinh^2 \left[\frac{S}{4c} - \frac{1}{2} \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh\left(S/2c\right)} \right] \right\}$$

Mid–span sag formula:

$$D(S/2)=\frac{h_2-h_1}{2}-2c\cdot\left\{\sinh^2\left[\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]-\right.$$
$$\left.-\sinh^2\left[\frac{S}{4c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]\right\}$$

Low point sag formula:

$$D(x_{MIN})=2c\cdot\left\{\frac{h_2-h_1}{2S}\left[\frac{S}{2c}-\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]+\right.$$
$$\left.+\sinh^2\left[\frac{S}{4c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\left(S/2c\right)}\right]\right\}\quad\forall\quad 0\le x_{MIN}\le S$$

I have demonstrated that the direction of the movement of the maximum sag from the mid–span, which occurs when the level span changes into an inclined one, can be determined analytically, not only numerically.

I have shown that if the conductor curve is considered as a catenary, then the sag function $D(x+S/2)$ is an *even* function in the case of a level span, while in inclined spans it is neither an *even* nor an *odd* function. The sag curve in a level span has the exact shape of an inverted catenary, while in an inclined span it slightly differs. The difference increases with the span inclination.

Publications connected to this thesis: [S1], [S4], [S7], [S8], [S13], [S15], [S17], [S24].

Thesis 2

I have developed a mathematical method, called *inclined span modelling by a given level span*, which using the given data (S, c, h_1) for a level span and a freely selected datum of the difference in the support points elevation ($h_2 - h_1$), creates equations for both the conductor and the sag curves in a modelled inclined span when the span length and the catenary parameter are common data in both spans.

$$y_{inc}(x)=2c\cdot\sinh\frac{x}{2c}\cdot\sinh\left(\frac{x-S}{2c}+\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\dfrac{S}{2c}}\right)+h_1\qquad x\in\left[0,S\right]$$

$$D_{inc}(x)=\frac{h_2-h_1}{S}x-2c\cdot\sinh\frac{x}{2c}\cdot\sinh\left(\frac{x-S}{2c}+\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\qquad x\in[0,S]$$

I have revealed that the quotient of the sag functions in inclined and level spans on the interval $(0,S)$ is not a constant in the case of the catenary as it is in the case of the parabola.

$$\frac{D_{inc}^{(\mathrm{cat})}(x)}{D_{lev}^{(\mathrm{cat})}(x)}=\left(\frac{D_{inc}^{(\mathrm{cat})}}{D_{lev}^{(\mathrm{cat})}}\right)(x)\neq\mathrm{const.}\qquad 0<x<S$$

Instead of the existing approximate relation I have derived a mathematically exact one between the catenary sags in inclined and level spans, usable at any point of the span.

$$\begin{aligned}D_{inc}(x)&=D_{lev}(x)+\frac{h_2-h_1}{S}x-\\&-4c\cdot\sinh\frac{x}{2c}\cdot\sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\cdot\cosh\left(\frac{S-x}{2c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\end{aligned}$$

The function, which describes the error along the span, resulted by the application of the approximate relation, changes sign near the middle of the span.

Instead of the existing approximate relation I have derived a mathematically exact one between the maximum sags of the catenary in inclined and level spans. The difference between the two mentioned sags increases with the span inclination.

$$\begin{aligned}D_{inc\max}&=D_{lev\max}+\frac{h_2-h_1}{S}\cdot\left(x_{MIN}+c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{S}\right)-2c\cdot\sinh^2\frac{S}{4c}+\\&+2c\cdot\sinh\left(\frac{1}{2c}\left(x_{MIN}+c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{S}\right)\right)\cdot\sinh\left(\frac{1}{2c}\left(x_{MIN}-c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{S}\right)\right)\end{aligned}$$

where x_{MIN} is the x–coordinate of the catenary's vertex point given as

$$x_{MIN}=\frac{S}{2}-c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh(S/2c)}$$

Publications connected to this thesis: [S2], [S8], [S9], [S10], [S14], [S17], [S25].

Thesis 3

I have derived a universal parabolic equation for the conductor curve by the given maximum sag and the coordinates of the support points, which is usable in level and inclined (classical and special) spans as well and from which the coordinates of the vertex point are directly readable.

$$y(x)=\frac{4D_{\max}}{S^2}\left[x-\frac{S}{2}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)\right]^2+h_1-D_{\max}\left(1-\frac{h_2-h_1}{4D_{\max}}\right)^2 \qquad x\in[0,S]$$

Subthesis 3.1

I have derived special parabolic equations for the conductor curve applicable strictly in inclined spans, by the given (x ; y) coordinates of the two support points and only one coordinate of the vertex point, x_{MIN} or y_{MIN}.

$$y(x)=\frac{h_2-h_1}{S(S-2x_{MIN})}(x-x_{MIN})^2+h_1-\frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})} \quad \forall \quad h_1\neq h_2 \quad \wedge \quad x\in[0,S]$$

$$y(x)=\left[\frac{h_2-h_1}{S\left(\sqrt{h_2-y_{MIN}}-\sqrt{h_1-y_{MIN}}\right)}\right]^2\cdot\left[x-\frac{S(h_1-y_{MIN})}{h_2-h_1}\left(\sqrt{\frac{h_2-y_{MIN}}{h_1-y_{MIN}}}-1\right)\right]^2+y_{MIN}$$
$$\forall \quad h_1\neq h_2 \quad \wedge \quad x\in[0,S]$$

Subthesis 3.2

I have created an analytical method for a parabolic approximation of the catenary in inclined spans. This method can also be applied in level ones.

$$y_{inc\psi}(x)=\frac{1}{2c\cdot\cos\psi}\left\{x-\frac{S}{2}\left[1-\frac{2c(h_2-h_1)}{S^2}\cos\psi\right]\right\}^2+h_1-\frac{1}{2c\cdot\cos\psi}\left\{\frac{S}{2}\left[1-\frac{2c(h_2-h_1)}{S^2}\cos\psi\right]\right\}^2$$
$$x\in[0,S]$$

I have provided a wide mathematical background which is related to $1/\cos\psi$ multiplier's effect. As the sag in an inclined span increases, $1/\cos\psi$ multiplier reduces the parabola's parameter and also its deviation from the catenary, resulting that the modified parabola resembles the catenary better than the basic (original) parabola.

$$\left|\, y_{\mathrm{par}\,\psi}(x)-y_{\mathrm{cat}}(x)\,\right| < \; y_{\mathrm{par}}(x)-y_{\mathrm{cat}}(x) \quad \forall \quad 0<x<S \quad \wedge \quad h_1\neq h_2$$

Subthesis 3.3

I have revealed that differently from the case of the catenary, the quotient of the sag functions in inclined and level spans on the interval $(0,S)$ is a constant in the case of the parabola (either basic or modified by $1/\cos\psi$), due to the two following relations:

$$\frac{D_{inc}^{(\mathrm{par})}(x)}{D_{lev}^{(\mathrm{par})}(x)} = \left(\frac{D_{inc}^{(\mathrm{par})}}{D_{lev}^{(\mathrm{par})}}\right)(x) = 1 \qquad 0 < x < S$$

$$\frac{D_{inc\psi}^{(\mathrm{par}\,\psi)}(x)}{D_{lev}^{(\mathrm{par})}(x)} = \left(\frac{D_{inc\psi}^{(\mathrm{par}\,\psi)}}{D_{lev}^{(\mathrm{par})}}\right)(x) = \frac{1}{\cos\psi} \qquad 0 < x < S$$

Publications connected to this thesis: [S4], [S5], [S6], [S15], [S16], [S17], [S18], [S20], [S22], [23], [S26], [S27], [S28], [S29].

Thesis 4

I have derived one universal formula for computing the length of the parabola and one for computing the length of the catenary, which are both usable in inclined and level spans as well, in full span and also in its arbitrarily chosen part.

Universal formula for the length of the parabola:

$$\begin{aligned}
L_{x_1x_2} = {} & \frac{S^2}{16D_{\max}}\operatorname{arcsinh}\left(\frac{8D_{\max}}{S^2}\left(x_2 - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right)\right) - \\
& - \frac{S^2}{16D_{\max}}\operatorname{arcsinh}\left(\frac{8D_{\max}}{S^2}\left(x_1 - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right)\right) + \\
& + \frac{1}{2}\left(x_2 - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right)\cdot\sqrt{1 + \left(\frac{8D_{\max}}{S^2}\left(x_2 - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right)\right)^2} - \\
& - \frac{1}{2}\left(x_1 - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right)\cdot\sqrt{1 + \left(\frac{8D_{\max}}{S^2}\left(x_1 - \frac{S}{2}\left(1 - \frac{h_2 - h_1}{4D_{\max}}\right)\right)\right)^2}
\end{aligned}$$

Universal formula for the length of the catenary:

$$L_{x_1x_2} = 2c\cdot\sinh\frac{x_2 - x_1}{2c}\cdot\cosh\left(\frac{x_1 + x_2 - S}{2c} + \operatorname{arcsinh}\frac{h_2 - h_1}{2c\cdot\sinh\left(S/2c\right)}\right)$$

Subthesis 4.1

Related to OHL practice, I have shown that when calculating the conductor length, the application of multiplier 1/cosψ for modifying the basic parabola in inclined spans ensures results closer to the catenary length in comparison to the case when the multiplier is not applied.

$$L_{\text{cat}}^{(inc)} > L_{\text{par}\,\psi}^{(inc)} > L_{\text{par}}^{(inc)}$$

Subthesis 4.2

Related to OHL practice, I have revealed that when the span inclination (or $|h_2 - h_1|$) increases, then the difference between the lengths of the catenary and its approximation by the modified parabola decreases, whereas the difference between the lengths of the catenary and its approximation by the basic parabola increases. It is expressed mathematically in the following two relations with the use of $|h_2 - h_1|$:

$$L_{\text{cat}\,2}^{(inc)} - L_{\text{par}\,\psi\,2}^{(inc)} < L_{\text{cat}\,1}^{(inc)} - L_{\text{par}\,\psi\,1}^{(inc)} \quad \forall \quad \left|h_2^{(2)} - h_1^{(2)}\right| > \left|h_2^{(1)} - h_1^{(1)}\right|$$

$$L_{\text{cat}\,2}^{(inc)} - L_{\text{par}\,2}^{(inc)} > L_{\text{cat}\,1}^{(inc)} - L_{\text{par}\,1}^{(inc)} \quad \forall \quad \left|h_2^{(2)} - h_1^{(2)}\right| > \left|h_2^{(1)} - h_1^{(1)}\right|$$

Publications connected to this thesis: [S3], [S4], [S11], [S12], [S19], [21], [S29].

Practical Application of the New Results

The practical usage of the new results is well presented and described through suitable numerical examples given in Chapters 2–5. The main application is the determination of the conductor height and the sag at any point of the span, by the universal equations for the conductor and the sag curves, which are applicable in any span type with any span inclination. New equations have been derived in the case of the parabola and the catenary as well. Drawing the conductor curve is another application of high importance. Besides computing all characteristic sags of the catenary, the inclined span modelling, the conductor length calculation and the parabolic approximation of the catenary in an inclined span are also very useful results presented in this work. The main new results were introduced in practice. My former company accepted my methods and implemented them in OHL design process after I had trained the other designers in the company. Representing the previous company, my project named *Designer Programme* successfully entered the *19th Hungarian Innovation Award Competition* and as a recognised innovation got into the *Innovation Award 2010* book under number 12.

The future application of the new mathematical equations and algorithms presented in the dissertation is their implementation in planning the trajectory of an autonomous drone used for inspection of overhead lines and for mounting, maintaining or replacing the smart sensors.

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LIST OF FREQUENTLY USED ABBREVIATIONS AND SYMBOLS

OHL Overhead Lines

LV Low Voltage

MV Medium Voltage

HV High Voltage

S – span length (horizontal distance between two towers/supports of a span)

h_1 – height of the left–hand side support point

h_2 – height of the right–hand side support point

$D_{\max}$ – maximum sag

w – weight of the conductor per unit length

H – horizontal tension

$c=w/H$ – catenary parameter

T – tension at any point of the conductor

$\Delta h=h_2–h_1$ – difference between the elevation of the support points

ψ – angle of the span inclination

$MIN(x_{MIN};y_{MIN})$ – vertex point

$y(x)$ – equation for the conductor curve

$y_{line}(x)$ – equation for the straight line connecting the support points

$D(x)$ – sag equation

$D(S/2)$ – mid–span sag

$D(x_{MIN})$ – low point sag

$\Delta D(x)$ – equation for the difference between the sags in inclined and level spans

$\Delta D(S/2)$ – difference between the mid–span sags in inclined and level spans

$\Delta D_{\max}$ – difference between the maximum sags in inclined and level spans

$E(x)$ – equation for the sag error

a, b, c – coefficients of the parabola according to $y(x)=ax^2+bx+c$

p – parameter of the parabola

L – conductor length in a full span

$L_{x_1 x_2}$ – conductor length in a span–part $[x_1, x_2] \subseteq [0, S]$.

Note:

Notation $_{\mathrm{cat}}$ or $^{(\mathrm{cat})}$ in the subscript or in the superscript refers to the catenary

Notation $_{\mathrm{par}}$ or $^{(\mathrm{par})}$ in the subscript or in the superscript refers to the parabola

Notation $_{\mathrm{par}\,\psi}$ or $^{(\mathrm{par}\,\psi)}$ in the subscript or in the superscript refers to the modified parabola

Notation $_{lev}$ or $^{(lev)}$ in the subscript or in the superscript refers to a level span

Notation $_{inc}$ or $^{(inc)}$ in the subscript or in the superscript refers to an inclined span.

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APPENDICES

Appendix 1:

$$y_{inc}(x) = (AQ)_{\text{curve}} = y_{lev}(x+q) + h_1 - y_{lev}(q)$$

$$y_{lev}(x+q) = c \cdot \cosh \frac{x+q-S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1$$

$$y_{lev}(q) = c \cdot \cosh \frac{q-S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1$$

$$y_{inc}(x) = \left(c \cdot \cosh \frac{x+q-S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1 \right) + h_1 - \left(c \cdot \cosh \frac{q-S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1 \right)$$

$$y_{inc}(x) = c \cdot \left(\cosh \frac{x+q-S/2}{c} - \cosh \frac{q-S/2}{c} \right) + h_1$$

$$\cosh(x) - \cosh(y) = 2 \sinh \frac{x+y}{2} \cdot \sinh \frac{x-y}{2}$$

$$y_{inc}(x) = 2c \cdot \sinh \frac{x+q-S/2+q-S/2}{2c} \cdot \sinh \frac{x+q-S/2-q+S/2}{2c} + h_1$$

$$y_{inc}(x) = 2c \cdot \sinh \frac{x+2q-S}{2c} \cdot \sinh \frac{x}{2c} + h_1 = 2c \cdot \sinh \frac{x-S+2q}{2c} \cdot \sinh \frac{x}{2c} + h_1$$

$$y_{inc}(x) = 2c \cdot \sinh \frac{x}{2c} \cdot \sinh \left(\frac{x-S}{2c} + \frac{q}{c} \right) + h_1$$

Appendix 2:

$$h_2 - h_1 = y_N - y_M = y_{lev}(S+q) - y_{lev}(q)$$

$$y_N = c \cdot \cosh \frac{S+q-S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1 = c \cdot \cosh \frac{S/2+q}{c} - c \cdot \cosh \frac{S}{2c} + h_1$$

$$y_M = c \cdot \cosh \frac{q-S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1$$

$$h_2 - h_1 = c \cdot \cosh \frac{S/2+q}{c} - c \cdot \cosh \frac{S}{2c} + h_1 - \left(c \cdot \cosh \frac{q-S/2}{c} - c \cdot \cosh \frac{S}{2c} + h_1 \right)$$

$$h_2 - h_1 = c \cdot \left(\cosh \frac{q+S/2}{c} - \cosh \frac{q-S/2}{c} \right)$$

$$\cosh(x) - \cosh(y) = 2 \sinh \frac{x+y}{2} \cdot \sinh \frac{x-y}{2}$$

$$h_2 - h_1 = 2c \cdot \sinh \frac{q+S/2+q-S/2}{2c} \cdot \sinh \frac{q+S/2-q+S/2}{2c} = 2c \cdot \sinh \frac{2q}{2c} \cdot \sinh \frac{S}{2c}$$

$$h_2 - h_1 = 2c \cdot \sinh \frac{q}{c} \cdot \sinh \frac{S}{2c}$$

$$\sinh \frac{q}{c} = \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}}$$

$$\operatorname{arcsinh} \left(\sinh \frac{q}{c} \right) = \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}}$$

$$\frac{q}{c} = \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}}$$

Appendix 3:

$$D_{lev}(x) = c \cdot \left(\cosh \frac{S}{2c} - \cosh \frac{x - S/2}{c} \right)$$

$$\cosh(x) - \cosh(y) = 2\sinh \frac{x+y}{2} \cdot \sinh \frac{x-y}{2}$$

$$D_{lev}(x) = 2c \cdot \sinh \frac{\frac{S}{2c} + \frac{x}{c} - \frac{S}{2c}}{2} \cdot \sinh \frac{\frac{S}{2c} - \frac{x}{c} + \frac{S}{2c}}{2} = 2c \cdot \sinh \frac{x}{2c} \cdot \sinh \frac{S-x}{2c}$$

$$D_{inc}(x) = \frac{h_2 - h_1}{S} x - 2c \cdot \sinh \frac{x}{2c} \cdot \sinh \left(\frac{x-S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} \right)$$

$$\Delta D(x) = D_{inc}(x) - D_{lev}(x)$$

$$\Delta D(x) = \frac{h_2 - h_1}{S} x - 2c \cdot \sinh \frac{x}{2c} \cdot \sinh \left(\frac{x-S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} \right) - 2c \cdot \sinh \frac{x}{2c} \cdot \sinh \frac{S-x}{2c}$$

$$\Delta D(x) = \frac{h_2 - h_1}{S} x - 2c \cdot \sinh \frac{x}{2c} \cdot \left(\sinh \left(\frac{x-S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} \right) + \sinh \frac{S-x}{2c} \right)$$

$$\sinh(x) + \sinh(y) = 2\sinh \frac{x+y}{2} \cdot \cosh \frac{x-y}{2}$$

$$\cosh(-x) = \cosh(x)$$

$$\sinh \left(\frac{x-S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} \right) + \sinh \frac{S-x}{2c} =$$

$$= 2\sinh \left(\frac{1}{2} \left(\frac{x-S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} + \frac{S-x}{2c} \right) \right) \cdot \cosh \left(\frac{1}{2} \left(\frac{x-S}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} - \frac{S-x}{2c} \right) \right) =$$

$$= 2\sinh \left(\frac{1}{2} \left(-\frac{S-x}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} + \frac{S-x}{2c} \right) \right) \cdot \cosh \left(\frac{1}{2} \left(-\frac{S-x}{2c} + \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} - \frac{S-x}{2c} \right) \right) =$$

$$= 2\sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\cdot\cosh\left(\frac{1}{2}\left(-\frac{S-x}{c}+\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\right)=$$

$$= 2\sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\cdot\cosh\left(-\left(\frac{S-x}{2c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\right)=$$

$$= 2\sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\cdot\cosh\left(\frac{S-x}{2c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)$$

$$\Delta D(x)=\frac{h_2-h_1}{S}x-4c\cdot\sinh\frac{x}{2c}\cdot\sinh\left(\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\cdot\cosh\left(\frac{S-x}{2c}-\frac{1}{2}\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)$$

Appendix 4:

$$D_{inc\max}=D_{inc}(x_C)$$

$$D_{inc\max}=D_{lev\max}+\frac{h_2-h_1}{S}x_C-2c\cdot\sinh^2\frac{S}{4c}+c\cdot\left(\cosh\frac{q-S/2}{c}-\cosh\frac{x_C-S/2+q}{c}\right)$$

$$\cosh(x)-\cosh(y)=2\sinh\frac{x+y}{2}\cdot\sinh\frac{x-y}{2}$$

$$c\cdot\left(\cosh\frac{q-S/2}{c}-\cosh\frac{x_C-S/2+q}{c}\right)=2c\cdot\sinh\frac{x_C-S+2q}{2c}\cdot\sinh\frac{-x_C}{2c}$$

$$\sinh(-x)=-\sinh(x)$$

$$c\cdot\left(\cosh\frac{q-S/2}{c}-\cosh\frac{x_C-S/2+q}{c}\right)=2c\cdot\sinh\frac{x_C}{2c}\cdot\sinh\frac{S-x_C-2q}{2c}$$

$$D_{inc\max}=D_{lev\max}+\frac{h_2-h_1}{S}\cdot\left(\frac{S}{2}-c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}+c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{S}\right)-2c\cdot\sinh^2\frac{S}{4c}+$$

$$+2c\cdot\sinh\left(\frac{1}{2c}\left(\frac{S}{2}-c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}+c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{S}\right)\right)\cdot$$

$$\cdot\sinh\left(\frac{1}{2c}\left(S-\left(\frac{S}{2}-c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}+c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{S}\right)-2c\cdot\operatorname{arcsinh}\frac{h_2-h_1}{2c\cdot\sinh\frac{S}{2c}}\right)\right)$$

$$D_{inc\max} = D_{lev\max} + \frac{h_2 - h_1}{S} \cdot \left(\frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) - 2c \cdot \sinh^2 \frac{S}{4c} +$$

$$+ 2c \cdot \sinh \left(\frac{1}{2c} \left(\frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) \right) \cdot$$

$$\cdot \sinh \left(\frac{1}{2c} \left(\frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) \right)$$

$$x_{MIN} = \frac{S}{2} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{2c \cdot \sinh \frac{S}{2c}}$$

$$D_{inc\max} = D_{lev\max} + \frac{h_2 - h_1}{S} \cdot \left(x_{MIN} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) - 2c \cdot \sinh^2 \frac{S}{4c} +$$

$$+ 2c \cdot \sinh \left(\frac{1}{2c} \left(x_{MIN} + c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) \right) \cdot \sinh \left(\frac{1}{2c} \left(x_{MIN} - c \cdot \operatorname{arcsinh} \frac{h_2 - h_1}{S} \right) \right)$$

$$D_{inc\max} = D_{lev\max} + \Delta D_{\max}$$

Appendix 5:

Deriving $y=y(S, h_1, h_2, x_{MIN}, x)$:

Expressing $D_{\max}=D_{\max}(S, h_1, h_2, x_{MIN}, y_{MIN})$ from (4.97):

$$x_{MIN} (h_2 - h_1 - 4D_{\max}) = -2S(h_1 - y_{MIN})$$

$$x_{MIN} [4D_{\max} - (h_2 - h_1)] = 2S(h_1 - y_{MIN})$$

$$4D_{\max} x_{MIN} = (h_2 - h_1) x_{MIN} + 2S(h_1 - y_{MIN})$$

$$D_{\max} = \frac{1}{4x_{MIN}} \cdot [2S(h_1 - y_{MIN}) + (h_2 - h_1) x_{MIN}]$$

Substituting $D_{\max}$ into (4.98) to get $y_{MIN}=y_{MIN}(S, h_1, h_2, x_{MIN})$:

$$y_{MIN} = h_1 - D_{\max} \left(\frac{2x_{MIN}}{S} \right)^2 = h_1 - \frac{1}{4x_{MIN}} \cdot [(h_2 - h_1) x_{MIN} + 2S(h_1 - y_{MIN})] \cdot \left(\frac{2x_{MIN}}{S} \right)^2$$

$$y_{MIN} = h_1 - \frac{x_{MIN}}{S^2} \cdot [(h_2 - h_1) x_{MIN} + 2S(h_1 - y_{MIN})]$$

$$y_{MIN} = h_1 - \frac{x_{MIN}^2}{S^2} \cdot (h_2 - h_1) - \frac{2x_{MIN}}{S} (h_1 - y_{MIN})$$

$$y_{MIN} = h_1 - \frac{x_{MIN}^2}{S^2} \cdot (h_2 - h_1) - \frac{2h_1 x_{MIN}}{S} + \frac{2x_{MIN}}{S} y_{MIN}$$

$$y_{MIN} \left(1 - \frac{2x_{MIN}}{S} \right) = h_1 - \frac{x_{MIN}^2}{S^2} \cdot (h_2 - h_1) - \frac{2h_1 x_{MIN}}{S}$$

$$y_{MIN} = \frac{1}{1-\frac{2x_{MIN}}{S}}\left[h_1 - \frac{x_{MIN}^2}{S^2}\cdot(h_2-h_1) - \frac{2h_1x_{MIN}}{S}\right]$$

$$y_{MIN} = \frac{S}{S-2x_{MIN}}\left[h_1 - \frac{x_{MIN}^2}{S^2}\cdot(h_2-h_1) - \frac{2h_1x_{MIN}}{S}\right]$$

$$y_{MIN} = \frac{Sh_1}{S-2x_{MIN}} - \frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})} - \frac{2h_1x_{MIN}}{S-2x_{MIN}}$$

$$y_{MIN} = \frac{Sh_1-2h_1x_{MIN}}{S-2x_{MIN}} - \frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})}$$

$$y_{MIN} = \frac{h_1(S-2x_{MIN})}{S-2x_{MIN}} - \frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})}$$

$$y_{MIN} = h_1 - \frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})} \qquad \forall \quad h_1 \neq h_2$$

Substituting y_{MIN} into expressions for coefficients a and b:

$$a = \frac{h_1-y_{MIN}}{x_{MIN}^2} = \frac{1}{x_{MIN}^2}\left[h_1-h_1+\frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})}\right] = \frac{1}{x_{MIN}^2}\cdot\frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})} = \frac{h_2-h_1}{S(S-2x_{MIN})}$$

$$b = -2\cdot\frac{h_1-y_{MIN}}{x_{MIN}} = \frac{-2}{x_{MIN}}\left[h_1-h_1+\frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})}\right] = \frac{-2}{x_{MIN}}\cdot\frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})} = \frac{-2(h_2-h_1)x_{MIN}}{S(S-2x_{MIN})}$$

Final equation in general form according to $y(x) = ax^2+bx+c$:

$$y(x) = \frac{h_2-h_1}{S(S-2x_{MIN})}x^2 - \frac{2(h_2-h_1)x_{MIN}}{S(S-2x_{MIN})}x + h_1 \qquad \forall \quad h_1 \neq h_2$$

Final equation in vertex form according to $y(x) = a(x-x_{MIN})^2 + y_{MIN}$:

$$y(x) = \frac{h_2-h_1}{S(S-2x_{MIN})}(x-x_{MIN})^2 + h_1 - \frac{(h_2-h_1)x_{MIN}^2}{S(S-2x_{MIN})} \qquad \forall \quad h_1 \neq h_2$$

Appendix 6:

Deriving y=$y(S, h_1, h_2, y_{MIN}, x)$:

Expressing $D_{\max}$=$D_{\max}(S, h_1, x_{MIN}, y_{MIN})$ from (4.98):

$$D_{\max}\left(\frac{2x_{MIN}}{S}\right)^2 = h_1 - y_{MIN} \quad \Rightarrow \quad D_{\max} = \frac{S^2(h_1-y_{MIN})}{4x_{MIN}^2}$$

Substituting $D_{\max}$ into (4.97) to get x_{MIN}=$x_{MIN}(S, h_1, h_2, y_{MIN})$:

$$x_{MIN} = \frac{-2S(h_1-y_{MIN})}{h_2-h_1-4D_{\max}} = \frac{-2S(h_1-y_{MIN})}{h_2-h_1-4\frac{S^2(h_1-y_{MIN})}{4x_{MIN}^2}} = \frac{-2S(h_1-y_{MIN})x_{MIN}^2}{(h_2-h_1)x_{MIN}^2 - S^2(h_1-y_{MIN})}$$

$$1 = \frac{-2S(h_1-y_{MIN})x_{MIN}}{(h_2-h_1)x_{MIN}^2 - S^2(h_1-y_{MIN})}$$

$$(h_2-h_1)x_{MIN}^2-S^2(h_1-y_{MIN})=-2S(h_1-y_{MIN})x_{MIN}$$

$$(h_2-h_1)x_{MIN}^2+2S(h_1-y_{MIN})x_{MIN}-S^2(h_1-y_{MIN})=0$$

$$a_z\cdot x_{MIN}^2+b_z\cdot x_{MIN}+c_z=0$$

$$x_{MIN1,2}=\frac{-b_z\pm\sqrt{{b_z}^2-4a_zc_z}}{2a_z}$$

$$a_z=h_2-h_1;\qquad b_z=2S(h_1-y_{MIN});\qquad c_z=-S^2(h_1-y_{MIN})$$

$$x_{MIN1,2}=\frac{-2S(h_1-y_{MIN})\pm\sqrt{4S^2(h_1-y_{MIN})^2+4S^2(h_2-h_1)(h_1-y_{MIN})}}{2(h_2-h_1)}$$

$$x_{MIN1,2}=\frac{-S(h_1-y_{MIN})\pm S\sqrt{(h_1-y_{MIN})^2+(h_2-h_1)(h_1-y_{MIN})}}{h_2-h_1}$$

The solution with ”+” sign is an appropriate one.

$$x_{MIN}=\frac{S}{h_2-h_1}\left[\sqrt{(h_1-y_{MIN})^2+(h_2-h_1)(h_1-y_{MIN})}-(h_1-y_{MIN})\right]$$

$$x_{MIN}=\frac{S}{h_2-h_1}\left[\sqrt{(h_1-y_{MIN})(h_1-y_{MIN}+h_2-h_1)}-(h_1-y_{MIN})\right]$$

$$x_{MIN}=\frac{S}{h_2-h_1}\left[\sqrt{(h_1-y_{MIN})(h_2-y_{MIN})}-(h_1-y_{MIN})\right]$$

or $$x_{MIN}=\frac{S(h_1-y_{MIN})}{h_2-h_1}\cdot\left(\sqrt{\frac{h_2-y_{MIN}}{h_1-y_{MIN}}}-1\right)\quad\forall\quad h_1\neq h_2$$

$$x_{MIN}^2=\left[\frac{S(h_1-y_{MIN})}{h_2-h_1}\cdot\left(\sqrt{\frac{h_2-y_{MIN}}{h_1-y_{MIN}}}-1\right)\right]^2\quad\forall\quad h_1\neq h_2$$

or $$x_{MIN}^2=\left[\frac{S(h_1-y_{MIN})}{h_2-h_1}\cdot\frac{\sqrt{h_2-y_{MIN}}-\sqrt{h_1-y_{MIN}}}{\sqrt{h_1-y_{MIN}}}\right]^2=\left[\frac{S\sqrt{h_1-y_{MIN}}}{h_2-h_1}\cdot\left(\sqrt{h_2-y_{MIN}}-\sqrt{h_1-y_{MIN}}\right)\right]^2$$

Substituting y_{MIN} into expressions for coefficients a and b:

$$a=\frac{h_1-y_{MIN}}{x_{MIN}^2}=\frac{h_1-y_{MIN}}{\left[\frac{S\sqrt{h_1-y_{MIN}}}{h_2-h_1}\cdot\left(\sqrt{h_2-y_{MIN}}-\sqrt{h_1-y_{MIN}}\right)\right]^2}=\left[\frac{h_2-h_1}{S\left(\sqrt{h_2-y_{MIN}}-\sqrt{h_1-y_{MIN}}\right)}\right]^2$$

$$b=-2\cdot\frac{h_1-y_{MIN}}{x_{MIN}}=\frac{-2(h_1-y_{MIN})}{\frac{S(h_1-y_{MIN})}{h_2-h_1}\cdot\left(\sqrt{\frac{h_2-y_{MIN}}{h_1-y_{MIN}}}-1\right)}=\frac{-2(h_2-h_1)}{S\left(\sqrt{\frac{h_2-y_{MIN}}{h_1-y_{MIN}}}-1\right)}$$

Final equation in general form according to $y(x)=ax^2+bx+c$:

$$y(x)=\left[\frac{h_2-h_1}{S\left(\sqrt{h_2-y_{MIN}}-\sqrt{h_1-y_{MIN}}\right)}\right]^2\cdot x^2-\frac{2(h_2-h_1)}{S\left(\sqrt{\frac{h_2-y_{MIN}}{h_1-y_{MIN}}}-1\right)}x+h_1\quad\forall\quad h_1\neq h_2$$

Final equation in vertex form according to $y(x) = a(x - x_{MIN})^2 + y_{MIN}$:

$$y(x) = \left[\frac{h_2 - h_1}{S\left(\sqrt{h_2 - y_{MIN}} - \sqrt{h_1 - y_{MIN}}\right)} \right]^2 \cdot \left[x - \frac{S(h_1 - y_{MIN})}{h_2 - h_1} \left(\sqrt{\frac{h_2 - y_{MIN}}{h_1 - y_{MIN}}} - 1 \right) \right]^2 + y_{MIN} \quad \forall \quad h_1 \neq h_2$$

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