Sobolev-type inequalities on Riemannian manifolds with applications

Thesis for the Degree of Doctor of Philosophy

Csaba Farkas

The Doctoral School of Applied Informatics and Applied Mathematics

Academic advisor:
prof. dr. Alexandru Kristály

Institute of Applied Mathematics
John von Neumann Faculty of Informatics
Óbuda University
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Preface

Sobolev-type inequalities or more generally functional inequalities are often manifestations of natural physical phenomena as they often express very general laws of nature formulated in physics, biology, economics and engineering problems. They also form the basis of fundamental mathematical structures such as the calculus of variations. In order to study some elliptic problems one needs to exploit various Sobolev-type embeddings, to prove the lower semi-continuity of the energy functional or to prove that the energy functional satisfies the Palais-Smale condition. This is one of the reasons why calculus of variations is one of the most powerful and far-reaching tools available for advancing our understanding of mathematics and its applications.

The main objective of calculus of variations is the minimization of functionals, which has always been present in the real world in one form or another. I have carried out my research activity over the last years in the calculus of variations. More precisely we combined with my coauthors, elements from calculus of variations with PDE and with geometrical analysis to study some elliptic problems on curved spaces, with various nonlinearities (sub-linear, oscillatory etc.), see [53, 54, 55, 56, 57, 58, 79]. Such problems deserve as models for nonlinear phenomena coming from mathematical physics (solitary waves in Schrödinger or Schrödinger-Maxwell equations, etc.).

The main purpose of the present thesis is to present the recent achievements obtained in the theory of functional inequalities, more precisely to present some new Sobolev-type inequalities on Riemannian manifolds. More precisely, in the first part of the present thesis we focus on the theoretical part of the functional inequalities, while in the second part we present some applications of the theoretical achievements. Such developments are highly motivated from practical point of view supported by various examples coming from physics.

The thesis is based on the following papers:

- F. Faraci and C. Farkas. *New conditions for the existence of infinitely many solutions for a quasi-linear problem.* Proc. Edinb. Math. Soc. (2), 59(3):655–669, 2016.

- F. Faraci and C. Farkas. *A characterization related to Schrödinger equations on Riemannian manifolds.* ArXiv e-prints, April 2017.

- F. Faraci, C. Farkas, and A. Kristály. *Multipolar Hardy inequalities on Riemannian manifolds.* ESAIM Control Optim. Calc. Var., accepted, 2017, DOI: 10.1051/cocv/2017057.

- C. Farkas, *Schrödinger-Maxwell systems on compact Riemannian manifolds.* preprint, 2017.

- C. Farkas, J. Fodor, and A. Kristály. *Anisotropic elliptic problems involving sublinear terms.* In 2015 IEEE 10th Jubilee International Symposium on Applied Computational Intelligence and Informatics, pages 141–146, May 2015.

- C. Farkas and A. Kristály. *Schrödinger-Maxwell systems on non-compact Riemannian manifolds.* Nonlinear Anal. Real World Appl., 31:473–491, 2016.

- C. Farkas, A. Kristály, and A. Szakál. *Sobolev interpolation inequalities on Hadamard manifolds.* In Applied Computational Intelligence and Informatics (SACI), 2016 IEEE 11th International Symposium on, pages 161–165, May 2016.

Most of the results of the present thesis is stated for Cartan-Hadamard manifolds, despite the fact that they are valid for other geometrical structures as well. Although, any Cartan-Hadamard manifold (M, g) is diffeomorphic to $\mathbb{R}^n$, $n = \dim M$ (cf. Cartan's theorem), this is a wide class of non-compact Riemannian manifolds including important geometric objects (as Euclidean spaces, hyperbolic spaces, the space of symmetric positive definite matrices endowed with a suitable Killing metric), see Bridson and Haefliger [24].

In the sequel we sketch the structure of the thesis. In the first Chapter (Chapter 1) of the thesis we enumerate the basic definitions and results from the theory of Sobolev spaces in Euclidean setting, the theory of calculus of variations, some recent results from the critical point theory, and finally from Riemannian geometry, which are indispensable in our study.

In the first part of the thesis we present some theoretical achievements. We present here some surprising phenomena. We start with Chapter 2, where we introduce the most important Sobolev inequalities both on the Euclidean and on Riemannian settings.

In Chapter 3 we prove Sobolev-type interpolation inequalities on Cartan-Hadamard manifolds and their optimality whenever the Cartan-Hadamard conjecture holds (e.g., in dimensions 2, 3 and 4). The existence of extremals leads to unexpected rigidity phenomena. This chapter is based on the paper [59].

In Chapter 4 we prove some multipolar Hardy inequalities on complete Riemannian manifolds, providing various curved counterparts of some Euclidean multipolar inequalities due to Cazacu and Zuazua [30]. We notice that our inequalities deeply depend on the curvature, providing (quantitative) information about the deflection from the flat case. This chapter is based on the recent paper [54].

In the second part of the thesis we present some applications, namely we study some PDE's on Riemannian manifolds. In Chapter 5 we study nonlinear Schrödinger-Maxwell systems on 3-dimensional compact Riemannian manifolds proving a new kind of multiplicity result with sublinear and superlinear nonlinearities. This chapter is based on [55].

In Chapter 6, we consider a Schrödinger-Maxwell system on n-dimensional Cartan-Hadamard manifolds, where $3 \leq n \leq 5$. The main difficulty resides in the lack of compactness of such manifolds which is recovered by exploring suitable isometric actions. By combining variational arguments, some existence, uniqueness and multiplicity of isometry-invariant weak solutions are established for such systems depending on the behavior of the nonlinear term. We also present a new set of assumptions ensuring the existence of infinitely many solutions for a quasilinear equation, which can be adapted easily for Schrödinger-Maxwell systems. This Chapter is based on the papers [52, 56].

In Chapter 7, by using inequalities presented in Chapter 4, together with variational methods, we also establish non-existence, existence and multiplicity results for certain Schrödinger-type problems involving the Laplace-Beltrami operator and bipolar potentials on Cartan-Hadamard manifolds. We also mention a multiplicity result for an anisotropic sub-linear elliptic problem with Dirichlet boundary condition, depending on a positive parameter λ. We prove that for enough large values of λ, our anisotropic problem has at least two non-zero distinct solutions. In particular, we show that at least one of the solutions provides a Wulff-type symmetry. This Chapter is based on the papers [54, 57].

In Chapter 8, we consider a Schrödinger type equation on non-compact Riemannian manifolds, depending on a positive parameter λ. By using variational methods we prove a characterization result for existence of solutions for this problem. This chapter is based on the paper [53].

Acknowledgements

> No one who achieves success does so without acknowledging the help of others. The wise and confident acknowledge this help with gratitude.
>
> *(Alfred North Whitehead)*

It is a pleasure for me to express my gratitude to the many people who have helped me or have been an important part of my life in the last years.

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I also want to thank two persons who accompanied me throughout all of my studies and gave me the taste for calculus of variations and critical point theory: my former advisor prof. Csaba Varga and my close collaborator prof. Francesca Faraci. I thank them for their support, their advice, their ideas, and their friendship.

Another person who explained to me many ideas and problems, is prof. Biagio Ricceri. He devoted a lot of time to me in the last several years and deserves a special thanks as well.

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I would like to thank my family for their love, understanding and endless patience. Without their support, I would have never finished this dissertation and I would have never found the courage to overcome all the difficulties during this work. Thank you.

1.

Preliminaries

> If you know the enemy and you know yourself you need not fear the results of a hundred battles.
>
> *(Sun Tzu)*

1.1. Sobolev Spaces: Introduction

Our understanding of the fundamental processes of the natural world is based on partial differential equations. Examples are the vibrations of solids, the flow of fluids, the diffusion of chemicals, the spread of heat, the structure of molecules, the interactions of photons and electrons, and the radiation of electromagnetic waves, some problems arising form biology, computer science (particularly in relation to image processing and graphics) and economics, see Brézis [23], Evans [51], Pinchover and Rubinstein [100], Strauss [116]. Partial differential equations also play a central role in modern mathematics, especially in geometry and analysis.

In studying partial differential equations, a differentiation can be regarded as an operator from one function space to another. Solving a (linear) differential equation is equivalent to find the inverse of an operator. In general, the natural function spaces involved are the Sobolev spaces. Indeed, the theory of Sobolev space has become inseparable with the study of partial differential equations.

1.1.1. Weak derivatives and Sobolev spaces

Assume that there is a function $u \in C^1(\Omega)$, where $\Omega \subset \mathbb{R}^n$. Then if $\varphi \in C_0^\infty(\Omega)$, we see from the integration by parts formula that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \frac{\partial u}{\partial x_i} \varphi dx \ (i = 1, 2, ..., n).$$

There are no boundary terms, since φ has compact support in Ω and thus vanishes near $\partial\Omega$. We see immediately that by repeated integration by parts we can generalize this result to a partial differential operator of arbitrary degree α, so long as we take $u \in C^{|\alpha|}(\Omega)$ and account for the parity dependence of the minus sign:

$$\int_\Omega u D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u \phi dx,$$

If $u \in C^{|\alpha|}(\Omega)$, the above formula is valid for every $\phi \in C_0^\infty(\Omega)$. The notion of the weak derivative asks if this formula is valid when u is not in $C^{|\alpha|}(\Omega)$. We insist that u has to be a locally integrable function (that is, integrable on compact sets), because otherwise the left hand side of the above equality is meaningless.

Definition 1.1.1. *Suppose u, $v \in L^1_{loc}(\Omega)$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index. We say that v is the α^{th} weak derivative of u and write $v = D^\alpha u$ if*

$$\int_\Omega u \, D^\alpha \phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \, \phi \, \mathrm{d}x \qquad \forall \phi \in C_c^\infty(U).$$

Note that the order of differentiation is irrelevant. For example, $u_{x_i x_j} = u_{x_j x_i}$ if one of them exists.

Theorem 1.1.1 (du Bois-Reymond lemma)**.** *Let $|\alpha| = 1$ and let $f \in C(\Omega)$ be such that $\partial^\alpha f \in C(\Omega)$. Then $D^\alpha f$ exists and $D^\alpha f = \partial^\alpha f$.*

Definition 1.1.2. *Assume $\Omega \subset \mathbb{R}^n$ and k is a non–negative integer and $p \in [1, \infty]$. The Sobolev space $W^{k,p}(\Omega)$ consists of those $L^p(\Omega)$ functions all of whose weak derivatives up to order k exist and are in $L^p(\Omega)$. Its norm is defined by*

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha| \le k} \|D^\alpha f\|_{L^p(\Omega)} \tag{1.1.1}$$

When $p \in [1, \infty)$, the space $W_0^{k,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the $\|\cdot\|_{W^{k,p}(\Omega)}$ norm.

When $p = 2$, $W^{k,2}$ and $W_0^{k,2}$ is often written as H^k and H_0^k respectively, which are Hilbert spaces. For $u \in W^{1,p}(\Omega)$ we will write

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n} \right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$||u|| = ||u||_p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p$$

or sometimes with the equivalent norm

$$\left(||u||_p^p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}} \text{ (if } 1 \le p < \infty).$$

We set $H^1(\Omega) = W^{1,2}(\Omega)$. The space $H^1(\Omega)$ is equipped with the scalar product

$$(u, v)_{H^1} = (u, v)_{L^2} + \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2} = \int_\Omega \left(uv + \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right) dx.$$

The associated norm is defined by

$$||u||_{H^1} = \left(||u||_2^2 + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{\frac{1}{2}}.$$

Theorem 1.1.2 (Sobolev spaces as function spaces, Brézis [23], Evans [51])**.** *The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Theorem 1.1.3 (Properties of weak derivatives, Brézis [23], Evans [51])**.** *Assume $u, v \in W^{k,p}(\Omega)$, $|\alpha| \le k$. Then*

(i) $D^{|\alpha|}u \in W^{k-|\alpha|,p}(\Omega)$ and $D^\beta (D^\alpha u) = D^\alpha \left(D^\beta u\right) = D^{\alpha+\beta} u$ for all multiindices α, β with $|\alpha| + |\beta| \le k$.

(ii) For each λ and μ, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v, \ |\alpha| \le k.$$

(iii) If Ω_0 is an open subset of Ω, then $u \in W^{k,p}(\Omega_0)$.

(iv) *If $\zeta \in C_0^\infty(\Omega)$, then $\zeta u \in W^{k,p}(\Omega)$ and*

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u,$$

where $\displaystyle\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

In what follows we enumerate certain properties of the Sobolev spaces:

Theorem 1.1.4. *Assume that $p \in [1, \infty)$.*

(a) $W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the $\|\cdot\|_{W^{k,p}(\mathbb{R})}$ norm.

(b) If $\Omega \subset \mathbb{R}^n$ is bounded with C^1 boundary, then $W^{k,p}(\Omega)$ is the completion of $C^\infty(\bar{\Omega})$ under $\|\cdot\|_{W^{k,p}(\Omega)}$ norm.

Theorem 1.1.5. *Assume that $\Omega \subset \mathbb{R}^n$ is bounded, $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then $\exists\,(u_m)$ in $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_m \to u$ in $W^{1,p}(\Omega)$. In addition, if $\partial\Omega \in C^1$ then we can choose the sequence $\{u_m\}$ such that $u_m \in C^\infty(\overline{\Omega})$ for all m.*

Theorem 1.1.6 (Friedrich theorem, Brézis [23], Evans [51])**.** *Assume that $\Omega \subset \mathbb{R}^n$ is bounded. Let $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then there exists a sequence (u_m) from $C_0^\infty(\mathbb{R}^n)$ such that*

(i) *$u_m|_\Omega \to u$ in $L^p(\Omega)$*

(ii) *$\nabla u_m|_\omega \to \nabla u|_\omega$ in $L^p(\omega)^n$ for each open set ω from $\mathbb{R}^n$ such that $\overline{\omega} \subset \Omega$, $\overline{\omega}$ is compact.*

In the case when $\Omega = \mathbb{R}^n$ and $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$ there exists a sequnece (u_m): u_m from $C_0^\infty(\Omega)$ such that

$$u_m \to u \text{ in } L^p(\Omega)$$

and

$$\nabla u_n \to \nabla u \text{ in } L^p(\Omega)^n.$$

Proposition 1.1.1 (Differentiation of a product, Brézis [23], Evans [51])**.** *Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \leq p \leq \infty$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and*

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i}, \quad i = 1, 2, ..., n.$$

Proposition 1.1.2 (Differentiation of a composition, Brézis [23], Evans [51])**.** *Let $G \in C^1(\Omega)$ be such that $G(0) = 0$ and $|G'(s)| \leq M$ $\forall s \in \mathbb{R}$ for some constant M. Let $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$. Then*

$$G \circ u \in W^{1,p}(\Omega) \text{ and } \frac{\partial}{\partial x_i}(G \circ u) = (G' \circ u)\frac{\partial u}{\partial x_i}, \quad i = 1, 2, ..., n.$$

1.2. Elements from the theory of calculus of variations

1.2.1. Dirichlet principle: Introduction

Laplace's equation is a useful approximation to the physical problem of determining the equilibrium displacement of an elastic membrane. (A membrane is a surface that resists stretching, but does not resist bending). The equation is also fundamental in mechanics, electromagnetism, probability, quantum mechanics, gravity, biology, etc. Poisson's equation, which are the generalization of the Laplace equation, represents the distribution of temperature u in a domain Ω at equilibrium. On physical grounds, one expects such equlibria to be characterized by least energy. We sketch a simple version of this issue, which represents a powerful general idea in the theory of calculus of variations: Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. We are looking $u : \overline{\Omega} \to \mathbb{R}$ satisfying the following equation:

$$\begin{cases} -\Delta u + u = f(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{$\mathscr{P}$}$$

where f is a given function on Ω.

Definition 1.2.1. *A classical solution of the problem ($\mathscr{P}$) is a function $u \in C^2(\overline{\Omega})$, satisfying ($\mathscr{P}$) in the usual sense. A weak solution of ($\mathscr{P}$) is a function $u \in H_0^1(\Omega)$ satisfying*

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx = \int_\Omega f v dx, \ \forall v \in H_0^1(\Omega).$$

In what follows we state a variational characterization for a solution of the problem ($\mathscr{P}$):

Theorem 1.2.1 (Dirichlet's principles)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Given any $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of ($\mathscr{P}$). Moreover, u is obtained by*

$$\min_{v \in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega \left(|\nabla v|^2 + v^2 \right) dx - \int_\Omega f v dx \right\}.$$

To prove that a weak solution is a classical solution, one needs also to prove a regularity of the weak solution.

1.2.2. Direct methods in calculus of variations

We assume that $E : X \to \mathbb{R}$ is functional defined on a Banach space $(X, \|\cdot\|)$. From the previous discussion, we want to find reasonable conditions that guarantees the existence of $\bar{u}$ such that

$$E(\bar{u}) = \min_X E(u).$$

The element $\bar{u}$ is called a minimizer of E, and the problem of finding a minimizer is called a variational problem. As we pointed out before, many phenomena arising in applications (such as geodesics or minimal surfaces) can be understood in terms of the minimization of an energy functional over an appropriate class of objects.

The first result reads as follows:

Theorem 1.2.2 (Compact case)**.** *Let X be a compact topological space and $E : X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous functional. Then E is bounded from below and its infimum is attained on X.*

Definition 1.2.2. *The functional E is sequentially weakly lower semicontinuous (s.w.l.s.c) if for every weakly convergent sequence $u_n \rightharpoonup u$ in X, $E(u) \le \liminf_{n\to\infty} E(u_n)$ holds. The functional E is coercive if $\|u_n\| \to \infty$ implies that $E(u_n) \to \infty$.*

The following theorem shows the main idea behind the direct method in the calculus of variations.

Theorem 1.2.3. *If X is a reflexive Banach space and $E : X \to \mathbb{R}$ is a s.w.l.s.c. and coercive functional, then there exists $\bar{u} \in X$ such that*

$$E(\bar{u}) = \inf_X E(u).$$

In general, the most difficult condition to deal with is the s.w.l.s.c. condition. An important class of functionals for which it is relatively easy to verify this condition is the class of convex functionals, see Zeidler [128]. We use this fact in Chapter 5:

Theorem 1.2.4. *If X is a normed space and $E : X \to \mathbb{R}$ is convex and lower semicontinuous, then E is s.w.l.c.*

Corollary 1.2.1. *If $E : X \to \mathbb{R}$ is a convex, lower semicontinuous, and coercive functional defined an a reflexive Banach space X, then E attains its minimum on X. Moreover, if the functional is strictly convex, the minimum is unique.*

1.2.3. Palais-Smale condition and the Mountain Pass theorem

The Palais-Smale condition is a condition that appears in the application part of the present thesis, so it deserves this place at the beginning. The original condition appears in the works of Palais and Smale:

Definition 1.2.3 (Palais-Smale condition, Willem [124]).
(a) A function $\varphi \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition) if every sequence $\{u_n\} \subset X$ such that

$$\lim_{n\to\infty} \varphi(u_n) = c \text{ and } \lim_{n\to\infty} \|\varphi'(u_n)\| = 0,$$

possesses a convergent subsequence.
(b) A function $\varphi \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition (shortly, (PS)-condition) if it satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.

Combining this compactness condition with the Ekeland's variational principle, one can obtain the following result:

Theorem 1.2.5. *Let X be a Banach space and a function $E \in C^1(X, \mathbb{R})$ which is bounded from below. If E satisfies the $(PS)_c$-condition at level $c = \inf_X f$, then c is a critical value of E, that is, there exists a point $u_0 \in X$ such that $E(u_0) = c$ and u_0 is a critical point of f, that is, $E'(u_0) = 0$.*

One of the simplest and most useful minimax theorems is the following, the so called Mountain-Pass theorem:

Theorem 1.2.6 (Ambrosetti-Rabinowitz, 1974, [4]). *Let X be a Banach space, and $E \in C^1(X, \mathbb{R})$ such that*

$$\inf_{\|u-e_0\|=\rho} E(u) \geq \alpha > \max\{E(e_0), E(e_1)\}$$

for some $\alpha \in \mathbb{R}$ and $e_0 \neq e_1 \in X$ with $0 < \rho < \|e_0 - e_1\|$. If E satisfies the $(PS)_c$-condition at level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E(\gamma(t)),$$

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = e_0,\ \gamma(1) = e_1\},$$

then c is a critical value of E with $c \geq \alpha$.

Sometimes we want to study solutions of the equations which are invariant under some symmetry group. Starting from the original equations, it is possible to calculate the symmetry-reduced equations:

Theorem 1.2.7 (Principle of symmetric criticality, Kristály, Varga and Rădulescu [87]). *Assume that the action of the topological group G on the reflexive and strictly convex Banach space X is isometric. If $\varphi \in C^1(X, \mathbb{R})$ is G-invariant and u is a critical point of φ restricted to the space of invariant points, then u is critical point of φ.*

Theorem 1.2.7 was proved by Palais for functionals of class C^1, later Kobayashi-Otani generalized to the case where the functional is not differentiable. Based on this result we recall the following form of the principle of symmetric criticality for Szulkin functionals which will be applied in the next chapters.

Theorem 1.2.8 (Kobayashi-Otani, [70]). *Let X be a reflexive Banach space and let $I = E + \zeta : X \to \mathbb{R} \cup \{\infty\}$ be a Szulkin-type functional on X. If a compact group G acts linearly and continuously on X, and the functionals E and ζ are G-invariant then u is a critical point of I restricted to the space of invariant points, then u is a critical point of I.*

We conclude this section with the following results:

Theorem 1.2.9 (Ricceri, [106]). *Let H be a separable and reflexive real Banach space, and let $\mathcal{N}, \mathcal{G} : H \to \mathbb{R}$ be two sequentially weakly lower semi-continuous and continuously Gâteaux differentiable functionals., with $\mathcal{N}$ coercive. Assume that the functional $\mathcal{N} + \lambda\mathcal{G}$ satisfies the Palais-Smale condition for every $\lambda > 0$ small enough and that the set of all global minima of $\mathcal{N}$ has at least m connected components in the weak topology, with $m \geq 2$. Then, for every $\eta > \inf_H \mathcal{N}$, there exists $\overline{\lambda} > 0$ such that for every $\lambda \in (0, \overline{\lambda})$ the functional $\mathcal{N} + \lambda\mathcal{G}$ has at least $m+1$ critical points, m of which are in $\mathcal{N}^{-1}((-\infty, \eta))$.*

Theorem 1.2.10 (Ricceri, [105]). *Let E be a reflexive real Banach space, and let $\Phi, \Psi : E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is sequentially weakly lower semi-continuous and coercive. Further, assume that Ψ is sequentially weakly continuous. In addition, assume that, for each $\mu > 0$, the functional $J_\mu := \mu\Phi - \Psi$ satisfies the classical compactness Palais-Smale condition. Then for each $\rho > \inf_E \Phi$ and each*

$$\mu > \inf_{u \in \Phi^{-1}((-\infty,\rho))} \frac{\sup_{v \in \Phi^{-1}((-\infty,\rho))} \Psi(v) - \Psi(u)}{\rho - \Phi(u)},$$

the following alternative holds: either the functional J_μ has a strict global minimum which lies in $\Phi^{-1}((-\infty, \rho))$, or J_μ has at least two critical points one of which lies in $\Phi^{-1}((-\infty, \rho))$.

Theorem 1.2.11 (Ricceri, [108]). *Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, $J : X \to \mathbb{R}$ a sequentially weakly upper semicontinuous and Gâteaux differentiable functional, with $J(0) = 0$. Assume that, for some $r > 0$, there exists a global maximum $\hat{x}$ of the restriction of J to $B_r = \{x \in X : \|x\|^2 \leq r\}$ such that*

$$J'(\hat{x})(\hat{x}) < 2J(\hat{x}). \tag{1.2.1}$$

Then, there exists an open interval $I \subseteq (0, +\infty)$ such that, for each $\lambda \in I$, the equation $x = \lambda J'(x)$ has a non-zero solution with norm less than r.

As it was already pointed out in Ricceri [109], the following remark adds some crucial information about the interval I:

Remark 1.2.1. *Set $\beta_r = \sup_{B_r} J$, $\delta_r = \sup_{x \in B_r \setminus \{0\}} \frac{J(x)}{\|x\|^2}$ and $\eta(s) = \sup_{y \in B_r} \frac{r - \|y\|^2}{s - J(y)}$, for all $s \in (\beta_r, +\infty)$. Then, η is convex and decreasing in $]\beta_r, +\infty[$. Moreover,*

$$I = \frac{1}{2}\eta((\beta_r, r\delta_r)).$$

1.3. Elements from Riemannian geometry

Let (M, g) be an $n-$dimensional complete Riemannian manifold ($n \geq 3$). As usual, T_xM denotes the tangent space at $x \in M$ and $TM = \bigcup_{x \in M} T_xM$ is the tangent bundle. Let $d_g : M \times M \to [0, \infty)$ be the distance function associated to the Riemannian metric g, and

$$B_r(x) = \{y \in M : d_g(x, y) < r\}$$

be the open geodesic ball with center $x \in M$ and radius $r > 0$. If $\mathrm{d}v_g$ is the canonical volume element on (M, g), the volume of a bounded open set $S \subset M$ is

$$\mathrm{Vol}_g(S) = \int_S \mathrm{d}v_g.$$

The behaviour of the volume of small geodesic balls can be expressed as follows, see Gallot, Hulin and Lafontaine [62]; for every $x \in M$ we have

$$\mathrm{Vol}_g(B_r(x)) = \omega_n r^n \left(1 + o(r)\right) \text{ as } r \to 0. \tag{1.3.1}$$

If $\mathrm{d}\sigma_g$ denotes the $(n-1)-$dimensional Riemannian measure induced on ∂S by g,

$$\mathrm{Area}_g(\partial S) = \int_{\partial S} \mathrm{d}\sigma_g$$

denotes the area of ∂S with respect to the metric g.

Let $u : M \to \mathbb{R}$ be a function of class C^1. If (x^i) denotes the local coordinate system on a coordinate neighbourhood of $x \in M$, and the local components of the differential of u are denoted by $u_i = \frac{\partial u}{\partial x_i}$, then the local components of the gradient $\nabla_g u$ are $u^i = g^{ij} u_j$. Here, g^{ij} are the local components of $g^{-1} = (g_{ij})^{-1}$. In particular, for every $x_0 \in M$ one has the eikonal equation

$$|\nabla_g d_g(x_0, \cdot)| = 1 \text{ a.e. on } M. \tag{1.3.2}$$

In fact, relation (1.3.2) is valid for every point $x \in M$ outside of the cut-locus of x_0 (which is a null measure set).

For enough regular $f : [0,\infty) \to \mathbb{R}$ one has the formula

$$-\Delta_g(f(d_g(x_0,x)) = -f''(d_g(x_0,x)) - f'(d_g(x_0,x))\Delta_g(d_g(x_0,x)) \text{ for a.e. } x \in M. \tag{1.3.3}$$

When no confusion arises, if $X, Y \in T_xM$, we simply write $|X|$ and $\langle X, Y\rangle$ instead of the norm $|X|_x$ and inner product $g_x(X,Y) = \langle X, Y\rangle_x$, respectively.

The $L^p(M)$ norm of $\nabla_g u(x) \in T_xM$ is given by

$$\|\nabla_g u\|_{L^p(M)} = \left(\int_M |\nabla_g u|^p \mathrm{d}v_g\right)^{1/p}.$$

The Laplace-Beltrami operator is given by $\Delta_g u = \mathrm{div}(\nabla_g u)$ whose expression in a local chart of associated coordinates (x^i) is $\Delta_g u = g^{ij}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k}\right)$, where Γ_{ij}^k are the coefficients of the Levi-Civita connection.

For every $c \le 0$, let $\mathbf{s}_c, \mathbf{ct}_c : [0,\infty) \to \mathbb{R}$ be defined by

$$\mathbf{s}_c(r) = \begin{cases} r & \text{if } \ c = 0, \\ \frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}} & \text{if } \ c < 0, \end{cases} \quad \text{and} \quad \mathbf{ct}_c(r) = \begin{cases} \frac{1}{r} & \text{if } \ c = 0, \\ \sqrt{-c}\coth(\sqrt{-c}r) & \text{if } \ c < 0, \end{cases} \tag{1.3.4}$$

and let $V_{c,n}(r) = n\omega_n \displaystyle\int_0^r \mathbf{s}_c(t)^{n-1} dt$ be the volume of the ball with radius $r > 0$ in the $n-$dimensional space form (i.e., either the hyperbolic space with sectional curvature c when $c < 0$ or the Euclidean space when $c = 0$), where $\mathbf{s}_c$ is given in (1.3.4). Note that for every $x \in M$ we have

$$\lim_{r \to 0^+} \frac{\mathrm{Vol}_g(B_r(x))}{V_{c,n}(r)} = 1. \tag{1.3.5}$$

The manifold (M,g) has Ricci curvature bounded from below if there exists $h \in \mathbb{R}$ such that $\mathrm{Ric}_{(M,g)} \ge hg$ in the sense of bilinear forms, i.e., $\mathrm{Ric}_{(M,g)}(X,X) \ge h|X|_x^2$ for every $X \in T_xM$ and $x \in M$, where $\mathrm{Ric}_{(M,g)}$ is the Ricci curvature, and $|X|_x$ denotes the norm of X with respect to the metric g at the point x. The notation $\mathbf{K} \le (\ge) c$ means that the sectional curvature is bounded from above(below) by c at any point and direction.

In the sequel, we shall explore the following comparison results (see Shen [112], Wu and Xin [125, Theorems 6.1 & 6.3]):

Theorem 1.3.1. [Volume comparison] *Let (M,g) be a complete, $n-$dimensional Riemannian manifold. Then the following statements hold.*

(a) *If (M,g) is a Cartan-Hadamard manifolds, the function $\rho \mapsto \frac{\mathrm{Vol}_g(B(x,\rho))}{\rho^n}$ is non-decreasing, $\rho > 0$. In particular, from (1.3.5) we have*

$$\mathrm{Vol}_g(B(x,\rho)) \ge \omega_n \rho^n \ \textit{for all } x \in M \textit{ and } \rho > 0. \tag{1.3.6}$$

If equality holds in (1.3.6), then the sectional curvature is identically zero.

(b) *If (M,g) has non-negative Ricci curvature, the function $\rho \mapsto \frac{\mathrm{Vol}_g(B(x,\rho))}{\rho^n}$ is non-increasing, $\rho > 0$. In particular, from (1.3.5) we have*

$$\mathrm{Vol}_g(B(x,\rho)) \leq \omega_n \rho^n \text{ for all } x \in M \text{ and } \rho > 0. \tag{1.3.7}$$

If equality holds in (1.3.7), then the sectional curvature is identically zero.

Theorem 1.3.2. [Laplacian comparison] *Let (M,g) be a complete, $n-$dimensional Riemannian manifold.*

(i) If $\mathbf{K} \leq k_0$ for some $k_0 \in \mathbb{R}$, then

$$\Delta_g d_g(x_0, x) \geq (n-1)\mathbf{ct}_{k_0}(d_g(x_0,x)), \tag{1.3.8}$$

(ii) if $\mathbf{K} \geq k_0$ for some $k_0 \in \mathbb{R}$, then

$$\Delta_g d_g(x_0, x) \leq (n-1)\mathbf{ct}_{k_0}(d_g(x_0,x)), \tag{1.3.9}$$

Note that in (1.3.9) it is enough to have the lower bound $(n-1)k_0$ for the Ricci curvature.

Consider now, a Riemannian manifold with asymptotically non-negative Ricci curvature with a base point $\tilde{x}_0 \in M$, i.e.,

(C) $\mathrm{Ric}_{(M,g)}(x) \geq -(n-1)H(d_g(\tilde{x}_0, x))$, a.e. $x \in M$, where $H \in C^1([0,\infty))$ is a non-negative bounded function satisfying $\displaystyle\int_0^\infty tH(t)dt = b_0 < +\infty$,

For an overview on such property see Adriano and Xia [3], Pigola, Rigoli and Setti [99].

Theorem 1.3.3 (Pigola, Rigoli and Setti [99])**.** *Let (M,g) be an $n-$dimensional complete Riemannian manifold. If (M,g) satisfies the curvature condition* **(C)***, then the following volume growth property holds true:*

$$\frac{\mathrm{Vol}_g(B_x(R))}{\mathrm{Vol}_g(B_x(r))} \leq e^{(n-1)b_0} \left(\frac{R}{r}\right)^n, \ 0 < r < R,$$

and

$$\mathrm{Vol}_g(B_x(\rho)) \leq e^{(n-1)b_0}\omega_n \rho^n, \ \rho > 0.$$

where b_0 is from condition (**C**)*.*

We present here a recent result by Poupaud [101] concerning the discreteness of the spectrum of the operator $-\Delta_g u + V(x)u$. Assume that $V : M \to \mathbb{R}$ is a measurable function satisfying the following conditions:

(V_1) $V_0 = \mathrm{essinf}_{x\in M} V(x) > 0$;

(V_2) $\displaystyle\lim_{d_g(x_0,x)\to\infty} V(x) = +\infty$, for some $x_0 \in M$.

Theorem 1.3.4 (Poupaud, [101])**.** *Let (M,g) be a complete, non-compact n-dimensional Riemannian manifold. Let $V : M \to \mathbb{R}$ be a potential verifying (V_1), (V_2). Assume the following on the manifold M:*

(A_1) there exists $r_0 > 0$ and $C_1 > 0$ such that for any $0 < r \leq \frac{r_0}{2}$, one has $\mathrm{Vol}_g(B_x(2r)) \leq C_1 \mathrm{Vol}_g(B_x(r))$ (doubling property);

(A_2) there exists $q > 2$ and $C_2 > 0$ such that for all balls $B_x(r)$, with $r \leq \frac{r_0}{2}$ and for all $u \in H_g^1(B_x(r))$

$$\left(\int_{B_x(r)} \left|u - u_{B_x(r)}\right|^q \mathrm{d}v_g\right)^{\frac{1}{q}} \leq C_2 r \mathrm{Vol}_g(B_x(r))^{\frac{1}{q}-\frac{1}{2}} \left(\int_{B_x(r)} |\nabla_g u|^2 \mathrm{d}v_g\right)^{\frac{1}{2}},$$

where $u_{B_x(r)} = \dfrac{1}{\mathrm{Vol}_g(B_x(r))} \displaystyle\int_{B_x(r)} u \mathrm{d}v_g$ *(Sobolev- Poincaré inequality).*

Then the spectrum of the operator $-\Delta_g + V(x)$ is discrete.

It is worth mentioning that such result was first obtained by Kondrat'ev and Shubin [73] for manifolds with bounded geometry and relies on the generalization of Molchanov's criterion. However, since the bounded geometry property is a strong assumption and implies the positivity of the radius of injectivity, many efforts have been made for improvement and generalizations. Later, Shen [113] characterized the discretness of the spectrum by using the basic length scale function and the effective potential function. For further recent studies in this topic, we invite the reader to consult the papers Cianchi and Mazya [33, 34] and Bonorino, Klaser and Telichevesky [21].

Part I.

Sobolev-type inequalities

2.

Sobolev-type inequalities

Life is really simple, but we insist on making it complicated.

(Confucius)

2.1. Euclidean case

Sobolev-type inequalities play an indispensable role in the study of certain elliptic problems. We will start our study with the following definition (see for instance Evans [51]):

Definition 2.1.1. *Let X and Y be Banach spaces and $X \subset Y$.*

(i) We say that X is embedded in Y, and written as

$$X \hookrightarrow Y$$

if there exists a constant C such that $\|u\|_Y \le C\|u\|_X$ for all $u \in X$.

(ii) We say that X is compactly embedded in Y and written as

$$X \overset{\text{cpt.}}{\hookrightarrow} Y$$

if (a) $X \hookrightarrow Y$ and (b) every bounded sequence in X is precompact in Y.

Given $1 \le p < n$. Sobolev [115], proved that there exists a constant $\mathtt{C} > 0$ such that for any $u \in C_0^\infty(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \mathrm{d}x\right)^{\frac{1}{p^*}} \le \mathtt{C}\left(\int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x\right)^{\frac{1}{p}}, \tag{2.1.1}$$

where ∇u is the gradient of the function u, and $p^* = \frac{pn}{n-p}$. Later, a more direct argument was applied by Gagliardo [61] and independently Nirenberg [96].

The approaches of Sobolev, Gagliardo and Nirenberg do not give the value of the best constant $\mathtt{C}$. A discussion of the sharp form of (2.1.1) when $n = 3$ and $p = 2$ appeared first in Rosen [110]. Then, in the works by Aubin and Talenti we find the sharp form of (2.1.1). If $\mathtt{C}(n,p)$ is the best constant in (2.1.1), it was shown by these authors, that for $p > 1$,

$$\mathtt{C}(n,p) = \frac{1}{n}\left(\frac{n(p-1)}{n-p}\right)^{1-\frac{1}{p}}\left(\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{p}\right)\Gamma\left(n+1-\frac{n}{p}\omega_{n-1}\right)}\right)^{\frac{1}{n}},$$

where ω_{n-1} is the volume of the unit sphere of $\mathbb{R}^n$. The sharp Sobolev inequality for $p > 1$ then reads as

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}} \le \mathtt{C}(n,p)^p \int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x. \tag{2.1.2}$$

It is easily seen that equality holds in (2.1.2) if u has the form

$$u = \left(\lambda + |x - x_0|^{\frac{p}{p-1}}\right)^{1-\frac{n}{p}}.$$

Theorem 2.1.1 (Sobolev-Gagliardo-Nirenberg, Evans [51])**.** *Assume* $1 \leq p < n$*. Then,*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \mathrm{d}x\right)^{\frac{p}{p^*}} \leq \mathtt{C}(n,p)^p \int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x,$$

for all $u \in C_0^\infty(\mathbb{R}^n)$*.*

An important role in the theory of geometric functional inequalities is played by the interpolation inequality and its limit cases. Sobolev interpolation inequalities, or Gagliardo-Nirenberg inequalities, can be used to establish a priori estimates in PDEs; the reader may consult the very recent paper by Sormani [1].

Fix $n \geq 2$, $p \in (1,n)$ and $\alpha \in (0, \frac{n}{n-p}] \setminus \{1\}$; for every $\lambda > 0$, let

$$h^\lambda_{\alpha,p}(x) = (\lambda + (\alpha-1)\|x\|^{p'})_+^{\frac{1}{1-\alpha}}, \ x \in \mathbb{R}^n,$$

where $p' = \frac{p}{p-1}$ is the conjugate to p, and $r_+ = \max\{0, r\}$ for $r \in \mathbb{R}$, and $\|\cdot\|$ is a norm on $\mathbb{R}^n$. Following Del Pino and Dolbeault [43] and Cordero-Erausquin, Nazaret and Villani [36], the sharp form in $\mathbb{R}^n$ of the Gagliardo-Nirenberg inequality can be states as follows:

Theorem 2.1.2. *Let* $n \geq 2$*,* $p \in (1,n)$*.*

(a) If $1 < \alpha \leq \frac{n}{n-p}$*, then*

$$\|u\|_{L^{\alpha p}} \leq \mathcal{G}_{\alpha,p,n} \|\nabla u\|^\theta_{L^p} \|u\|^{1-\theta}_{L^{\alpha(p-1)+1}}, \ \forall u \in \dot{W}^{1,p}(\mathbb{R}^n), \tag{2.1.3}$$

where

$$\theta = \frac{p^\star(\alpha-1)}{\alpha p(p^\star - \alpha p + \alpha - 1)}, \tag{2.1.4}$$

and the best constant

$$\mathcal{G}_{\alpha,p,n} = \left(\frac{\alpha-1}{p'}\right)^\theta \frac{\left(\frac{p'}{n}\right)^{\frac{\theta}{p}+\frac{\theta}{n}} \left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}\right)^{\frac{1}{\alpha p}} \left(\frac{\alpha(p-1)+1}{\alpha-1}\right)^{\frac{\theta}{p}-\frac{1}{\alpha p}}}{\left(\omega_n \mathsf{B}\left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}, \frac{n}{p'}\right)\right)^{\frac{\theta}{n}}}$$

is achieved by the family of functions $h^\lambda_{\alpha,p}$*,* $\lambda > 0$*;*

(b) If $0 < \alpha < 1$*, then*

$$\|u\|_{L^{\alpha(p-1)+1}} \leq \mathcal{N}_{\alpha,p,n} \|\nabla u\|^\gamma_{L^p} \|u\|^{1-\gamma}_{L^{\alpha p}}, \ \forall u \in \dot{W}^{1,p}(\mathbb{R}^n), \tag{2.1.5}$$

where

$$\gamma = \frac{p^\star(1-\alpha)}{(p^\star - \alpha p)(\alpha p + 1 - \alpha)}, \tag{2.1.6}$$

and the best constant

$$\mathcal{N}_{\alpha,p,n} = \left(\frac{1-\alpha}{p'}\right)^\gamma \frac{\left(\frac{p'}{n}\right)^{\frac{\gamma}{p}+\frac{\gamma}{n}} \left(\frac{\alpha(p-1)+1}{1-\alpha} + \frac{n}{p'}\right)^{\frac{\gamma}{p}-\frac{1}{\alpha(p-1)+1}} \left(\frac{\alpha(p-1)+1}{1-\alpha}\right)^{\frac{1}{\alpha(p-1)+1}}}{\left(\omega_n \mathsf{B}\left(\frac{\alpha(p-1)+1}{1-\alpha}, \frac{n}{p'}\right)\right)^{\frac{\gamma}{n}}}$$

is achieved by the family of functions $h^\lambda_{\alpha,p}$*,* $\lambda > 0$*.*

The original proof of these inequalities for $p > 1$ was based on a symmetrization process, itself based on the isoperimetric inequality, to reduce the problem to the one-dimensional case, which is easier to handle. In [36], the authors give a new proof (which is rather simple and elegant in the Euclidean space) of the optimal Sobolev inequalities above based on the mass transportation and on the Brenier map. Their technique also make it possible to recover the subfamily of Gagliardo-Nirenberg inequalities treated by del Pino and Dolbeault [43] by more standard methods.

2.2. Riemannian case

In the sequel we follow Hebey[66] and Kristály Rădulescu and Varga [87]. Let (M,g) be a Riemannian manifold of dimension n. For $k \in \mathbb{N}$ and $u \in C^\infty(M), \nabla^k u$ denotes the k-th covariant derivative of u (with the convection $\nabla^0 u = u$.) The component of ∇u in the local coordinates $(x^1, \cdots, x^n)$ are given by

$$(\nabla^2)_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}.$$

By definition one has

$$|\nabla^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \cdots i_k} (\nabla^k u)_{j_1 \dots j_k}.$$

For $m \in \mathbb{N}$ and $p \geq 1$ real, we denote by $\mathcal{C}_k^m(M)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla^j u| \in L^p(M)$ for any $j = 0, \cdots, k$. Hence,

$$\mathcal{C}_k^p = \left\{ u \in C^\infty(M) : \ \forall\ j = 0, ..., k, \quad \int_M |\nabla^j u|^p dv(g) < \infty \ \right\}$$

where, in local coordinates, $\mathrm{d}v_g = \sqrt{det(g_{ij})}dx$, and where dx stands for the Lebesque's volume element of $\mathbb{R}^n$. If M is compact, on has that $\mathcal{C}_k^p(M) = C^\infty(M)$ for all k and $p \geq 1$.

Definition 2.2.1. The Sobolev space $H_k^p(M)$ is the completion of $\mathcal{C}_k^p(M)$ with respect the norm

$$||u||_{H_k^p} = \sum_{j=0}^{k} \left(\int_M |\nabla^j u|^p \mathrm{d}v_g \right)^{\frac{1}{p}}.$$

More precisely, one can look at $H_k^p(M)$ as the space of functions $u \in L^p(M)$ which are limits in $L^p(M)$ of a Cauchy sequence $(u_m) \subset \mathcal{C}_k$, and define the norm $||u||_{H_k^p}$ as above where $|\nabla^j u|, 0 \leq j \leq k$, is now the limit in $L^p(M)$ of $|\nabla^j u_m|$. These space are Banach spaces, and if $p > 1$, then H_k^p is reflexive. We note that, if M is compact, $H_k^p(M)$ does not depend on the Riemannian metric. If $p = 2$, $H_k^2(M)$ is a Hilbert space when equipped with the equivalent norm

$$||u|| = \sqrt{\sum_{j=0}^{k} \int_M |\nabla^j u|^2 \mathrm{d}v_g}. \tag{2.2.1}$$

The scalar product $\langle \cdot, \cdot \rangle$ associated to $|| \cdot ||$ is defined in local coordinates by

$$\langle u, v \rangle = \sum_{m=0}^{k} \int_M \left(g^{i_1 j_1} \cdots g^{i_m j_m} (\nabla^m u)_{i_1 \dots i_m} (\nabla^m v)_{j_1 \dots j_m} \right) \mathrm{d}v_g. \tag{2.2.2}$$

We denote by $C^k(M)$ the set of k times continuously differentiable functions, for which the norm

$$\|u\|_{C^k} = \sum_{i=1}^{n} \sup_M |\nabla^i u|$$

is finite. The Hölder space $C^{k,\alpha}(M)$ is defined for $0 < \alpha < 1$ as the set of $u \in C^k(M)$ for which the norm

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \sup_{x,y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}$$

is finite, where the supremum is over all $x \neq y$ such that y is contained in a normal neighborhood of x, and $\nabla^k u(y)$ is taken to mean the tensor at x obtained by parallel transport along the radial geodesics from x to y.

As usual, $C^\infty(M)$ and $C_0^\infty(M)$ denote the spaces of smooth functions and smooth compactly supported functions on M respectively.

Definition 2.2.2. The Sobolev space $\overset{\circ}{H}{}^p_k(M)$ is the closure of $C_0^\infty(M)$ in $H_k^p(M)$.

If (M,g) is a complete Riemannian manifold, then for any $p \geq 1$, we have $\overset{\circ}{H}{}^p_k(M) = H_k^p(M)$.

We finish this section with the Sobolev embedding theorem and the Rellich–Kondrachov result for compact manifolds without and with boundary.

Theorem 2.2.1. *(Sobolev embedding theorems for compact manifolds) Let M be a compact Riemannian manifold of dimension n.*

a) If $\frac{1}{r} \geq \frac{1}{p} - \frac{k}{n}$, then the embedding $H_k^p(M) \hookrightarrow L^r(M)$ is continuous.

b) (Rellich-Kondrachov theorem) Suppose that the inequality in a) is strict, then the embedding $H_k^p(M) \hookrightarrow L^r(M)$ is compact.

It was proved by Aubin [8] and independently by Cantor [26] that the Sobolev embedding $H_g^1(M) \hookrightarrow L^{2^*}(M)$ is continuous for complete manifolds with bounded sectional curvature and positive injectivity radius. The above result was generalized (see Hebey, [66]) for manifolds with Ricci curvature bounded from below and positive injectivity radius. Taking into account that, if (M,g) is an n-dimensional complete non-compact Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius, then $\inf_{x \in M} \mathrm{Vol}_g(B_x(1)) > 0$ (see Croke [37]), we have the following result:

Theorem 2.2.2 (Hebey [66], Varaopoulos[122])**.** *Let (M,g) be a complete, non-compact n-dimensional Riemannian manifold such that its Ricci curvature is bounded from below and $\inf_{x \in M} \mathrm{Vol}_g(B_x(1)) > 0$. Then the embedding $H_1^q(M) \subset L^p(M)$ is continuous for $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$.*

We conclude this section, recalling some rigidity results:

If (M,g) is a complete Riemannian manifold, with $\dim M = n$, we may introduce the Sobolev constant

$$K(p,M) = \inf\left\{ \frac{\|\nabla u\|_{L^p}}{\|u\|_{L^{pn/(n-p)}}} : u \in C_0^\infty(M) \right\}.$$

M. Ledoux [88] proved the following result: if (M,g) is a complete Riemannian manifold with non-negative Ricci curvature such that $K(p,M) = K(p,\mathbb{R}^n)$, then (M,g) is the Euclidean space.

Further first-order Sobolev-type inequalities on Riemannian/Finsler manifolds can be found in Bakry, Concordet and Ledoux [11], Druet, Hebey and Vaugon [50], do Carmo and Xia [47], Kristály [78]; moreover, similar Sobolev inequalities are also considered on 'nonnegatively' curved metric measure spaces formulated in terms of the Lott-Sturm-Villani-type curvature-dimension condition or the Bishop-Gromov-type doubling measure condition, see Kristály [80] and Kristály and Ohta [83]. Also, Barbosa and Kristály [13] proved that if (M,g) is an $n-$dimensional complete open Riemannian manifold with nonnegative Ricci curvature verifying $\rho\Delta_g\rho \geq n-5 \geq 0$, supports the second-order Sobolev inequality with the euclidean constant if and only if (M,g) is isometric to the Euclidean space $\mathbb{R}^n$.

For simplicity reason, we denote by $H_g^1(M)$ the completion of $C_0^\infty(M)$ with respect to the norm

$$\|u\|_{H_g^1(M)} = \sqrt{\|u\|^2_{L^2(M)} + \|\nabla_g u\|^2_{L^2(M)}}.$$

Consider $V : M \to \mathbb{R}$. We assume that:

(V_1) $V_0 = \inf_{x \in M} V(x) > 0$;

(V_2) $\lim_{d_g(x_0,x) \to \infty} V(x) = +\infty$ for some $x_0 \in M$, Let us consider now, the functional space

$$H^1_V(M) = \left\{ u \in H^1_g(M) : \int_M \left(|\nabla_g u|^2 + V(x)u^2\right) \mathrm{d}v_g < +\infty \right\}$$

endowed with the norm

$$\|u\|_V = \left(\int_M |\nabla_g u|^2 \mathrm{d}v_g + \int_M V(x)u^2 \mathrm{d}v_g \right)^{1/2}.$$

Lemma 2.2.1. *Let (M, g) be a complete, non-compact $n-$dimensional Riemannian manifold. If V satisfies (V_1) and (V_2), the embedding $H^1_V(M) \hookrightarrow L^p(M)$ is compact for all $p \in [2, 2^*)$.*

Proof. Let $\{u_k\}_k \subset H^1_V(M)$ be a bounded sequence in $H^1_V(M)$, i.e., $\|u_k\|_V \leq \eta$ for some $\eta > 0$. Let $q > 0$ be arbitrarily fixed; by (V_2), there exists $R > 0$ such that $V(x) \geq q$ for every $x \in M \setminus B_R(x_0)$. Thus,

$$\int_{M\setminus B_R(x_0)} (u_k - u)^2 \mathrm{d}v_g \leq \frac{1}{q} \int_{M\setminus B_R(x_0)} V(x)|u_k - u|^2 \leq \frac{(\eta + \|u\|_V)^2}{q}.$$

On the other hand, by (V_1), we have that $H^1_V(M) \hookrightarrow H^1_g(M) \hookrightarrow L^2_{\mathrm{loc}}(M)$; thus, up to a subsequence we have that $u_k \to u$ in $L^2_{\mathrm{loc}}(M)$. Combining the above two facts and taking into account that $q > 0$ can be arbitrary large, we deduce that $u_k \to u$ in $L^2(M)$; thus the embedding follows for $p = 2$. Now, if $p \in (2, 2^*)$, by using an interpolation inequality and the Sobolev inequality on Cartan-Hadamard manifolds (see Hebey [66, Chapter 8]), one has

$$\begin{aligned} \|u_k - u\|^p_{L^p(M)} &\leq \|u_k - u\|^{n(p-2)/2}_{L^{2^*}(M)} \|u_k - u\|^{n(1-p/2^*)}_{L^2(M)} \\ &\leq \mathcal{C}_n \|\nabla_g(u_k - u)\|^{n(p-2)/2}_{L^2(M)} \|u_k - u\|^{n(1-p/2^*)}_{L^2(M)}, \end{aligned}$$

where $\mathcal{C}_n > 0$ depends on n. Therefore, $u_k \to u$ in $L^p(M)$ for every $p \in (2, 2^*)$. □

3. Sobolev interpolation inequalities on Cartan-Hadamard manifolds

> The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.
>
> *(Henri Poincaré)*

3.1. Statement of main results

The Gagliardo-Nirenberg interpolation inequality reduces to the optimal Sobolev inequality when[1] $\alpha = \frac{n}{n-p}$, see Talenti [119] and Aubin [8]. We also note that the families of extremal functions in Theorem 2.1.2 (with $\alpha \in \left(\frac{1}{p}, \frac{n}{n-p}\right] \setminus \{1\}$) are *uniquely* determined up to translation, constant multiplication and scaling, see Cordero-Erausquin, Nazaret and Villani [36], Del Pino and Dolbeault [43]. In the case $0 < \alpha \leq \frac{1}{p}$, the uniqueness of $h^{\lambda}_{\alpha,p}$ is not known.

Recently, Kristály [80] studied Gagliardo-Nirenberg inequalities on a generic metric measure space which satisfies the Lott-Sturm-Villani curvature-dimension condition $\mathsf{CD}(K, n)$ for some $K \geq 0$ and $n \geq 2$, by establishing some global non-collapsing $n-$dimensional volume growth properties.

A similar study can be found also in Kristály and Ohta [83] for a class of Caffarelli-Kohn-Nirenberg inequalities.

The purpose of the present chapter is study the counterpart of the aforementioned papers; namely, we shall consider spaces which are non-positively curved.

To be more precise, let (M, g) be an $n(\geq 2)-$dimensional Cartand-Hadamard manifold (i.e., a complete, simply connected Riemannian manifold with non-positive sectional curvature) endowed with its canonical volume form dv_g. We say that the *Cartan-Hadamard conjecture holds on* (M, g) if

$$\mathrm{Area}_g(\partial D) \geq n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g(D)^{\frac{n-1}{n}} \tag{3.1.1}$$

for any bounded domain $D \subset M$ with smooth boundary ∂D and equality holds in (3.1.1) if and only if D is isometric to the $n-$dimensional Euclidean ball with volume $\mathrm{Vol}_g(D)$, see Aubin [8]. Note that $n\omega_n^{\frac{1}{n}}$ is precisely the isoperimetric ratio in the Euclidean setting. Hereafter, $\mathrm{Area}_g(\partial D)$ stands for the area of ∂D with respect to the metric induced on ∂D by g, and $\mathrm{Vol}_g(D)$ is the volume of D with respect to g. We note that the Cartan-Hadamard conjecture is true in dimension 2 (cf. Beckenbach and Radó [17]) in dimension 3 (cf. Kleiner [68]); and in dimension 4 (cf. Croke [37]), but it is open for higher dimensions.

For $n \geq 3$, Croke [37] proved a general isoperimetric inequality on Hadamard manifolds:

$$\mathrm{Area}_g(\partial D) \geq C(n) \mathrm{Vol}_g(D)^{\frac{n-1}{n}} \tag{3.1.2}$$

[1] Based on the paper [59]

for any bounded domain $D \subset M$ with smooth boundary ∂D, where

$$C(n) = (n\omega_n)^{1-\frac{1}{n}} \left((n-1)\omega_{n-1} \int_0^{\frac{\pi}{2}} \cos^{\frac{n}{n-2}}(t) \sin^{n-2}(t) dt \right)^{\frac{2}{n}-1}. \tag{3.1.3}$$

Note that $C(n) \leq n\omega_n^{\frac{1}{n}}$ for every $n \geq 3$ while equality holds if and only if $n = 4$. Let $C(2) = 2\sqrt{\pi}$.

By suitable symmetrization on Cartan-Hadamard manifolds, inspired by Hebey [66], Ni [95] and Perelman [98], our main results can be stated as follows:

Theorem 3.1.1. *Let (M, g) be an $n(\geq 2)-$dimensional Cartan-Hadamard manifold, $p \in (1, n)$ and $\alpha \in (1, \frac{n}{n-p}]$. Then we have:*

(i) *The Gagliardo-Nirenberg inequality*

$$\|u\|_{L^{\alpha p}(M)} \leq \mathcal{C} \|\nabla_g u\|^{\theta}_{L^p(M)} \|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)}, \quad \forall u \in C_0^\infty(M) \tag{$\mathbf{(GN1)}^{\alpha,p}_{\mathcal{C}}$}$$

holds for $\mathcal{C} = \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)} \right)^{\theta} \mathcal{G}_{\alpha,p,n}$;

(ii) *If the Cartan-Hadamard conjecture holds on (M, g), then the optimal Gagliardo-Nirenberg inequality $\mathbf{(GN1)}^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ holds on (M, g), i.e.,*

$$\mathcal{G}^{-1}_{\alpha,p,n} = \inf_{u \in C_0^\infty(M) \setminus \{0\}} \frac{\|\nabla_g u\|^{\theta}_{L^p(M)} \|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)}}{\|u\|_{L^{\alpha p}(M)}}. \tag{3.1.4}$$

In almost similar way, we can prove the following result:

Theorem 3.1.2. *Let (M, g) be an $n(\geq 2)-$dimensional Cartan-Hadamard manifold, $p \in (1, n)$ and $\alpha \in (0, 1)$. Then we have:*

(i) *The Gagliardo-Nirenberg inequality*

$$\|u\|_{L^{\alpha(p-1)+1}(M)} \leq \mathcal{C} \|\nabla_g u\|^{\gamma}_{L^p(M)} \|u\|^{1-\gamma}_{L^{\alpha p}(M)}, \quad \forall u \in \mathrm{Lip}_0(M) \tag{$\mathbf{(GN2)}^{\alpha,p}_{\mathcal{C}}$}$$

holds for $\mathcal{C} = \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)} \right)^{\gamma} \mathcal{N}_{\alpha,p,n}$;

(ii) *If the Cartan-Hadamard conjecture holds on (M, g), then the optimal Gagliardo-Nirenberg inequality $\mathbf{(GN2)}^{\alpha,p}_{\mathcal{N}_{\alpha,p,n}}$ holds on (M, g), i.e.,*

$$\mathcal{N}^{-1}_{\alpha,p,n} = \inf_{u \in C_0^\infty(M) \setminus \{0\}} \frac{\|\nabla_g u\|^{\gamma}_{L^p(M)} \|u\|^{1-\gamma}_{L^{\alpha p}(M)}}{\|u\|_{L^{\alpha(p-1)+1}(M)}}.$$

Remark 3.1.1. Optimal Sobolev-type inequalities (Nash's inequality, Morrey-Sobolev inequality, and L^2-logarithmic Sobolev inequality) have been obtained on Cartan-Hadamard manifolds whenever (3.1.1) holds, see Druet, Hebey and Vaugon [50], Hebey [66], Kristály [77], Ni [95], and indicated in Perelman [98, p. 26].

Although in Theorems 3.1.1-3.1.2 we stated optimal Gagliardo-Nirenberg-type inequalities, the *existence of extremals* is *not* guaranteed. In fact, we prove that the existence of extremals, having similar geometric features as their Euclidean counterparts, implies novel rigidity results.

Before to state this result, we need one more notion (see Kristály [77]): a function $u : M \to [0, \infty)$ is *concentrated around* $x_0 \in M$ if for every $0 < t < \|u\|_{L^\infty}$ the level set $\{x \in M : u(x) > t\}$ is a geodesic ball $B_{x_0}(r_t)$ for some $r_t > 0$. Note that in $\mathbb{R}^n$ (cf. Theorem 2.1.2) the extremal function $h^\lambda_{\alpha,p}$ is concentrated around the origin.

We can state the following characterization concerning the extremals:

Theorem 3.1.3. *Let (M,g) be an $n(\geq 2)-$dimensional Cartan-Hadamard manifold which satisfies the Cartan-Hadamard conjecture, $p \in (1,n)$ and $x_0 \in M$. The following statements are equivalent:*

(i) *For a fixed $\alpha \in \left(1, \dfrac{n}{n-p}\right]$, there exists a bounded positive extremal function in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ concentrated around x_0;*

(ii) *For a fixed $\alpha \in \left(\dfrac{1}{p}, 1\right)$, to every $\lambda > 0$ there exists a non-negative extremal function $u_\lambda \in C_0^\infty(M)$ in $(\mathbf{GN2})^{\alpha,p}_{\mathcal{N}_{\alpha,p,n}}$ concentrated around x_0 and $\mathrm{Vol}_g(\mathrm{supp}(u_\lambda)) = \lambda$;*

(iii) *(M,g) is isometric to the Euclidean space $\mathbb{R}^n$.*

Remark 3.1.2. The proof of Theorem 3.1.3 deeply exploits the uniqueness of the family of extremal functions in the Gagliardo-Nirenberg-type inequalities; this is the reason why the case $\alpha \in (0, \frac{1}{p}]$ in Theorem 3.1.3 (ii) is not considered.

3.2. Proof of main results

In this section we shall prove Theorems 3.1.1-3.1.3; before to do this, we recall some elements from symmetrization arguments on Riemannian manifolds, following Druet, Hebey and Vaugon, see [48], [50] and [66], and Ni [95, p. 95].

We first recall the following Aubin-Hebey-type result, see Kristály [80]:

Proposition 3.2.1. *Let (M,g) be a complete $n-$dimensional Riemannian manifold and $\mathcal{C} > 0$. The following statements hold:*

(i) *If $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$ holds on (M,g) for some $p \in (1,n)$ and $\alpha \in \left(1, \dfrac{n}{n-p}\right]$ then $\mathcal{C} \geq \mathcal{G}_{\alpha,p,n}$;*

(ii) *If $(\mathbf{GN2})^{\alpha,p}_{\mathcal{C}}$ holds on (M,g) for some $p \in (1,n)$ and $\alpha \in (0,1)$ then $\mathcal{C} \geq \mathcal{N}_{\alpha,p,n}$;*

Let (M,g) be an $n-$dimensional Cartan-Hadamard manifold $(n \geq 2)$ endowed with its canonical form dv_g. By using classical Morse theory and density arguments, in order to handle Gagliardo-Nirenberg-type inequalities (and generic Sobolev inequalities), it is enough to consider continuous test functions $u : M \to [0,\infty)$ having compact support $S \subset M$, where S is smooth enough, u being of class C^2 in S and having only non-degenerate critical points in S. Due to Druet, Hebey and Vaugon [50], we associate to such a function $u : M \to [0,\infty)$ its *Euclidean rearrangement* function $u^* : \mathbb{R}^n \to [0,\infty)$ which is radially symmetric, non-increasing in $|x|$, and for every $t > 0$ is defined by

$$\mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\}) = \mathrm{Vol}_g(\{x \in M : u(x) > t\}). \tag{3.2.1}$$

Here, Vol_e denotes the usual $n-$dimensional Euclidean volume. The following properties are crucial in the proof of Theorems 3.1.1-3.1.3:

Theorem 3.2.1. *Let (M,g) be an $n(\geq 2)-$dimensional Cartan-Hadamard manifold. Let $u : M \to [0,\infty)$ be a non-zero function with the above properties and $u^* : \mathbb{R}^n \to [0,\infty)$ its Euclidean rearrangement function. Then the following properties hold:*

(i) Volume-preservation:

$$\mathrm{Vol}_g(\mathrm{supp}(u)) = \mathrm{Vol}_e(\mathrm{supp}(u^*));$$

(ii) Norm-preservation: *for every* $q \in (0, \infty]$,

$$\|u\|_{L^q(M)} = \|u^*\|_{L^q(\mathbb{R}^n)};$$

(iii) Pólya-Szegő inequality: *for every* $p \in (1, n)$,

$$\frac{n\omega_n^{\frac{1}{n}}}{C(n)}\|\nabla_g u\|_{L^p(M)} \geq \|\nabla u^*\|_{L^p(\mathbb{R}^n)},$$

where $C(n)$ *is from* (3.1.3). *Moreover, if the Cartan-Hadamard conjecture holds, then*

$$\|\nabla_g u\|_{L^p(M)} \geq \|\nabla u^*\|_{L^p(\mathbb{R}^n)}. \tag{3.2.2}$$

Proof. (i)&(ii) It is clear that u^* is a Lipschitz function with compact support, and by definition, one has

$$\|u\|_{L^\infty(M)} = \|u^*\|_{L^\infty(\mathbb{R}^n)}, \tag{3.2.3}$$

$$\mathrm{Vol}_g(\mathrm{supp}(u)) = \mathrm{Vol}_e(\mathrm{supp}(u^*)). \tag{3.2.4}$$

Let $q \in (0, \infty)$. By the layer cake representation easily follows that

$$\begin{aligned}
\|u\|_{L^q(M)}^q &= \int_M u^q dv_g \\
&= \int_0^\infty \mathrm{Vol}_g(\{x \in M : u(x) > t^{\frac{1}{q}}\})dt \\
&\overset{(3.2.1)}{=} \int_0^\infty \mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t^{\frac{1}{q}}\})dt \\
&= \int_{\mathbb{R}^n} (u^*(x))^q dx \\
&= \|u^*\|_{L^q(\mathbb{R}^n)}^q.
\end{aligned}$$

(iii) We follow the arguments from Hebey [66], Ni [95] and Perelman [98]. For every $0 < t < \|u\|_{L^\infty}$, we consider the level sets

$$\Gamma_t = u^{-1}(t) \subset S \subset M, \quad \Gamma_t^* = (u^*)^{-1}(t) \subset \mathbb{R}^n,$$

which are the boundaries of the sets $\{x \in M : u(x) > t\}$ and $\{x \in \mathbb{R}^n : u^*(x) > t\}$, respectively. Since u^* is radially symmetric, the set Γ_t^* is an $(n-1)-$dimensional sphere for every $0 < t < \|u\|_{L^\infty(M)}$. If Area_e denotes the usual $(n-1)-$dimensional Euclidean area, the Euclidean isoperimetric relation gives that

$$\mathrm{Area}_e(\Gamma_t^*) = n\omega_n^{\frac{1}{n}}\mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\})^{\frac{n-1}{n}}.$$

Due to Croke's estimate (see relation (3.1.2)) and (3.2.1), it follows that

$$\begin{aligned}
\mathrm{Area}_g(\Gamma_t) &\geq C(n)\mathrm{Vol}_g(\{x \in M : u(x) > t\})^{\frac{n-1}{n}} \\
&= C(n)\mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\})^{\frac{n-1}{n}} \\
&= \frac{C(n)}{n\omega_n^{\frac{1}{n}}}\mathrm{Area}_e(\Gamma_t^*).
\end{aligned} \tag{3.2.5}$$

If we introduce the notation

$$\begin{aligned}
V(t) &:= \mathrm{Vol}_g(\{x \in M : u(x) > t\}) \\
&= \mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\}),
\end{aligned}$$

the co-area formula (see Chavel [32, pp. 302-303]) gives

$$V'(t) = -\int_{\Gamma_t} \frac{1}{|\nabla_g u|} d\sigma_g = -\int_{\Gamma_t^*} \frac{1}{|\nabla u^*|} d\sigma_e, \tag{3.2.6}$$

where $d\sigma_g$ (resp. $d\sigma_e$) denotes the natural $(n-1)-$dimensional Riemannian (resp. Lebesgue) measure induced by dv_g (resp. dx). Since $|\nabla u^*|$ is constant on the sphere Γ_t^*, by the second relation of (3.2.6) it turns out that

$$V'(t) = -\frac{\mathrm{Area}_e(\Gamma_t^*)}{|\nabla u^*(x)|},\ x \in \Gamma_t^*. \tag{3.2.7}$$

Hölder's inequality and the first relation of (3.2.6) imply that

$$\mathrm{Area}_g(\Gamma_t) = \int_{\Gamma_t} d\sigma_g \leq (-V'(t))^{\frac{p-1}{p}} \left(\int_{\Gamma_t} |\nabla_g u|^{p-1} d\sigma_g \right)^{\frac{1}{p}}.$$

Therefore, by (3.2.5) and (3.2.7), for every $0 < t < \|u\|_{L^\infty(M)}$ we have $(x \in \Gamma_t^*)$

$$\begin{aligned}
\int_{\Gamma_t} |\nabla_g u|^{p-1} d\sigma_g &\geq \mathrm{Area}_g(\Gamma_t)^p \left(-V'(t)\right)^{1-p} \\
&\geq \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}} \right)^p \mathrm{Area}_e(\Gamma_t^*)^p \left(\frac{\mathrm{Area}_e(\Gamma_t^*)}{|\nabla u^*(x)|} \right)^{1-p} \\
&= \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}} \right)^p \int_{\Gamma_t^*} |\nabla u^*|^{p-1} d\sigma_e.
\end{aligned}$$

The latter estimate and the co-area formula give

$$\begin{aligned}
\int_M |\nabla_g u|^p \mathrm{d}v_g &= \int_0^\infty \int_{\Gamma_t} |\nabla_g u|^{p-1} d\sigma_g dt \\
&\geq \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}} \right)^p \int_0^\infty \int_{\Gamma_t^*} |\nabla u^*|^{p-1} d\sigma_e dt \\
&= \left(\frac{C(n)}{n\omega_n^{\frac{1}{n}}} \right)^p \int_{\mathbb{R}^n} |\nabla u^*|^p dx,
\end{aligned} \tag{3.2.8}$$

which concludes the first part of the proof.

If the Cartan-Hadamard conjecture holds, we can apply (3.1.1) instead of (3.1.2), obtaining in place of (3.2.5) that

$$\mathrm{Area}_g(\Gamma_t) \geq \mathrm{Area}_e(\Gamma_t^*) \ \text{ for every } 0 < t < \|u\|_{L^\infty(M)}, \tag{3.2.9}$$

and subsequently,

$$\int_M |\nabla_g u|^p \mathrm{d}v_g \geq \int_{\mathbb{R}^n} |\nabla u^*|^p dx,$$

which ends the proof. □

Remark 3.2.1. Relation (3.2.8) is a kind of quantitative Pólya-Szegő inequality on generic Cartan-Hadamard manifolds which becomes optimal whenever the Cartan-Hadamard conjecture holds. For another type of quantitative Pólya-Szegő inequality (in the Euclidean setting) the reader may consult Cianchi, Esposito, Fusco and Trombetti [35] where the gap between $\|\nabla u\|_{L^p}$ and $\|\nabla u^*\|_{L^p}$ is estimated.

Proof of Theorem 3.1.1. (i) Let $u : M \to [0, \infty)$ be an arbitrarily fixed test function with the above properties (i.e., it is continuous with a compact support $S \subset M$, S being smooth enough and u of class C^2 in S with only non-degenerate critical points in S). According to Theorem A, the Euclidean rearrangement $u^* : \mathbb{R}^n \to [0, \infty)$ of u satisfies the optimal Gagliardo-Nirenberg inequality (2.1.3), thus Theorem 3.2.1 (ii)&(iii) implies that

$$\begin{aligned}
\|u\|_{L^{\alpha p}(M)} &= \|u^*\|_{L^{\alpha p}(\mathbb{R}^n)} \\
&\leq \mathcal{G}_{\alpha,p,n}\|\nabla u^*\|^{\theta}_{L^p(\mathbb{R}^n)}\|u^*\|^{1-\theta}_{L^{\alpha(p-1)+1}(\mathbb{R}^n)} \\
&\leq \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)}\right)^{\theta}\mathcal{G}_{\alpha,p,n}\|\nabla_g u\|^{\theta}_{L^p(M)}\|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)},
\end{aligned}$$

which means that the inequality $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$ holds on (M, g) for $\mathcal{C} = \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)}\right)^{\theta}\mathcal{G}_{\alpha,p,n}$.

(ii) If the Cartan-Hadamard conjecture holds, then a similar argument as above and (3.2.2) imply that

$$\begin{aligned}
\|u\|_{L^{\alpha p}(M)} &= \|u^*\|_{L^{\alpha p}(\mathbb{R}^n)} \\
&\leq \mathcal{G}_{\alpha,p,n}\|\nabla u^*\|^{\theta}_{L^p(\mathbb{R}^n)}\|u^*\|^{1-\theta}_{L^{\alpha(p-1)+1}(\mathbb{R}^n)} \\
&\leq \mathcal{G}_{\alpha,p,n}\|\nabla_g u\|^{\theta}_{L^p(M)}\|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)},
\end{aligned} \tag{3.2.10}$$

i.e., $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ holds on (M, g). Moreover, Proposition 3.2.1 shows that $(\mathbf{GN1})^{\alpha,p}_{\mathcal{C}}$ cannot hold with $\mathcal{C} < G_{\alpha,p,n}$, which ends the proof of the optimality in (3.1.4). □

Proof of Theorem 3.1.2. One can follow step by step the line of the proof of Theorem 3.1.1, combining Theorem 3.2.1 with Theorem 2.1.2 and Proposition 3.2.1, respectively. □

Proof of Theorem 3.1.3. We assume that the Cartan-Hadamard manifold (M, g) satisfies the Cartan-Hadamard conjecture.

(iii)⇒(i)∧(ii). These implications easily follow from Theorem 2.1.2, taking into account the shapes of extremal functions $h^{\lambda}_{\alpha,p}$ in the Euclidean case.

(i)⇒(iii) Let us fix $\alpha \in \left(1, \dfrac{n}{n-p}\right]$, and assume that there exists a bounded positive extremal function $u : M \to [0, \infty)$ in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ concentrated around x_0. By rescaling, we may assume that $\|u\|_{L^\infty(M)} = 1$. Since u is an extremal function, we have equalities in relation (3.2.10) which implies that the Euclidean rearrangement $u^* : \mathbb{R}^n \to [0, \infty)$ of u is an extremal function in the optimal Euclidean Gagliardo-Nirenberg inequality (2.1.3). Thus, the uniqueness (up to translation, constant multiplication and scaling) of the extremals in (2.1.3) and

$$\|u^*\|_{L^\infty(\mathbb{R}^n)} = \|u\|_{L^\infty(M)} = 1$$

determine the shape of u^* which is given by

$$u^*(x) = (1 + c_0|x|^{p'})^{\frac{1}{1-\alpha}},\ x \in \mathbb{R}^n,$$

for some $c_0 > 0$. By construction, u^* is concentrated around the origin and for every $0 < t < 1$, we have

$$\{x \in \mathbb{R}^n : u^*(x) > t\} = B_0(r_t), \tag{3.2.11}$$

where $r_t = c_0^{-\frac{1}{p'}}(t^{1-\alpha} - 1)^{\frac{1}{p'}}$.

We claim that

$$\{x \in M : u(x) > t\} = B_{x_0}(r_t),\ 0 < t < 1. \tag{3.2.12}$$

Here, $B_{x_0}(r)$ denotes the geodesic ball in (M, g) with center x_0 and radius $r > 0$. By assumption, the function u is concentrated around x_0, thus there exists $r_t' > 0$ such that $\{x \in M : u(x) > t\} = B_{x_0}(r_t')$. We are going to prove that $r_t' = r_t$, which proves the claim.

According to (3.2.1) and (3.2.11), one has

$$\begin{aligned}
\mathrm{Vol}_g(B_{x_0}(r_t')) &= \mathrm{Vol}_g(\{x \in M : u(x) > t\}) \\
&= \mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\}) \qquad (3.2.13)\\
&= \mathrm{Vol}_e(B_0(r_t)). \qquad (3.2.14)
\end{aligned}$$

Furthermore, since u is an extremal function in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$, by the equalities in (3.2.10) and Theorem 3.2.1 (ii), it turns out that we have actually equality also in the Pólya-Szegő inequality, i.e.,

$$\|\nabla_g u\|_{L^p(M)} = \|\nabla u^*\|_{L^p(\mathbb{R}^n)}.$$

A closer inspection of the proof of Pólya-Szegő inequality (see Theorem 3.2.1 (iii)) applied for the functions u and u^* shows that we have also equality in (3.2.9), i.e.,

$$\mathrm{Area}_g(\Gamma_t) = \mathrm{Area}_e(\Gamma_t^*),\ \ 0 < t < 1.$$

In particular, the latter relation, the isoperimetric equality for the pair $(\Gamma_t^*, B_0(r_t))$ and relation (3.2.1) imply that

$$\begin{aligned}
\mathrm{Area}_g(\partial B_{x_0}(r_t')) &= \mathrm{Area}_g(\Gamma_t) = \mathrm{Area}_e(\Gamma_t^*) \\
&= n\omega_n^{\frac{1}{n}} \mathrm{Vol}_e(\{x \in \mathbb{R}^n : u^*(x) > t\})^{\frac{n-1}{n}} \\
&= n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g(\{x \in M : u(x) > t\})^{\frac{n-1}{n}} \\
&= n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g(B_{x_0}(r_t'))^{\frac{n-1}{n}}.
\end{aligned}$$

From the validity of the Cartan-Hadamard conjecture (in particular, from the equality case in (3.1.1)), the above relation implies that the open geodesic ball

$$\{x \in M : u(x) > t\} = B_{x_0}(r_t')$$

is isometric to the $n-$dimensional Euclidean ball with volume $\mathrm{Vol}_g(B_{x_0}(r_t'))$. On the other hand, by relation (3.2.13) we actually have that the balls $B_{x_0}(r_t')$ and $B_0(r_t)$ are isometric, thus $r_t' = r_t$, proving the claim (3.2.12).

On account of (3.2.12) and (3.2.1), it follows that

$$\mathrm{Vol}_g(B_{x_0}(r_t)) = \omega_n r_t^n,\ \ 0 < t < 1.$$

Since $\lim\limits_{t\to 1} r_t = 0$ and $\lim\limits_{t\to 0} r_t = +\infty$, the continuity of $t \mapsto r_t$ on $(0, 1)$ and the latter relation imply that

$$\mathrm{Vol}_g(B_{x_0}(\rho)) = \omega_n \rho^n \ \text{ for all } \rho > 0. \qquad (3.2.15)$$

Standard comparison arguments in Riemannian geometry imply that the sectional curvature on the Cartan-Hadamard manifold (M, g) is identically zero, thus (M, g) is isometric to the Euclidean space $\mathbb{R}^n$.

(ii)⇒(iii) Fix $\alpha \in (\frac{1}{p}, 1)$. By assumption, to every $\lambda > 0$ there exists a non-negative extremal function $u_\lambda \in \mathrm{Lip}_0(M)$ in $(\mathbf{GN2})^{\alpha,p}_{\mathcal{N}_{\alpha,p,n}}$ concentrated around x_0 with

$$\mathrm{Vol}_g(\mathrm{supp}(u_\lambda)) = \lambda.$$

For the Euclidean rearrangement u_λ^* of u_λ, we clearly has (see Theorem 3.1.2) that

$$\begin{aligned}
\|u_\lambda\|_{L^{\alpha(p-1)+1}(M)} &= \|u_\lambda^*\|_{L^{\alpha(p-1)+1}(\mathbb{R}^n)} \\
&\leq \mathcal{N}_{\alpha,p,n} \|\nabla u_\lambda^*\|^{\gamma}_{L^p(\mathbb{R}^n)} \|u_\lambda^*\|^{1-\gamma}_{L^{\alpha p}(\mathbb{R}^n)} \\
&\leq \mathcal{N}_{\alpha,p,n} \|\nabla u_\lambda\|^{\gamma}_{L^p(M)} \|u_\lambda\|^{1-\gamma}_{L^{\alpha p}(M)}.
\end{aligned}$$

Since u_λ is an extremal in $(\mathbf{GN2})^{\alpha,p}_{\mathcal{N}_{\alpha,p,n}}$, the function u^*_λ is also extremal in the optimal Gagliardo-Nirenberg inequality (2.1.5). Note that u^*_λ is uniquely determined (up to translation, constant multiplication and scaling) together with the condition $\mathrm{Vol}_g(\mathrm{supp}(u_\lambda)) = \lambda$; thus, we may assume that it has the form

$$u^*_\lambda(x) = \left(1 - c_\lambda |x|^{p'}\right)_+^{\frac{1}{1-\alpha}}, \ x \in \mathbb{R}^n,$$

where $c_\lambda = \omega_n^{\frac{p'}{n}} \lambda^{-\frac{p'}{n}}$. In a similar manner as in the previous proof, one has that

$$\{x \in M : u_\lambda(x) > t\} = B_{x_0}(r_t^\lambda), \ 0 < t < 1,$$

where $r_t^\lambda = c_\lambda^{-\frac{1}{p'}} (1 - t^{1-\alpha})^{\frac{1}{p'}}$ and by (3.2.1),

$$\mathrm{Vol}_g(B_{x_0}(r_t^\lambda)) = \omega_n (r_t^\lambda)^n, \ 0 < t < 1.$$

If $t \to 0$ in the latter relation, it yields that

$$\mathrm{Vol}_g(B_{x_0}(\omega_n^{-\frac{1}{n}} \lambda^{\frac{1}{n}})) = \lambda.$$

By the arbitrariness of $\lambda > 0$, we arrive to (3.2.15), concluding the proof. $\square$

4. Multipolar Hardy inequalities on Riemannian manifolds

True pleasure lies not in the discovery of truth, but in the search for it.

(Tolstoy)

4.1. Introduction and statement of main results

The classical *unipolar Hardy inequality* (or, uncertainty principle) states that if $n \geq 3$, then[1]

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \mathrm{d}x, \ \forall u \in C_0^\infty(\mathbb{R}^n);$$

here, the constant $\frac{(n-2)^2}{4}$ is sharp and not achieved. Many efforts have been made over the last two decades to improve/extend Hardy inequalities in various directions. One of the most challenging research topics in this direction is the so-called *multipolar Hardy inequality*. Such kind of extension is motivated by molecular physics and quantum chemistry/cosmology. Indeed, by describing the behavior of electrons and atomic nuclei in a molecule within the theory of Born-Oppenheimer approximation or Thomas-Fermi theory, particles can be modeled as certain singularities/poles $x_1, ..., x_m \in \mathbb{R}^n$, producing their effect within the form $x \mapsto |x - x_i|^{-1}$, $i \in \{1, ..., m\}$. Having such mathematical models, several authors studied the behavior of the operator with inverse square potentials with multiple poles, namely

$$\mathscr{L} := -\Delta - \sum_{i=1}^{m} \frac{\mu_i^+}{|x - x_i|^2},$$

see Bosi, Dolbeaut and Esteban [22], Cao and Han [27], Felli, Marchini and Terracini [60], Guo, Han and Niu [65], Lieb [90], Adimurthi [2], and references therein. Very recently, Cazacu and Zuazua [30] proved an optimal multipolar counterpart of the above (unipolar) Hardy inequality, i.e.,

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2} u^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{4.1.1}$$

where $n \geq 3$, and $x_1, ..., x_m \in \mathbb{R}^n$ are different poles; moreover, the constant $\frac{(n-2)^2}{m^2}$ is optimal. By using the paralelogrammoid law, (4.1.1) turns to be equivalent to

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \left| \frac{x - x_i}{|x - x_i|^2} - \frac{x - x_j}{|x - x_j|^2} \right|^2 u^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{4.1.2}$$

[1] Based on the paper [54]

All of the aforementioned works considered the flat/isotropic setting where no external force is present. Once the ambient space structure is perturbed, coming for instance by a magnetic or gravitational field, the above results do not provide a full description of the physical phenomenon due to the presence of the *curvature*.

In order to discuss such a curved setting, we put ourselves into the Riemannian realm, i.e., we consider an $n(\geq 3)$-dimensional complete Riemannian manifold (M,g), $d_g : M\times M \to [0,\infty)$ is its usual distance function associated to the Riemannian metric g, $\mathrm{d}v_g$ is its canonical volume element, $\exp_x : T_xM \to M$ is its standard exponential map, and $\nabla_g u(x)$ is the gradient of a function $u : M \to \mathbb{R}$ at $x \in M$, respectively. Clearly, in the curved setting of (M,g), the vector $x - x_i$ and distance $|x - x_i|$ should be reformulated into a geometric context by considering $\exp_{x_i}^{-1}(x)$ and $d_g(x,x_i)$, respectively. Note that

$$\nabla_g d_g(\cdot,y)(x) = -\frac{\exp_x^{-1}(y)}{d_g(x,y)} \text{ for every } y\in M,\ x\in M\setminus(\{y\}\cup \mathrm{cut}(y)),$$

where $\mathrm{cut}(y)$ denotes the cut-locus of y on (M,g). In this setting, a natural question arises: if $\Omega \subseteq M$ is an open domain and $S = \{x_1,...,x_m\}\subset\Omega$ is the set of distinct poles, can we prove

$$\int_\Omega |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{(n-2)^2}{m^2}\sum_{1\leq i<j\leq m}\int_\Omega V_{ij}(x)u^2\mathrm{d}x,\quad \forall u\in C_0^\infty(\Omega), \tag{4.1.3}$$

where

$$V_{ij}(x) = \frac{d_g(x_i,x_j)^2}{d_g(x,x_i)^2 d_g(x,x_j)^2} \quad\text{or}\quad V_{ij}(x) = \left|\frac{\nabla_g d_g(x,x_i)}{d_g(x,x_i)} - \frac{\nabla_g d_g(x,x_j)}{d_g(x,x_j)}\right|^2?$$

Clearly, in the Euclidean space $\mathbb{R}^n$, inequality (4.1.3) corresponds to (4.1.1) and (4.1.2), for the above choices of V_{ij}, respectively. It turns out that the answer deeply depends on the curvature of the Riemannian manifold (M,g). Indeed, if the Ricci curvature verifies $\mathrm{Ric}(M,g)\geq c_0(n-1)g$ for some $c_0>0$ (as in the case of the n-dimensional unit sphere $\mathbb{S}^n$), we know by the theorem of Bonnet-Myers that (M,g) is compact; thus, we may use the constant functions $u\equiv c\in\mathbb{R}$ as test-functions in (4.1.3), and we get a contradiction. However, when (M,g) is a Cartan-Hadamard manifold (i.e., complete, simply connected Riemannian manifold with non-positive sectional curvature), we can expect the validity of (4.1.3), see Theorems 4.1.1 & 4.1.2 and suitable Laplace comparison theorems, respectively.

Accordingly, the primary aim of the present chapter is to investigate *multipolar Hardy inequalities on complete Riemannian manifolds*. We emphasize that such a study requires new technical and theoretical approaches. In fact, we need to explore those geometric and analytic properties which are behind of the theory of multipolar Hardy inequalities in the flat context, formulated now in terms of curvature, geodesics, exponential map, etc. We notice that striking results were also achieved recently in the theory of unipolar Hardy-type inequalities on curved spaces. The pioneering work of Carron [29], who studied Hardy inequalities on complete non-compact Riemannian manifolds, opened new perspectives in the study of functional inequalities with singular terms on curved spaces. Further contributions have been provided by D'Ambrosio and Dipierro [38], Kristály [81], Kombe and Özaydin [71, 72], Xia [126], and Yang, Su and Kong [127], where various improvements of the usual Hardy inequality is presented on complete, non-compact Riemannian manifolds. Moreover, certain unipolar Hardy and Rellich type inequalities were obtained on non-reversible Finsler manifolds by Farkas, Kristály and Varga [58], and Kristály and Repovs [86].

In the sequel we shall present our results; for further use, let Δ_g be the Laplace-Beltrami operator on (M,g). Let $m\geq 2$, $S=\{x_1,...,x_m\}\subset M$ be the set of poles with $x_i\neq x_j$ if $i\neq j$, and for simplicity of notation, let $d_i = d_g(\cdot,x_i)$ for every $i\in\{1,...,m\}$. Our first result reads as follows.

Theorem 4.1.1 (Multipolar Hardy inequality I). *Let (M,g) be an n-dimensional complete Riemannian manifold and $S = \{x_1, ..., x_m\} \subset M$ be the set of distinct poles, where $n \geq 3$ and $m \geq 2$. Then*

$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \mathrm{d}v_g + \frac{n-2}{m} \sum_{i=1}^m \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \mathrm{d}v_g, \quad \forall u \in C_0^\infty(M). \tag{4.1.4}$$

Moreover, in the bipolar case (i.e., $m = 2$), the constant $\frac{(n-2)^2}{m^2} = \frac{(n-2)^2}{4}$ is optimal in (4.1.4).

Remark 4.1.1. (a) The proof of inequality (4.1.4) is based on a direct calculation. If $m = 2$, the local behavior of geodesic balls implies the optimality of the constant $\frac{(n-2)^2}{m^2} = \frac{(n-2)^2}{4}$; in particular, the second term is a lower order perturbation of the first one of the RHS (independently of the curvature).

(b) The optimality of $\frac{(n-2)^2}{m^2}$ seems to be a hard nut to crack. A possible approach could be a fine Agmon-Allegretto-Piepenbrink-type spectral estimate developed by Devyver [44] and Devyver, Fraas and Pinchover [45] whenever (M,g) has asymptotically non-negative Ricci curvature (see Pigola, Rigoli and Setti [99, Corollary 2.17, p. 44]). Indeed, under this curvature assumption one can prove that the operator $-\Delta_g - W$ is critical (see [45, Definition 4.3]), where

$$W = \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 + \frac{n-2}{m} \sum_{i=1}^m \frac{d_i \Delta_g d_i - (n-1)}{d_i^2}.$$

Although expected, we have no full control on the second summand with respect to the first one in W, i.e., the latter term could compete with the 'leading' one; clearly, in the Euclidean setting no such competition is present, thus the optimality of $\frac{(n-2)^2}{m^2}$ immediately follows by the criticality of W. It remains to investigate this issue in a forthcoming study.

(c) We emphasize that the second term in the RHS of (4.1.4) has a crucial role. Indeed, on one hand, when the Ricci curvature verifies $\mathrm{Ric}(M,g) \geq c_0(n-1)g$ for some $c_0 > 0$, one has that $d_i(x) = g_d(x, x_i) \leq \pi/\sqrt{c_0}$ for every $x \in M$ and by the Laplace comparison theorem, we have that $d_i \Delta_g d_i - (n-1) \leq (n-1)(\sqrt{c_0} d_i \cot(\sqrt{c_0} d_i) - 1) < 0$ for $d_i > 0$, i.e. for every $x \neq x_i$. Thus, this term modifies the original problem (4.1.3) by filling the gap in a suitable way. On the other hand, when (M,g) is a Cartan-Hadamard manifold, one has $d_i \Delta_g d_i - (n-1) \geq 0$, and inequality (4.1.4) implies (4.1.3). This result will be resumed in Corollary 4.3.1 (i). In particular, when $M = \mathbb{R}^n$ is the Euclidean space, then $\exp_x(y) = x + y$ for every $x, y \in \mathbb{R}^n$ and $|x|\Delta|x| = n-1$ for every $x \neq 0$; therefore, Theorem 4.1.1 and the criticality of $-\Delta - W$ immediately yield the main result of Cazacu and Zuazua [30], i.e., inequality (4.1.2) (and equivalently (4.1.1)).

Although the paralelogrammoid law in the Euclidean setting provides the equivalence between (4.1.1) and (4.1.2), this property is no longer valid on generic manifolds. However, a curvature-based quantitative paralelogrammoid law and a Toponogov-type comparison result provide a suitable counterpart of inequality (4.1.1):

Theorem 4.1.2 (Multipolar Hardy inequality II). *Let (M,g) be an n-dimensional complete Riemannian manifold with $\mathbf{K} \geq k_0$ for some $k_0 \in \mathbb{R}$ and let $S = \{x_1, ..., x_m\} \subset M$ be the set of distinct poles belonging to a strictly convex open set $\tilde{S} \subset M$, where $n \geq 3$ and $m \geq 2$. Then we have the following inequality:*

$$\int_{\tilde{S}} |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{4(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_{\tilde{S}} \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i) \mathbf{s}_{k_0}(d_j)} u^2 \mathrm{d}v_g + \sum_{1\leq i<j\leq m} \int_{\tilde{S}} R_{ij}(k_0) u^2 \mathrm{d}v_g + \frac{n-2}{m} \sum_{i=1}^m \int_{\tilde{S}} \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \mathrm{d}v_g, \quad \forall u \in C_0^\infty(\tilde{S}), \tag{4.1.5}$$

where $d_{ij} = d_g(x_i, x_j)$ *and*

$$R_{ij}(k_0) = \begin{cases} \frac{1}{d_i^2} + \frac{1}{d_j^2} - \frac{2}{k_0 d_i d_j}\left(\frac{1}{\mathbf{s}_{k_0}(d_i)\mathbf{s}_{k_0}(d_j)} - \mathbf{ct}_{k_0}(d_i)\mathbf{ct}_{k_0}(d_j)\right), & \text{if} \quad k_0 \neq 0, \\ 0, & \text{if} \quad k_0 = 0. \end{cases}$$

Remark 4.1.2. When (M, g) is a Cartan-Hadamard manifold and $k_0 \leq 0$, one has that $R_{ij}(k_0) \geq 0$; thus we obtain a similar result as in (4.1.3); the precise statement will be given in Corollary 4.3.1 (ii).

4.2. Proof of Theorems 4.1.1 and 4.1.2

4.2.1. Multipolar Hardy inequality: influence of curvature

Proof of Theorem 4.1.1. Let $E = \prod_{i=1}^{m} d_i^{2-n}$ and fix $u \in C_0^\infty(M)$ arbitrarily. A direct calculation on the set $M \setminus \bigcup_{i=1}^m (\{x_i\} \cup \mathrm{cut}(x_i))$ yields that

$$\nabla_g \left(uE^{-\frac{1}{m}}\right) = E^{-\frac{1}{m}} \nabla_g u + \frac{n-2}{m} u E^{-\frac{1}{m}} \sum_{i=1}^{m} \frac{\nabla_g d_i}{d_i}.$$

Integrating the latter relation, the divergence theorem and eikonal equation (1.3.2) give that

$$\begin{aligned} \int_M \left|\nabla_g \left(uE^{-\frac{1}{m}}\right)\right|^2 E^{\frac{2}{m}} \mathrm{d}v_g &= \int_M |\nabla_g u|^2 \mathrm{d}v_g + \frac{(n-2)^2}{m^2} \int_M \left|\sum_{i=1}^{m} \frac{\nabla_g d_i}{d_i}\right|^2 u^2 \mathrm{d}v_g \\ &\quad + \frac{n-2}{m} \sum_{i=1}^{m} \int_M \left\langle \nabla_g u^2, \frac{\nabla_g d_i}{d_i} \right\rangle \mathrm{d}v_g \\ &= \int_M |\nabla_g u|^2 \mathrm{d}v_g + \frac{(n-2)^2}{m^2} \int_M \left|\sum_{i=1}^{m} \frac{\nabla_g d_i}{d_i}\right|^2 u^2 \mathrm{d}v_g \\ &\quad - \frac{n-2}{m} \sum_{i=1}^{m} \int_M \mathrm{div}\left(\frac{\nabla_g d_i}{d_i}\right) u^2 \mathrm{d}v_g. \end{aligned}$$

Due to (1.3.2), we have

$$\mathrm{div}\left(\frac{\nabla_g d_i}{d_i}\right) = \frac{d_i \Delta_g d_i - 1}{d_i^2}, \; i \in \{1, ..., m\}.$$

Thus, an algebraic reorganization of the latter relation provides an Agmon-Allegretto-Piepenbrink-type multipolar representation

$$\begin{aligned} \int_M |\nabla_g u|^2 \mathrm{d}v_g - \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_M \left|\frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j}\right|^2 u^2 \mathrm{d}v_g \quad &= \quad \int_M \left|\nabla_g \left(uE^{-1/m}\right)\right|^2 E^{2/m} \mathrm{d}v_g \\ &\quad + \frac{n-2}{m} \sum_{i=1}^{m} \mathcal{K}_i(u), \end{aligned} \tag{4.2.1}$$

where $\mathcal{K}_i(u) = \displaystyle\int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \mathrm{d}v_g$. Inequality (4.1.4) directly follows by (4.2.1).

In the sequel, we deal with the optimality of the constant $\frac{(n-2)^2}{m^2}$ in (4.1.4) when $m = 2$. In this case the right hand side of (4.1.4) behaves as $\frac{(n-2)^2}{4} d_g(x, x_i)^{-2}$ whenever $x \to x_i$ and by the local behavior of the geodesic balls (see (1.3.1)) we may expect the optimality of $\frac{(n-2)^2}{4}$. In order to be more explicit, let $A_i[r,R]=\{x\in M: r\leq d_i(x)\leq R\}$ for $r<R$ and $i\in\{1,...,m\}$. If $0<r<<R$ are within the range of (1.3.1), a layer cake representation yields for every $i\in\{1,...,m\}$ that

$$\begin{aligned}\int_{A_i[r,R]} d_i^{-n}\mathrm{d}v_g &= \frac{\mathrm{Vol}_g(B_R(x_i))}{R^n}-\frac{\mathrm{Vol}_g(B_r(x_i))}{r^n}+n\int_r^R \mathrm{Vol}_g(B_\rho(x_i))\rho^{-1-n}\mathrm{d}\rho \\ &= o(R)+n\omega_n\log\frac{R}{r}. \end{aligned} \tag{4.2.2}$$

Let $S=\{x_1,x_2\}$ be the set of poles, $x_1\neq x_2$. Let $\varepsilon\in(0,1)$ be small enough such that it belongs to the range of (1.3.1), and $B_{2\sqrt{\varepsilon}}(x_1)\cap B_{2\sqrt{\varepsilon}}(x_2)=\emptyset$. Let

$$u_\varepsilon(x)=\begin{cases}\frac{\log\left(\frac{d_i(x)}{\varepsilon^2}\right)}{\log\left(\frac{1}{\varepsilon}\right)}d_i(x)^{\frac{2-n}{2}}, & \text{if } \ x\in A_i[\varepsilon^2,\varepsilon];\\ \frac{2\log\left(\frac{\sqrt{\varepsilon}}{d_i(x)}\right)}{\log\left(\frac{1}{\varepsilon}\right)}d_i(x)^{\frac{2-n}{2}}, & \text{if } \ x\in A_i[\varepsilon,\sqrt{\varepsilon}];\\ 0, & \text{otherwise},\end{cases}$$

with $i\in\{1,2\}$. Note that $u_\varepsilon\in C^0(M)$, having compact support $\bigcup_{i=1}^2 A_i[\varepsilon^2,\sqrt{\varepsilon}]\subset M$; in fact, u_ε can be used as a test function in (4.1.4). For later use let us denote by $\varepsilon^*=\frac{1}{\log\left(\frac{1}{\varepsilon}\right)^2}$,

$$\mathcal{I}_\varepsilon=\int_M|\nabla_g u_\varepsilon|^2\mathrm{d}v_g,\ \ \mathcal{L}_\varepsilon=\int_M\frac{\langle\nabla_g d_1,\nabla_g d_2\rangle}{d_1d_2}u_\varepsilon^2\mathrm{d}v_g,\ \ \mathcal{K}_\varepsilon=\sum_{i=1}^2\int_M\frac{d_i\Delta_g d_i-(n-1)}{d_i^2}u_\varepsilon^2\mathrm{d}v_g$$

and

$$\mathcal{J}_\varepsilon=\int_M\left[\frac{1}{d_1^2}+\frac{1}{d_2^2}\right]u_\varepsilon^2\mathrm{d}v_g.$$

The proof is based on the following claims:

$$\mathcal{I}_\varepsilon-\mu_H\mathcal{J}_\varepsilon=\mathcal{O}(1),\ \ \mathcal{L}_\varepsilon=\mathcal{O}(\sqrt[4]{\varepsilon})\ \textit{and}\ \mathcal{K}_\varepsilon=\mathcal{O}(\sqrt[4]{\varepsilon})\ \ \text{as}\ \ \varepsilon\to0, \tag{4.2.3}$$

and

$$\lim_{\varepsilon\to0}\mathcal{J}_\varepsilon=+\infty. \tag{4.2.4}$$

Indeed, taking into account that $u_\varepsilon\equiv0$ on $M\setminus\bigcup_{i=1}^2 B_{\sqrt{\varepsilon}}(x_i)$, the previous step implies that

$$\begin{aligned}\mathcal{K}_\varepsilon^i:=\left|\int_{B_{\sqrt{\varepsilon}}(x_i)}\frac{d_i\Delta_g d_i-(n-1)}{d_i^2}u_\varepsilon^2\mathrm{d}v_g\right| &\leq\int_{B_{\sqrt{\varepsilon}}(x_i)}\left|\frac{d_i\Delta_g d_i-(n-1)}{d_i^2}\right|u_\varepsilon^2\mathrm{d}v_g\\ \leq\int_{B_{\sqrt{\varepsilon}}(x_i)}\frac{u_\varepsilon^2}{d_i}\mathrm{d}v_g=\int_{A_i[\varepsilon^2,\varepsilon]}\frac{u_\varepsilon^2}{d_i}\mathrm{d}v_g &+\int_{A_i[\varepsilon,\sqrt{\varepsilon}]}\frac{u_\varepsilon^2}{d_i}\mathrm{d}v_g=\mathcal{O}(\sqrt[4]{\varepsilon})\text{ as }\varepsilon\to0,\end{aligned}$$

which concludes the proof of the first part of (4.2.3).

A direct calculation yields that

$$\nabla_g u_\varepsilon=\begin{cases}\sqrt{\varepsilon^*}d_i^{-\frac{n}{2}}\left(1-\sqrt{\mu_H}\log\left(\frac{d_i}{\varepsilon^2}\right)\right)\nabla_g d_i & \text{on}\ \ A_i[\varepsilon^2,\varepsilon];\\ -2\sqrt{\varepsilon^*}d_i^{-\frac{n}{2}}\left(1+\sqrt{\mu_H}\log\left(\frac{\sqrt{\varepsilon}}{d_i}\right)\right)\nabla_g d_i & \text{on}\ \ A_i[\varepsilon,\sqrt{\varepsilon}];\\ 0, & \text{otherwise}.\end{cases}$$

Then, by the eikonal equation (1.3.2), we have

$$|\nabla_g u_\varepsilon|^2=\begin{cases}\frac{u_\varepsilon^2}{d_i^2}\left[\frac{1}{\log\left(\frac{d_i}{\varepsilon^2}\right)}-\sqrt{\mu_H}\right]^2 & \text{on}\ \ A_i[\varepsilon^2,\varepsilon];\\ \frac{u_\varepsilon^2}{d_i^2}\left[\frac{1}{\log\left(\frac{\sqrt{\varepsilon}}{d_i}\right)}+\sqrt{\mu_H}\right]^2 & \text{on}\ \ A_i[\varepsilon,\sqrt{\varepsilon}];\\ 0, & \text{otherwise}.\end{cases}$$

By the above computation it turns out that

$$\mathcal{I}_\varepsilon - \mu_H \mathcal{J}_\varepsilon = \sum_{i=1}^m \int_{A_i[\varepsilon^2,\varepsilon]} \left[\frac{1}{\log^2\left(\frac{d_i}{\varepsilon^2}\right)} - \frac{2\sqrt{\mu_H}}{\log\left(\frac{d_i}{\varepsilon^2}\right)} \right] \frac{u_\varepsilon^2}{d_i^2} \mathrm{d}v_g + \sum_{i=1}^m \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} \left[\frac{1}{\log^2\left(\frac{\sqrt{\varepsilon}}{d_i}\right)} + \frac{2\sqrt{\mu_H}}{\log\left(\frac{\sqrt{\varepsilon}}{d_i}\right)} \right] \frac{u_\varepsilon^2}{d_i^2} \mathrm{d}v_g.$$

By (4.2.2) one has

$$\begin{aligned}
\mathcal{I}_\varepsilon^{i,1} &:= \int_{A_i[\varepsilon^2,\varepsilon]} \left| \frac{1}{\log^2\left(\frac{d_i}{\varepsilon^2}\right)} - \frac{2\sqrt{\mu_H}}{\log\left(\frac{d_i}{\varepsilon^2}\right)} \right| \frac{u_\varepsilon^2}{d_i^2} \mathrm{d}v_g \leq \varepsilon^* \int_{A_i[\varepsilon^2,\varepsilon]} d_i^{-n} \left[1 + 2\sqrt{\mu_H} \log\left(\frac{d_i}{\varepsilon^2}\right) \right] \mathrm{d}v_g \\
&\leq \varepsilon^* \left[1 + \frac{2\sqrt{\mu_H}}{\sqrt{\varepsilon^*}} \right] \int_{A_i[\varepsilon^2,\varepsilon]} d_i^{-n} \mathrm{d}v_g = \varepsilon^* \left[1 + \frac{2\sqrt{\mu_H}}{\sqrt{\varepsilon^*}} \right] \left[o(\varepsilon) + \frac{n\omega_n}{\sqrt{\varepsilon^*}} \right] \\
&= \mathcal{O}(1) \text{ as } \varepsilon \to 0.
\end{aligned}$$

In a similar way, it yields

$$\begin{aligned}
0 < \mathcal{I}_\varepsilon^{i,2} &:= \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} \left[\frac{1}{\log^2\left(\frac{\sqrt{\varepsilon}}{d_i}\right)} + \frac{2\sqrt{\mu_H}}{\log\left(\frac{\sqrt{\varepsilon}}{d_i}\right)} \right] \frac{u_\varepsilon^2}{d_i^2} \mathrm{d}v_g \leq 4\varepsilon^* \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} d_i^{-n} \left[1 + 2\sqrt{\mu_H} \log\left(\frac{\sqrt{\varepsilon}}{d_i}\right) \right] \mathrm{d}v_g \\
&\leq 4\varepsilon^* \left[1 + \frac{\sqrt{\mu_H}}{\sqrt{\varepsilon^*}} \right] \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} d_i^{-n} \mathrm{d}v_g = 4\varepsilon^* \left[1 + \frac{\sqrt{\mu_H}}{\sqrt{\varepsilon^*}} \right] \left[o(\sqrt{\varepsilon}) + \frac{n\omega_n}{2\sqrt{\varepsilon^*}} \right] \\
&= \mathcal{O}(1) \text{ as } \varepsilon \to 0.
\end{aligned}$$

We observe that $|\mathcal{L}_\varepsilon| \leq \int_M \frac{u_\varepsilon^2}{d_1 d_2} \mathrm{d}v_g$. Moreover, for some $C > 0$ (independent of $\varepsilon > 0$), we have

$$\begin{aligned}
\int_{A_i[\varepsilon^2,\varepsilon]} \frac{u_\varepsilon^2}{d_i d_j} \mathrm{d}v_g &\leq C \int_{A_i[\varepsilon^2,\varepsilon]} \frac{u_\varepsilon^2}{d_i} \mathrm{d}v_g = C\varepsilon^* \int_{A_i[\varepsilon^2,\varepsilon]} d_i^{1-n} \log^2\left(\frac{d_i}{\varepsilon^2}\right) \mathrm{d}v_g \\
&\leq C \int_{A_i[\varepsilon^2,\varepsilon]} d_i^{1-n} \mathrm{d}v_g \leq C\varepsilon \int_{A_i[\varepsilon^2,\varepsilon]} d_i^{-n} \mathrm{d}v_g \\
&= C\varepsilon \left[o(\varepsilon) + \frac{n\omega_n}{\sqrt{\varepsilon^*}} \right] = \mathcal{O}(\sqrt{\varepsilon}) \text{ as } \varepsilon \to 0.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\int_{A_i[\varepsilon,\sqrt{\varepsilon}]} \frac{u_\varepsilon^2}{d_i d_j} \mathrm{d}v_g &\leq C \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} \frac{u_\varepsilon^2}{d_i} \mathrm{d}v_g = 4C\varepsilon^* \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} d_i^{1-n} \log^2\left(\frac{\sqrt{\varepsilon}}{d_i}\right) \mathrm{d}v_g \\
&\leq C \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} d_i^{1-n} \mathrm{d}v_g \leq C\sqrt{\varepsilon} \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} d_i^{-n} \mathrm{d}v_g \\
&= C\sqrt{\varepsilon} \left[o(\sqrt{\varepsilon}) + \frac{n\omega_n}{2\sqrt{\varepsilon^*}} \right] = \mathcal{O}(\sqrt[4]{\varepsilon}) \text{ as } \varepsilon \to 0,
\end{aligned}$$

which is the second part of (4.2.3). Finally, we know that for $\varepsilon > 0$ small enough one has

$$\left| \Delta_g d_i - \frac{n-1}{d_i} \right| \leq 1 \quad \text{a.e. in } B_{\sqrt{\varepsilon}}(x_i),$$

see Kristály and Repovs [86]. Thus, taking into account that $u_\varepsilon \equiv 0$ on $M \setminus \bigcup_{i=1}^2 B_{\sqrt{\varepsilon}}(x_i)$, the previous step implies that

$$\begin{aligned}
\mathcal{K}_\varepsilon^i &:= \left| \int_{B_{\sqrt{\varepsilon}}(x_i)} \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u_\varepsilon^2 \mathrm{d}v_g \right| \leq \int_{B_{\sqrt{\varepsilon}}(x_i)} \left| \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} \right| u_\varepsilon^2 \mathrm{d}v_g \\
&\leq \int_{B_{\sqrt{\varepsilon}}(x_i)} \frac{u_\varepsilon^2}{d_i} \mathrm{d}v_g = \int_{A_i[\varepsilon^2,\varepsilon]} \frac{u_\varepsilon^2}{d_i} \mathrm{d}v_g + \int_{A_i[\varepsilon,\sqrt{\varepsilon}]} \frac{u_\varepsilon^2}{d_i} \mathrm{d}v_g = \mathcal{O}(\sqrt[4]{\varepsilon}) \text{ as } \varepsilon \to 0,
\end{aligned}$$

which concludes the proof of (4.2.3).

Now, we are going to prove (4.2.4).

Indeed, by the layer cake representation (see for instance Lieb and Loss [91, Theorem 1.13]) one has

$$
\begin{aligned}
\mathcal{J}_\varepsilon &\geq \int_{A_i[\varepsilon^2,\varepsilon]} \frac{u_\varepsilon^2}{d_i^2} \mathrm{d}v_g = \varepsilon^* \int_{A_i[\varepsilon^2,\varepsilon]} d_i^{-n} \log^2\left(\frac{d_i}{\varepsilon^2}\right) \mathrm{d}v_g \\
&= \varepsilon^* \int_0^\infty \mathrm{Vol}_g(\{x \in A_i[\varepsilon^2,\varepsilon] : d_i^{-n}\log^2(d_i/\varepsilon^2) > t\}) \mathrm{d}t \\
&\geq \varepsilon^* \int_{\varepsilon^2 e^{\frac{2}{n}}}^{\varepsilon} \left(\mathrm{Vol}_g(B_\rho(x_i)) - \mathrm{Vol}_g(B_{\varepsilon^2 e^{\frac{2}{n}}}(x_i))\right) \rho^{-n-1} \log\left(\frac{\rho}{\varepsilon^2}\right)\left(n\log\left(\frac{\rho}{\varepsilon^2}\right) - 2\right) \mathrm{d}\rho.
\end{aligned}
$$

Note that by (1.3.1), we have

$$
\begin{aligned}
\mathcal{J}_\varepsilon^1 &:= \varepsilon^* \int_{\varepsilon^2 e^{\frac{2}{n}}}^{\varepsilon} \mathrm{Vol}_g(B_\rho(x_i))\rho^{-n-1} \log\left(\frac{\rho}{\varepsilon^2}\right)\left(n\log\left(\frac{\rho}{\varepsilon^2}\right) - 2\right) \mathrm{d}\rho \\
&= \varepsilon^*\omega_n \int_{\varepsilon^2 e^{\frac{2}{n}}}^{\varepsilon} (1+o(\rho))\rho^{-1} \log\left(\frac{\rho}{\varepsilon^2}\right)\left(n\log\left(\frac{\rho}{\varepsilon^2}\right) - 2\right) \mathrm{d}\rho \\
&\geq \frac{\varepsilon^*\omega_n}{2} \int_{\varepsilon^2 e^{\frac{2}{n}}}^{\varepsilon} \rho^{-1} \log\left(\frac{\rho}{\varepsilon^2}\right)\left(n\log\left(\frac{\rho}{\varepsilon^2}\right) - 2\right)) \mathrm{d}\rho \\
&= \frac{\varepsilon^*\omega_n}{2}\left[\frac{n}{6(\varepsilon^*)^{\frac{3}{2}}} - \frac{1}{\varepsilon^*} + \mathcal{O}(1)\right] \to +\infty \text{ as } \varepsilon \to 0,
\end{aligned}
$$

while

$$
\begin{aligned}
\mathcal{J}_\varepsilon^2 &:= \varepsilon^* \mathrm{Vol}_g(B_{\varepsilon^2 e^{\frac{2}{n}}}(x_i)) \int_{\varepsilon^2 e^{\frac{2}{n}}}^{\varepsilon} \rho^{-n-1} \log\left(\frac{\rho}{\varepsilon^2}\right)\left(n\log\left(\frac{\rho}{\varepsilon^2}\right) - 2\right) \mathrm{d}\rho \\
&\leq 2n\omega_n \varepsilon^{2n} e^2 \int_{\varepsilon^2 e^{\frac{2}{n}}}^{\varepsilon} \rho^{-n-1} \mathrm{d}\rho \\
&= 2\omega_n\left[1 - \varepsilon^n e^2\right] = \mathcal{O}(1) \text{ as } \varepsilon \to 0.
\end{aligned}
$$

Since $\mathcal{J}_\varepsilon \geq \mathcal{J}_\varepsilon^1 - \mathcal{J}_\varepsilon^2$, relation (4.2.4) holds. Combining relations (4.2.3) and (4.2.4) with inequality (4.1.4), we have that

$$
\mu_H \leq \frac{\mathcal{I}_\varepsilon - \frac{n-2}{2}\mathcal{K}_\varepsilon}{\mathcal{J}_\varepsilon - 2\mathcal{L}_\varepsilon} \leq \frac{\mathcal{I}_\varepsilon + \frac{n-2}{2}|\mathcal{K}_\varepsilon|}{\mathcal{J}_\varepsilon - 2|\mathcal{L}_\varepsilon|} = \frac{\mu_H \mathcal{J}_\varepsilon + \mathcal{O}(1)}{\mathcal{J}_\varepsilon + \mathcal{O}(\sqrt[4]{\varepsilon})} \to \mu_H \text{ as } \varepsilon \to 0,
$$

which concludes the proof. □

Remark 4.2.1. Let us assume that in Theorem 4.1.1, (M,g) is a Riemannian manifold with sectional curvature verifying $\mathbf{K} \leq c$. By the Laplace comparison theorem I (see (**??**)) we have:

$$
\begin{aligned}
\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq\ & \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \left|\frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j}\right|^2 u^2 \mathrm{d}v_g \\
& + \frac{(n-2)(n-1)}{m} \sum_{i=1}^m \int_M \frac{\mathbf{D}_c(d_i)}{d_i^2} u^2 \mathrm{d}v_g, \quad \forall u \in C_0^\infty(M), \qquad (4.2.5)
\end{aligned}
$$

where $\mathbf{D}_c(r) = r\mathbf{ct}_c(r) - 1$, $r \geq 0$. In addition, if (M,g) is a Cartan-Hadamard manifold with $\mathbf{K} \leq c \leq 0$, then $\mathbf{D}_c(r) \geq \frac{3|c|r^2}{\pi^2+|c|r^2}$ for all $r \geq 0$. Accordingly, stronger curvature of the Cartan-Hadamard manifold implies improvement in the multipolar Hardy inequality (4.2.5).

4.2.2. Multipolar Hardy inequality with Topogonov-type comparision

Proof of Theorem 4.1.2. It is clear that

$$\left|\frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j}\right|^2 = \frac{1}{d_i^2} + \frac{1}{d_j^2} - 2\frac{\langle \nabla_g d_i, \nabla_g d_j\rangle}{d_i d_j}. \tag{4.2.6}$$

Let us fix two arbitrary poles x_i and x_j $(i \neq j)$, and a point $x \in \tilde{S}$. We consider the Alexandrov comparison triangle with vertexes $\tilde{x}_i$, $\tilde{x}_j$ and $\tilde{x}$ in the space M_0 of constant sectional curvature k_0, associated to the points x_i, x_j and x, respectively. More precisely, M_0 is the n-dimensional hyperbolic space of curvature k_0 when $k_0 < 0$, the Euclidean space when $k_0 = 0$, and the sphere with curvature k_0 when $k_0 > 0$.

We first prove that the perimeter $L(x_i x_j x)$ of the geodesic triangle $x_i x_j x$ is strictly less than $\frac{2\pi}{\sqrt{k_0}}$; clearly, when $k_0 \leq 0$ we have nothing to prove. Due to the strict convexity of $\tilde{S}$, the unique geodesic segments joining pairwisely the points x_i, x_j and x belong entirely to $\tilde{S}$ and as such, these points are not conjugate to each other. Thus, due to do Carmo [46, Proposition 2.4, p. 218], every side of the geodesic triangle has length $\leq \frac{\pi}{\sqrt{k_0}}$. By Klingenberg [69, Theorem 2.7.12, p. 226] we have that

$$L(x_i x_j x) \leq \frac{2\pi}{\sqrt{k_0}}.$$

Moreover, by the same result of Klingenberg, if

$$L(x_i x_j x) = \frac{2\pi}{\sqrt{k_0}},$$

it follows that either $x_i x_j x$ forms a closed geodesic, or $x_i x_j x$ is a geodesic biangle (one of the sides has length $\frac{\pi}{\sqrt{k_0}}$ and the two remaining sides form together a minimizing geodesic of length $\frac{\pi}{\sqrt{k_0}}$). In both cases we find points on the sides of the geodesic triangle $x_i x_j x$ which can be joined by two minimizing geodesics, contradicting the strict convexity of $\tilde{S}$.

We are now in the position to apply a Toponogov-type comparison result, see Klingenberg [69, Proposition 2.7.7, p. 220]; namely, we have the comparison of angles

$$\gamma_{M_0} = m(\widehat{\tilde{x}_i \tilde{x} \tilde{x}_j}) \leq \gamma_M = m(\widehat{x_i x x_j}).$$

Therefore,

$$\langle \nabla_g d_i, \nabla_g d_j\rangle = \cos(\gamma_M) \leq \cos(\gamma_{M_0}).$$

On the other hand, by the cosine-law on the space form M_0, see Bridson and Haefliger [24, p. 24], we have

$$\begin{cases} \cosh(\sqrt{-k_0}d_{ij}) = \cosh(\sqrt{-k_0}d_i)\cosh(\sqrt{-k_0}d_j) - \sinh(\sqrt{-k_0}d_i)\sinh(\sqrt{-k_0}d_j)\cos(\gamma_{M_0}), & \text{if} \quad k_0 < 0; \\ \cos(\sqrt{k_0}d_{ij}) = \cos(\sqrt{k_0}d_i)\cos(\sqrt{k_0}d_j) + \sin(\sqrt{k_0}d_i)\sin(\sqrt{k_0}d_j)\cos(\gamma_{M_0}), & \text{if} \quad k_0 > 0; \\ d_{ij}^2 = d_i^2 + d_j^2 - 2d_i d_j \cos(\gamma_{M_0}), & \text{if} \quad k_0 = 0. \end{cases}$$

Consequently,

$$\begin{cases} \cos(\gamma_M) \leq \frac{\cosh(\sqrt{-k_0}d_i)\cosh(\sqrt{-k_0}d_j) - \cosh(\sqrt{-k_0}d_{ij})}{\sinh(\sqrt{-k_0}d_i)\sinh(\sqrt{-k_0}d_j)}, & \text{if} \quad k_0 < 0; \\ \cos(\gamma_M) \leq \frac{\cos(\sqrt{k_0}d_{ij}) - \cos(\sqrt{k_0}d_i)\cos(\sqrt{k_0}d_j)}{\sin(\sqrt{k_0}d_i)\sin(\sqrt{k_0}d_j)}, & \text{if} \quad k_0 > 0; \\ \cos(\gamma_M) \leq \frac{d_i^2 + d_j^2 - d_{ij}^2}{2d_i d_j}, & \text{if} \quad k_0 = 0, \end{cases}$$

which implies

$$\frac{1}{d_i^2} + \frac{1}{d_j^2} - \frac{2\cos(\gamma_M)}{d_i d_j} \geq \begin{cases} \frac{4}{d_i d_j}\frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{\mathbf{s}_{k_0}(d_i)\mathbf{s}_{k_0}(d_j)} + R_{ij}(k_0), & \text{if} \quad k_0 \neq 0; \\ \frac{d_{ij}^2}{d_i^2 d_j^2}, & \text{if} \quad k_0 = 0, \end{cases}$$

where the expression $R_{ij}(k_0)$ is given in the statement of the theorem. Relation (4.2.6), the above inequality and (4.1.4) imply together (4.1.5). $\square$

Remark 4.2.2. Let us assume that (M,g) is a *Hadamard manifold* in Theorem 4.1.2. In particular, a Laplace comparison principle yields that

(b) Limiting cases:

- If $k_0 \to 0$, then
$$\frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i)\mathbf{s}_{k_0}(d_j)} \to \frac{d_{ij}^2}{4d_i^2 d_j^2} \text{ and } R_{ij}(k_0) \to 0,$$
thus (4.1.5) reduces to (4.1.2).

- If $k_0 \to -\infty$, then basic properties of the sinh function shows that for a.e. on M we have
$$\frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i)\mathbf{s}_{k_0}(d_j)} \to 0 \text{ and } R_{ij}(k_0) \to \left(\frac{1}{d_i} - \frac{1}{d_j}\right)^2;$$
therefore, (4.1.5) reduces to
$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \sum_{1\leq i<j\leq m} \int_M u^2 \left(\frac{1}{d_i} - \frac{1}{d_j}\right)^2 \mathrm{d}v_g, \quad \forall u \in C_0^\infty(M).$$

4.3. A bipolar Schrödinger-type equation on Cartan-Hadamard manifolds

In this section we present an application in Cartan-Hadamard manifolds.

By using inequalities (4.1.4) and (4.1.5), we obtain the following non-positively curved versions of Cazacu and Zuazua's inequalities (4.1.2) and (4.1.1) for multiple poles, respectively:

Corollary 4.3.1. *Let (M,g) be an n-dimensional Cartan-Hadamard manifold and let $S = \{x_1, ..., x_m\} \subset M$ be the set of distinct poles, with $n \geq 3$ and $m \geq 2$. Then we have the following inequality:*
$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \left|\frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j}\right|^2 u^2 \mathrm{d}v_g, \quad \forall u \in H_g^1(M). \tag{4.3.1}$$
Moreover, if $\mathbf{K} \geq k_0$ for some $k_0 \in \mathbb{R}$, then
$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{4(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i)\mathbf{s}_{k_0}(d_j)} u^2 \mathrm{d}v_g, \quad \forall u \in H_g^1(M). \tag{4.3.2}$$

Proof. Since (M,g) is a Cartan-Hadamard manifold, by using inequality (4.1.4) and the Laplace comparison theorem I (i.e., inequality (**??**) for $c = 0$), standard approximation procedure based on the density of $C_0^\infty(M)$ in $H_g^1(M)$ and Fatou's lemma immediately imply (4.3.1). Moreover, elementary properties of hyperbolic functions show that $R_{ij}(k_0) \geq 0$ (since $k_0 \leq 0$). Thus, the latter inequality and (4.1.5) yield (4.3.2). $\square$

Remark 4.3.1. *A positively curved counterpart of (4.3.1) can be stated as follows by using (4.1.4) and a Mittag-Leffler expansion (the interested reader can establish a similar inequality to (4.3.2) as well):*

Corollary 4.3.2. *Let $\mathbb{S}^n_+$ be the open upper hemisphere and let $S = \{x_1, ..., x_m\} \subset \mathbb{S}^n_+$ be the set of distinct poles, with $n \geq 3$ and $m \geq 2$. Let $\beta = \max_{i=\overline{1,m}} d_g(x_0, x_i)$, where $x_0 = (0, ..., 0, 1)$ is the north pole of the sphere $\mathbb{S}^n$ and g is the natural Riemannian metric of $\mathbb{S}^n$ inherited by $\mathbb{R}^{n+1}$. Then we have the following inequality:*

$$\|u\|^2_{\mathtt{C}(n,\beta)} \geq \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_{\mathbb{S}^n_+} \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \,\mathrm{d}v_g, \quad \forall u \in H^1_g(\mathbb{S}^n_+), \tag{4.3.3}$$

where

$$\|u\|^2_{\mathtt{C}(n,\beta)} = \int_{\mathbb{S}^n_+} |\nabla_g u|^2 \mathrm{d}v_g + \mathtt{C}(n,\beta) \int_{\mathbb{S}^n_+} u^2 \mathrm{d}v_g$$

and

$$\mathtt{C}(n,\beta) = (n-1)(n-2) \frac{7\pi^2 - 3\left(\beta + \frac{\pi}{2}\right)^2}{2\pi^2 \left(\pi^2 - \left(\beta + \frac{\pi}{2}\right)^2\right)}.$$

Part II.

Applications

5.

Schrödinger-Maxwell systems: the compact case

Whatever you do may seem insignificant to you, but it is most important that you do it.

(Gandhi)

5.1. Introduction and motivation

The Schrödinger-Maxwell system[1]

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + eu\phi = f(x,u) & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = 4\pi e u^2 & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{5.1.1}$$

describes the statical behavior of a charged non-relativistic quantum mechanical particle interacting with the electromagnetic field. More precisely, the unknown terms $u : \mathbb{R}^3 \to \mathbb{R}$ and $\phi : \mathbb{R}^3 \to \mathbb{R}$ are the fields associated to the particle and the electric potential, respectively. Here and in the sequel, the quantities m, e, ω and $\hbar$ are the mass, charge, phase, and Planck's constant, respectively, while $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying some growth conditions.

In fact, system (5.1.1) comes from the evolutionary nonlinear Schrödinger equation by using a Lyapunov-Schmidt reduction.

The Schrödinger-Maxwell system (or its variants) has been the object of various investigations in the last two decades. Without sake of completeness, we recall in the sequel some important contributions to the study of system (5.1.1). Benci and Fortunato [20] considered the case of $f(x,s) = |s|^{p-2}s$ with $p \in (4,6)$ by proving the existence of infinitely many radial solutions for (5.1.1); their main step relies on the reduction of system (5.1.1) to the investigation of critical points of a "one-variable" energy functional associated with (5.1.1). Based on the idea of Benci and Fortunato, under various growth assumptions on f further existence/multiplicity results can be found in Ambrosetti and Ruiz [5], Azzolini [9], Azzollini, d'Avenia and Pomponio [10], d'Avenia [41], d'Aprile and Mugnai [39], Cerami and Vaira [31], Kristály and Repovs [85], Ruiz [111], Sun, Chen and Nieto [117], Wang and Zhou [123], Zhao and Zhao [129], and references therein. By means of a Pohozaev-type identity, d'Aprile and Mugnai [40] proved the non-existence of non-trivial solutions to system (5.1.1) whenever $f \equiv 0$ or $f(x,s) = |s|^{p-2}s$ and $p \in (0,2] \cup [6,\infty)$.

In recent years considerable efforts have been done to describe various nonlinear phenomena in *curves spaces* (which are mainly understood in linear structures), e.g. optimal mass transportation on metric measure spaces, geometric functional inequalities and optimization problems on Riemannian/Finsler manifolds, etc. In particular, this research stream reached as well the study of Schrödinger-Maxwell systems. Indeed, in the last five years Schrödinger-Maxwell systems has been studied on $n-$dimensional *compact Riemannian manifolds* $(2 \leq n \leq 5)$ by Druet and Hebey

[1]Based on the paper [55]

[49], Hebey and Wei [67], Ghimenti and Micheletti [63, 64] and Thizy [120, 121]. More precisely, in the aforementioned papers various forms of the system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u+\omega u+eu\phi=f(u) & \text{in} \quad M, \\ -\Delta_g\phi+\phi=4\pi eu^2 & \text{in} \quad M, \end{cases} \tag{5.1.2}$$

has been considered, where (M,g) is a compact Riemannian manifold and Δ_g is the Laplace-Beltrami operator, by proving existence results with further qualitative property of the solution(s). As expected, the compactness of (M,g) played a crucial role in these investigations.

5.2. Statement of main results

In this section we are focusing to the following Schrödinger-Maxwell system:

$$\begin{cases} -\Delta_g u+\beta(x)u+eu\phi=\Psi(\lambda,x)f(u) & \text{in} \quad M, \\ -\Delta_g\phi+\phi=qu^2 & \text{in} \quad M, \end{cases} \tag{$\mathcal{SM}^e_{\Psi(\lambda,\cdot)}$}$$

where (M,g) is 3-dimensional compact Riemannian manifold without boundary, $e,q>0$ are positive numbers, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, $\beta\in C^\infty(M)$ and $\Psi\in C^\infty(\mathbb{R}_+\times M)$ are positive functions. The solutions (u,ϕ) of $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ are sought in the Sobolev space $H^1_g(M)\times H^1_g(M)$.

The aim of this section is threefold.

First we consider the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ with $\Psi(\lambda,x)=\lambda\alpha(x)$, where α is a suitable function and we assume that f is a sublinear nonlinearity (see the assumptions $(f_1)-(f_3)$ below). In this case we prove that if the parameter λ is small enough the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has only the trivial solution, while if λ is large enough then the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has at least two solutions. It is natural to ask what happens between this two threshold values. In this gap interval we have no information on the number of solutions $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$; in the case when $q\to 0$ these two threshold values may be arbitrary close to each other.

Second, we consider the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ with $\Psi(\lambda,x)=\lambda\alpha(x)+\mu_0\beta(x)$, where α and β are suitable functions. In order to prove a new kind of multiplicity for the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$, we show that certain properties of the nonlinearity, concerning the set of all global minima's, can be reflected to the energy functional associated to the problem, see Theorem 5.2.2.

Third, as a counterpart of Theorem 5.2.1 we will consider the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ with $\Psi(\lambda,x)=\lambda$, and f here satisfies the so called Ambrosetti-Rabinowitz condition. This type of result is motivated by the result of G. Anello [6] and the result of B. Ricceri [105], where the authors studied the classical Ambrosetti - Rabinowitz result without the assumption $\lim\limits_{t\to 0}\dfrac{f(t)}{t}=0$, i.e., the authors proved that if the nonlinearity f satisfies the Ambrosetti-Rabinowitz condition (see $(\tilde{f}_2)$ below) and a subcritical growth condition (see $(\tilde{f}_1)$ below), then if λ is small enough the problem

$$\begin{cases} -\Delta u=\lambda f(u) & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases}$$

has at least two weak solutions in $H^1_0(\Omega)$.

As we mentioned before, we first consider a continuous function $f:[0,\infty)\to\mathbb{R}$ which verifies the following assumptions:

(f_1) $\frac{f(s)}{s}\to 0$ as $s\to 0^+$;

(f_2) $\frac{f(s)}{s}\to 0$ as $s\to\infty$;

(f_3) $F(s_0)>0$ for some $s_0>0$, where $F(s)=\displaystyle\int_0^s f(t)\mathrm{d}t$, $s\geq 0$.

Due to the assumptions $(f_1)-(f_3)$, the numbers

$$c_f = \max_{s>0} \frac{f(s)}{s}$$

and

$$c_F = \max_{s>0} \frac{4F(s)}{2s^2 + eqs^4}$$

are well-defined and positive. Now, we are in the position to state the first result of the thesis. In order to do this, first we recall the definition of the weak solutions of the problem $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$: The pair $(u,\phi) \in H^1_g(M) \times H^1_g(M)$ is a *weak solution* to the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ if

$$\int_M (\langle \nabla_g u, \nabla_g v\rangle + \beta(x)uv + eu\phi v)\mathrm{d}v_g = \int_M \Psi(\lambda,x)f(u)v\mathrm{d}v_g \text{ for all } v \in H^1_g(M), \tag{5.2.1}$$

$$\int_M (\langle \nabla_g \phi, \nabla_g \psi\rangle + \phi\psi)\mathrm{d}v_g = q\int_M u^2\psi\mathrm{d}v_g \text{ for all } \psi \in H^1_g(M). \tag{5.2.2}$$

Theorem 5.2.1. *Let (M,g) be a $3-$dimensional compact Riemannian manifold without boundary, and let $\beta \equiv 1$. Assume that $\Psi(\lambda,x) = \lambda\alpha(x)$ and $\alpha \in C^\infty(M)$ is a positive function. If the continuous function $f:[0,\infty) \to \mathbb{R}$ satisfies assumptions (f_1)-(f_3), then*

(a) *if $0 \le \lambda < c_f^{-1}\|\alpha\|^{-1}_{L^\infty}$, system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has only the trivial solution;*

(b) *for every $\lambda \ge c_F^{-1}\|\alpha\|^{-1}_{L^1}$, system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has at least two distinct non-zero, non-negative weak solutions in $H^1_g(M) \times H^1_g(M)$.*

Similar multiplicity results were obtained by Kristály [76].

Remark 5.2.1. (a) Due to (f_1), it is clear that $f(0) = 0$, thus we can extend continuously the function $f:[0,\infty) \to \mathbb{R}$ to the whole $\mathbb{R}$ by $f(s) = 0$ for $s \le 0$; thus, $F(s) = 0$ for $s \le 0$.

(b) (f_1) and (f_2) mean that f is superlinear at the origin and sublinear at infinity, respectively. The function $f(s) = \ln(1+s^2)$, $s \ge 0$, verifies hypotheses $(f_1)-(f_3)$.

In order to obtain new kind of multiplicity result for the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ instead of the assumption (f_1) we require the following one:

(f_4) There exists $\mu_0 > 0$ such that the set of all global minima of the function

$$t \mapsto \Phi_{\mu_0}(t) := \frac{1}{2}t^2 - \mu_0 F(t)$$

has at least $m \ge 2$ connected components.

In this case we can state the following result:

Theorem 5.2.2. *Let (M,g) be a $3-$dimensional compact Riemannian manifold without boundary. Let $f:[0,\infty) \to \mathbb{R}$ be a continuous function which satisfies (f_2) and (f_4), $\beta \in C^\infty(M)$ is a positive function. Assume that $\Psi(\lambda,x) = \lambda\alpha(x) + \mu_0\beta(x)$, where $\alpha \in C^\infty(M)$ is a positive function. Then for every $\tau > \max\left\{0, \|\alpha\|_{L^1(M)} \max_t \Phi_{\mu_0}(t)\right\}$ there exists $\lambda_\tau > 0$ such that for every $\lambda \in (0,\lambda_\tau)$ the problem $(\mathcal{SM}^\lambda_{\Psi(\lambda,\cdot)})$ has at least $m+1$ weak solutions.*

Similar multiplicity results was obtained by Kristály and Rădulescu [84].

As a counterpart of the Theorem 5.2.1 we consider the case when the continuous function $f:[0,+\infty) \to \mathbb{R}$ satisfies the following assumptions:

$(\tilde{f_1})$ $|f(s)| \le C(1+|s|^{p-1})$, for all $s \in \mathbb{R}$, where $p \in (2,6)$;

($\tilde{f}_2$) there exists $\eta > 4$ and $\tau_0 > 0$ such that

$$0 < \eta F(s) \leq sf(s), \forall |s| \geq \tau_0.$$

Theorem 5.2.3. *Let (M, g) be a $3-$dimensional compact Riemannian manifold without boundary, and let $\beta \equiv 1$. Assume that $\Psi(\lambda, x) = \lambda$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, which satisfies hypotheses ($\tilde{f}_1$), ($\tilde{f}_2$). Then there exists λ_0 such that for every $0 < \lambda < \lambda_0$ the problem $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has at least two weak solutions.*

5.3. Proof of the main results

Using the Lax-Milgram theorem one can see that the equation

$$-\Delta_g \phi + \phi = qu^2, \quad \text{in } M$$

has a unique solution. Let us denote this solution by ϕ_u. In the sequel we present some basic properties of the map $u \mapsto \phi_u$:

Lemma 5.3.1. *Let (M, g) be a compact Riemannian manifold without boundary. The map $u \mapsto \phi_u : H^1_g(M) \to H^1_g(M)$ has the following properties:*

(a) $\|\phi_u\|^2_{H^1_g} = q \int_M \phi_u u^2 \mathrm{d}v_g$, $\phi_u \geq 0$;

(b) *if* $u_n \rightharpoonup u$ *in* $H^1_g(M)$, *then* $\int_M \phi_{u_n} u_n^2 \mathrm{d}v_g \to \int_M \phi_u u^2 \mathrm{d}v_g$;

(c) The map $u \mapsto \int_M \phi_u u^2 \mathrm{d}v_g$ *is convex;*

(d) We have that $\int_M (u\phi_u - v\phi_v)(u - v) \mathrm{d}v_g \geq 0$ *for all* $u, v \in H^1_g(M)$;

(e) If $v(x) \leq u(x)$ *a.e.* $x \in M$, *then* $\phi_v \leq \phi_u$.

For the proof of the previous lemma, one can consult the following references Ambrosetti and Ruiz [5], d'Avenia [41], D'Aprile and Mugnai [39, 40] and Kristály and Repovs [85].

Lemma 5.3.2. *The energy functional $\mathcal{E}_\lambda$ is coercive for every $\lambda \geq 0$.*

Proof. Indeed, due to (f_2), we have that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|F(s)| \leq \varepsilon |s|^2$, for every $|s| > \delta$. Thus, since $\Psi(x, \lambda) \in L^\infty(M)$ we have that

$$\begin{aligned}\mathcal{F}(u) &= \int_{\{u>\delta\}} \Psi(x,\lambda) F(u) \mathrm{d}v_g + \int_{\{u\leq\delta\}} \Psi(x,\lambda) F(u) \mathrm{d}v_g \\ &\leq \varepsilon \|\Psi(\cdot,\lambda)\|_{L^\infty(M)} \kappa_2^2 \|u\|^2_\beta + \|\Psi(\cdot,\lambda)\|_{L^\infty(M)} \mathrm{Vol}_g M \max_{|s|\leq\delta} |F(s)|.\end{aligned}$$

Therefore,

$$\mathcal{E}_\lambda(u) \geq \left(\frac{1}{2} - \varepsilon\kappa_2^2 \|\Psi(\cdot,\lambda)\|_{L^\infty(M)}\right) \|u\|^2_\beta - \mathrm{Vol_g} M \cdot \|\Psi(\cdot,\lambda)\|_{L^\infty(M)} \max_{|s|\leq\delta} |F(s)|.$$

In particular, if $0 < \varepsilon < (2\kappa_2^2 \|\Psi(\cdot,\lambda)\|_{L^\infty(M)})^{-1}$, then $\mathcal{E}_\lambda(u) \to \infty$ as $\|u\|_\beta \to \infty$. □

Lemma 5.3.3. *The energy functional $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition for every $\lambda \geq 0$.*

Proof. Let $\{u_j\}_j \subset H^1_\beta(M)$ be a Palais-Smale sequence, i.e., $\{\mathcal{E}_\lambda(u_j)\}_j$ is bounded and

$$\|(\mathcal{E}_\lambda)'(u_j)\|_{H^1_\beta(M)^*} \to 0$$

as $j \to \infty$. Since $\mathcal{E}_\lambda$ is coercive, the sequence $\{u_j\}_j$ is bounded in $H^1_\beta(M)$. Therefore, up to a subsequence, then $\{u_j\}_j$ converges weakly in $H^1_\beta(M)$ and strongly in $L^p(M)$, $p \in (2, 2^*)$, to an element $u \in H^1_\beta(M)$. Note that

$$\int_M |\nabla_g u_j - \nabla_g u|^2 \mathrm{d}v_g + \int_M \beta(x)\,(u_j - u)^2\, \mathrm{d}v_g =$$

$$(\mathcal{E}_\lambda)'(u_j)(u_j - u) + (\mathcal{E}_\lambda)'(u)(u - u_j) + \int_M \Psi(x,\lambda)[f(u_j(x)) - f(u(x))](u_j - u)\mathrm{d}v_g.$$

Since $\|(\mathcal{E}_\lambda)'(u_j)\|_{H^1_\beta(M)^*} \to 0$, and $u_j \rightharpoonup u$ in $H^1_\beta(M)$, the first two terms at the right hand side tend to 0. Let $p \in (2, 2^*)$.

By the assumptions on f, for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon |s|^{p-1},$$

for every $s \in \mathbb{R}$. The latter relation, Hölder inequality and the fact that $u_j \to u$ in $L^p(M)$ imply that

$$\left|\int_M \Psi(x,\lambda)[f(u_j) - f(u)](u_j - u)\mathrm{d}v_g\right| \to 0,$$

as $j \to \infty$. Therefore, $\|u_j - u\|^2_{H^1_\beta(M)} \to 0$ as $j \to \infty$, which proves our claim. $\square$

5.3.1. Schrödinger-Maxwell systems involving sublinear nonlinearity

Proof of Theorem 5.2.1. First recall that, in this case, $\beta(x) \equiv 1$ and $\Psi(\lambda, x) = \lambda\alpha(x)$, and $\alpha \in C^\infty(M)$ is a positive function.

(a) Let $\lambda \geq 0$. If we choose $v = u$ in (6.1.1) we obtain that

$$\int_M \left(|\nabla_g u|^2 + u^2 + e\phi_u u^2\right) \mathrm{d}v_g = \lambda \int_M \alpha(x) f(u) u \mathrm{d}v_g.$$

As we already mentioned, due to the assumptions $(f_1) - (f_3)$, the number $c_f = \max\limits_{s>0} \dfrac{f(s)}{s}$ is well-defined and positive. Thus, since $\|\phi_u\|^2_{H^1_g(M)} = \displaystyle\int_M \phi_u u^2 \mathrm{d}v_g \geq 0$, we have that

$$\|u\|^2_{H^1_g(M)} \leq \|u\|^2_{H^1_g(M)} + \int_M \phi_u u^2 \mathrm{d}v_g \leq \lambda c_f \|\alpha\|_{L^\infty(M)} \int_M u^2 \mathrm{d}v_g \leq \lambda c_f \|\alpha\|_{L^\infty(M)} \|u\|^2_{H^1_g(M)}.$$

Therefore, if $\lambda < c_f^{-1}\,\|\alpha\|^{-1}_{L^\infty(M)}$, then the last inequality gives $u = 0$. By the Maxwell's equation we also have that $\phi = 0$, which concludes the proof of (a).

(b) By using assumptions (f_1) and (f_2), one has

$$\lim_{\mathscr{H}(u) \to 0} \frac{\mathcal{F}(u)}{\mathscr{H}(u)} = \lim_{\mathscr{H}(u) \to \infty} \frac{\mathcal{F}(u)}{\mathscr{H}(u)} = 0,$$

where $\mathscr{H}(u) = \dfrac{1}{2}\|u\|^2_\beta + \dfrac{e}{4}\displaystyle\int_M \phi_u u^2 \mathrm{d}v_g$. Since $\alpha \in C^\infty(M)_+ \setminus \{0\}$, on account of (f_3), one can guarantee the existence of a suitable truncation function $u_T \in H^1_g(M) \setminus \{0\}$ such that $\mathcal{F}(u_T) > 0$. Therefore, we may define

$$\lambda_0 = \inf_{\substack{u \in H^1_g(M)\setminus\{0\} \\ \mathcal{F}(u) > 0}} \frac{\mathscr{H}(u)}{\mathcal{F}(u)}.$$

The above limits imply that $0 < \lambda_0 < \infty$. Since $H_g^1(M)$ contains the positive constant functions on M, we have

$$\lambda_0 = \inf_{\substack{u \in H_g^1(M)\setminus\{0\} \\ \mathcal{F}(u)>0}} \frac{\mathscr{H}(u)}{\mathcal{F}(u)} \leq \max_{s>0} \frac{2s^2 + eqs^4}{4F(s)\|\alpha\|_{L^1(M)}} = c_F^{-1}\|\alpha\|_{L^1(M)}^{-1}.$$

For every $\lambda > \lambda_0$, the functional $\mathcal{E}_\lambda$ is bounded from below, coercive and satisfies the Palais-Smale condition. If we fix $\lambda > \lambda_0$ one can choose a function $w \in H_g^1(M)$ such that $\mathcal{F}(w) > 0$ and

$$\lambda > \frac{\mathscr{H}(w)}{\mathcal{F}(w)} \geq \lambda_0.$$

In particular,

$$c_1 := \inf_{H_g^1(M)} \mathcal{E}_\lambda \leq \mathcal{E}_\lambda(w) = \mathscr{H}(w) - \lambda\mathcal{F}(w) < 0.$$

The latter inequality proves that the global minimum $u_\lambda^1 \in H_g^1(M)$ of $\mathcal{E}_\lambda$ on $H_g^1(M)$ has negative energy level.

In particular, $(u_\lambda^1, \phi_{u_\lambda^1}) \in H_g^1(M) \times H_g^1(M)$ is a nontrivial weak solution to $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$.
Let $q \in (2,6)$ be fixed. By assumptions, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that

$$0 \leq |f(s)| \leq \frac{\varepsilon}{\|\alpha\|_{L^\infty(M)}}|s| + C_\varepsilon |s|^{q-1} \text{ for all } s \in \mathbb{R}.$$

Thus

$$\begin{aligned} 0 \leq |\mathcal{F}(u)| &\leq \int_M \alpha(x)|F(u(x))|\mathrm{d}v_g \\ &\leq \int_M \alpha(x)\left(\frac{\varepsilon}{2\|\alpha\|_{L^\infty(M)}}u^2(x) + \frac{C_\varepsilon}{q}|u(x)|^q\right)\mathrm{d}v_g \\ &\leq \frac{\varepsilon}{2}\|u\|^2_{H_g^1(M)} + \frac{C_\varepsilon}{q}\|\alpha\|_{L^\infty(M)}\widetilde{\kappa}_q^q\|u\|^q_{H_g^1(M)}, \end{aligned}$$

where $\widetilde{\kappa}_q$ is the embedding constant in the compact embedding $H_g^1(M) \hookrightarrow L^p(M),\ p \in [1,6)$.

Therefore,

$$\mathcal{E}_\lambda(u) \geq \frac{1}{2}(1-\lambda\varepsilon)\|u\|^2_{H_g^1(M)} - \frac{\lambda C_\varepsilon}{q}\|\alpha\|_{L^\infty(M)}\widetilde{\kappa}_q^q\|u\|^q_{H_g^1(M)}.$$

Bearing in mind that $q > 2$, for enough small $\rho > 0$ and $\varepsilon < \lambda^{-1}$ we have that

$$\inf_{\|u\|_{H_g^1(M)}=\rho} \mathcal{E}_\lambda(u) \geq \frac{1}{2}\left(1-\varepsilon\lambda\right)\rho - \frac{\lambda C_\varepsilon}{q}\|\alpha\|_{L^\infty(M)}\widetilde{\kappa}_q^q\rho^{\frac{q}{2}} > 0.$$

A standard mountain pass argument (see for instance, Willem [124]) implies the existence of a critical point $u_\lambda^2 \in H_g^1(M)$ for $\mathcal{E}_\lambda$ with positive energy level. Thus $(u_\lambda^2, \phi_{u_\lambda^2}) \in H_g^1(M) \times H_g^1(M)$ is also a nontrivial weak solution to $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$. Clearly, $u_\lambda^1 \neq u_\lambda^2$. $\square$

It is clear that $c_f > c_F$. Indeed, let $s_0 > 0$ be a maximum point for the function $s \mapsto \frac{4F(s)}{2s^2+eqs^4}$, therefore

$$c_F = \frac{4F(s_0)}{2s_0^2 + eqs_0^4} = \frac{f(s_0)}{s_0 + eqs_0^3} \leq \frac{f(s_0)}{s_0} \leq c_f.$$

Now we assume that $c_f = c_F$. Let

$$\widetilde{s}_0 := \inf\left\{s > 0 : C = \frac{4F(s)}{2s^2 + eqs^4}\right\}.$$

Note that $\widetilde{s}_0 > 0$. Fix $t_0 \in (0, \widetilde{s}_0)$, in particular $4F(t_0) < C(2t_0^3 + eqt_0^4)$. On the other hand, from the definition of c_f, one has $f(t) \leq C(s + eqs^3)$. Therefore

$$0 = 4F(\widetilde{s}_0) - C(2\widetilde{s}_0 + eq\widetilde{s}_0^4) = \left(4F(t_0) - C(2t_0^2 + eqt_0^4)\right) + 4\int_{t_0}^{\widetilde{s}_0} (f(t) - C(s + eqs^3))ds < 0,$$

which is a contradiction, thus $c_f > c_F$.

It is also clear that the function $q \mapsto \max_{s>0} \dfrac{4F(s)}{2s^2 + eqs^4}$ is non-increasing.

Let $a > 1$ be a real number. Now, consider the following function

$$f(s) = \begin{cases} 0, & 0 \leq s < 1, \\ s + g(s), & 1 \leq s < a, \\ a + g(a), & s \geq a, \end{cases}$$

where $g : [1, +\infty) \to \mathbb{R}$ is a continuous function with the following properties

(g_1) $g(1) = -1$;

(g_2) the function $s \mapsto \dfrac{g(s)}{s}$ is non-decreasing on $[1, +\infty)$;

(g_3) $\lim_{s\to\infty} g(s) < \infty$.

In this case the

$$F(s) = \begin{cases} 0, & 0 \leq s < 1, \\ \dfrac{s^2}{2} + G(s) - \dfrac{1}{2}, & 1 \leq s < a, \\ (a + g(a))s - \dfrac{a^2}{2} + G(a) - ag(a) - \dfrac{1}{2}, & s \geq a, \end{cases}$$

where $G(s) = \int_1^s g(t)dt$. It is also clear that f satisfies the assumptions $(f_1) - (f_3)$.

Thus, a simple calculation shows that

$$c_f = \frac{a + g(a)}{a},$$

and

$$\widehat{c}_F = \lim_{q\to 0} c_F = \frac{(a + g(a))^2}{a^2 + 2ag(a) - 2G(a) + 1}.$$

One can see that, from the assumptions on g, that the values c_f and $\widehat{c}_F$ may be arbitrary close to each other. Indeed, when

$$\lim_{a\to\infty} c_f = \lim_{a\to\infty} \widehat{c}_F = 1.$$

Therefore, if $\alpha \equiv 1$ then the threshold values are c_f^{-1} and c_F^{-1} (which are constructed independently), i.e. if $\lambda \in (0, c_f^{-1})$ we have just the trivial solution, while if $\lambda \in (c_F^{-1}, +\infty)$ we have at least two solutions. λ lying in the gap-interval $[c_f^{-1}, c_F^{-1}]$ we have no information on the number of solutions for $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$.

Taking into account the above example we see that if the "impact" of the Maxwell equation is small ($q \to 0$), then the values c_f and c_F may be arbitrary close to each other.

Remark 5.3.1. *Typical examples for function g can be:*

(a) $g(s) = -1$. *In this case* $c_f = \frac{a-1}{a}$ *and* $\widehat{c}_F = \frac{a-1}{a+1}$.

(b) $g(s) = \frac{1}{s} - 2$. *In this case* $c_f = \frac{(a-1)^2}{a^2}$ *and* $\widehat{c}_F = \frac{(a-1)^4}{a^2(a^2-2\ln a-1)}$.

Proof of Theorem 5.2.2. Let us denote by

$$\|u\|_\beta^2 = \int_M |\nabla_g u|^2 + \beta(x)u^2 \mathrm{d}v_g.$$

First we claim that the set of all global minima's of the functional $\mathcal{N} : H_g^1(M) \to \mathbb{R}$,

$$\mathcal{N}(u) = \frac{1}{2}\|u\|_\beta^2 - \mu_0 \int_M \beta(x)F(u)\mathrm{d}v_g$$

has at least m connected components in the weak topology on $H_g^1(M)$. Indeed, for every $u \in H_g^1(M)$ one has

$$\begin{aligned}
\mathcal{N}(u) &= \frac{1}{2}\|u\|_\beta^2 - \mu_0 \int_M \beta(x)F(u)\mathrm{d}v_g \\
&= \frac{1}{2}\int_M |\nabla_g u|^2 \mathrm{d}v_g + \int_M \beta(x)\Phi_{\mu_0}(u)\mathrm{d}v_g \\
&\geq \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t).
\end{aligned}$$

Moreover, if we consider $u = \tilde{t}$ for a.e. $x \in M$, where $\tilde{t} \in \mathbb{R}$ is the minimum point of the function $t \mapsto \Phi_{\mu_0}(t)$, then we have equality in the previous estimate. Thus,

$$\inf_{u \in H_g^1(M)} \mathcal{N}(u) = \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t).$$

On the other hand if $u \in H_g^1(M)$ is not a constant function, then $|\nabla_g u|^2 > 0$ on a positive measure set in M, i.e., $\mathcal{N}(u) > \|\beta\|_{L^1(M)} \inf_t \Phi_{\mu_0}(t)$. Consequently, there is a one-to-one correspondence between the sets

$$\mathrm{Min}(\mathcal{N}) = \left\{ u \in H_g^1(M) : \ \mathcal{N}(u) = \inf_{u \in H_g^1(M)} \mathcal{N}(u) \right\}$$

and

$$\mathrm{Min}\left(\Phi_{\mu_0}\right) = \left\{ t \in \mathbb{R} : \ \Phi_{\mu_0}(t) = \inf_{t \in \mathbb{R}} \Phi_{\mu_0}(t) \right\}.$$

Let ξ be the function that associates to every $t \in \mathbb{R}$ the equivalence class of those functions which are a.e. equal to t on the whole M. Then $\xi : \mathrm{Min}(\mathcal{N}) \to \mathrm{Min}\left(\Phi_{\mu_0}\right)$ is actually a homeomorphism, where $\mathrm{Min}(\mathcal{N})$ is considered with the relativization of the weak topology on $H_g^1(M)$. On account of (f_4), the set $\mathrm{Min}\left(\Phi_{\mu_0}\right)$ has at least $m \geq 2$ connected components. Therefore, the same is true for the set $\mathrm{Min}(\mathcal{N})$, which proves the claim.

Now we are in the position to apply Theorem 1.2.9 with $H = H_g^1(M)$, $\mathcal{N}$and

$$\mathcal{G} = \frac{1}{4}\int_M \phi_u u^2 \mathrm{d}v_g - \int_M \alpha(x)F(u)\mathrm{d}v_g.$$

It is clear that the functionals $\mathcal{N}$ and $\mathcal{G}$ satisfies all the hypotheses of Theorem 1.2.9. Therefore for every $\tau > \max\{0, \|\alpha\|_1 \max_t \Phi_{\mu_0}(t)\}$ there exists $\lambda_\tau > 0$ such that for every $\lambda \in (0, \lambda_\tau)$ the problem $(\mathcal{SM}^\lambda_{\Psi(\lambda,\cdot)})$ has at least $m+1$ solutions. $\square$

5.3.2. Schrödinger-Maxwell systems involving superlinear nonlinearity

In the sequel we prove Theorem 5.2.3. Recall that $\Psi(\lambda, x) = \lambda\alpha(x)$ and $\beta \equiv 1$. The energy functional associated with the problem $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ is defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|^2_{H^1_g(M)} + \frac{e}{4}\int_M \phi_u u^2 \mathrm{d}v_g - \lambda\int_M F(u)\mathrm{d}v_g.$$

Lemma 5.3.4. *Every (PS) sequence for the functional $\mathcal{E}_\lambda$ is bounded in $H^1_g(M)$.*

Proof. We consider a Palais-Smale sequence $(u_j)_j \subset H^1_g(M)$ for $\mathcal{E}_\lambda$, i.e., $\{\mathcal{E}_\lambda(u_j)\}$ is bounded and

$$\|(\mathcal{E}_\lambda)'(u_j)\|_{H^1_g(M)^*} \to 0 \text{ as } j \to \infty.$$

We claim that $(u_j)_j$ is bounded in $H^1_g(M)$. We argue by contradiction, so suppose the contrary. Passing to a subsequence if necessary, we may assume that

$$\|u_j\|_{H^1_g(M)} \to \infty, \quad \text{as } j \to \infty.$$

It follows that there exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$ we have that

$$\begin{aligned}\mathcal{E}_\lambda(u_j) - \frac{\langle \mathcal{E}'_\lambda(u_j), u_j\rangle}{\eta} &= \frac{1}{2}\left(\frac{\eta-2}{\eta}\right)\|u_j\|^2_{H^1_g(M)} + e\left(\frac{\eta-4}{\eta}\right)\int_M \phi_{u_j}u_j^2\mathrm{d}v_g \\ &+ \lambda\int_M\left(\frac{f(u_j)u_j}{\eta} - F(u_j)\right)\mathrm{d}v_g.\end{aligned}$$

Thus, bearing in mind that $\displaystyle\int_M \phi_u u^2 \mathrm{d}v_g \geq 0$ and $(\tilde{f}_2)$ one has that

$$\frac{1}{2}\left(\frac{\eta-2}{\eta}\right)\|u_j\|^2_{H^1_g(M)} \leq \mathcal{E}_\lambda(u_j) - \frac{\langle \mathcal{E}'_\lambda(u_j), u_j\rangle}{\eta} + \chi\mathrm{Vol_g}(M),$$

where

$$\chi = \sup\left\{\left|\frac{tf(t)}{\eta} - F(t)\right| : t \leq \tau_0\right\}.$$

Therefore, for every $j \geq j_0$ we have that

$$\frac{1}{2}\left(\frac{\eta-2}{\eta}\right)\|u_j\|^2_{H^1_g(M)} \leq \mathcal{E}_\lambda(u_j) + \frac{1}{\eta}\|(\mathcal{E}_\lambda)'(u_j)\|_{H^1_g*}\|u_j\|_{H^1_g(M)} + \chi\mathrm{Vol_g}(M).$$

Dividing by $\|u_j\|_{H^1_g(M)}$ and letting $j \to \infty$ we get a contradiction, which implies the boundedness of the sequence $\{u_j\}_j$ in $H^1_g(M)$. $\square$

Proof of the Theorem 5.2.3. Let us consider as before the following functionals:

$$\mathscr{H}(u) = \frac{1}{2}||u||^2_{H^1_g(M)} + \frac{e}{4}\int_M \phi_u u^2\mathrm{d}v_g \text{ and } \mathcal{F}(u) = \int_M F(u)\mathrm{d}v_g.$$

Form the positivity and the convexity of functional $u \mapsto \displaystyle\int_M \phi_u u^2$ it follows that the functional $\mathscr{H}$ is sequentially weakly semicontinuous and coercive functional. It is also clear that $\mathcal{F}$ is sequentially weakly continuous. Then, for $\mu = \frac{1}{2\lambda}$, we define the functional

$$J_\mu(u) = \mu\mathscr{H}(u) - \mathcal{F}(u).$$

Integrating, we get from $(\tilde{f}_2)$ that,

$$F(ts) \geq t^\eta F(s), \quad t \geq 1 \text{ and } |s| \geq \tau_0.$$

Now, let us consider a fixed function $u_0 \in H_g^1(M)$ such that

$$\mathrm{Vol_g}\left(\{x \in M : |u_0(x)| \geq \tau_0\}\right) > 0,$$

and using the previous inequality and the fact that $\phi_{tu} = t^2\phi_u$, we have that:

$$\begin{aligned}
J_\mu(tu_0) &= \mu\mathscr{H}(tu_0) - \mathcal{F}(tu_0) \\
&= \mu\frac{t^2}{2}||u_0||^2_{H_g^1(M)} + \mu\frac{e}{4}t^4\int_M \phi_{u_0}u_0^2 - \int_M F(tu_0) \\
&\leq \mu t^2||u_0||^2_{H_g^1(M)} + \mu\frac{e}{2}t^4\int_M \phi_{u_0}u_0^2 - t^\eta\int_{\{x\in M::|u_0|\geq\tau_0\}} F(u_0) + \chi_2\mathrm{Vol}_g(M) \overset{\eta>4}{\to} -\infty,
\end{aligned}$$

as $t \to \infty$, where

$$\chi_2 = \sup\left\{|F(t)| : |t| \leq \tau_0\right\}.$$

Thus, the functional J_μ is unbounded from below. A similar argument as before shows that (taking eventually a subsequence), one has that the functional J_μ satisfies the (PS) condition.

Let us denote by $K_\tau = \left\{x \in M : \|u\|^2_{H_g^1(M)} < \tau\right\}$ and by

$$h(\tau) = \inf_{u\in K_\tau}\frac{\sup\limits_{v\in K_\tau}\mathcal{F}(v) - \mathcal{F}(u)}{\tau - \mathscr{H}(u)}$$

Since $0 \in K_\tau$, we have that

$$h(\tau) \leq \frac{\sup_{v\in K_\tau}\mathcal{F}(v)}{\tau}.$$

On the other hand bearing in mind the assumption $(\tilde{f}_1)$, we have that,

$$\mathcal{F}(v) \leq C\|v\|_{H_g^1(M)} + \frac{C}{p}\kappa_p^p\|v\|^p_{H_g^1(M)}.$$

Therefore

$$h(\tau) \leq \frac{C}{2}\tau^{\frac{1}{2}} + \frac{C\kappa_p^p}{p}\tau^{\frac{p-2}{2}}.$$

Thus, if

$$\lambda < \lambda_0 := \frac{p\tau^{\frac{1}{2}}}{2pC + 2C\kappa_p^p\tau^{\frac{p-1}{2}}}$$

one has $\mu = \frac{1}{2\lambda} > h(\tau)$. Therefore, we are in the position to apply Ricceri's result, i.e., Theorem 1.2.10, which concludes our proof. □

Remark 5.3.2. *Form the proof of Theorem 5.2.3 one can see that*

$$\lambda_0 \leq \frac{p}{2C}\max_{\tau>0}\frac{\tau^{\frac{1}{2}}}{p + \kappa_p^p\tau^{\frac{p-1}{2}}}.$$

Since $p > 2$, $\max\limits_{\tau>0}\dfrac{p\tau^{\frac{1}{2}}}{2pC + 2C\kappa_p^p\tau^{\frac{p-1}{2}}} < \infty.$

6.

Schrödinger-Maxwell systems: the non-compact case

It does not matter how slowly you go as long as you do not stop.

(Confucius)

6.1. Statement of main results

As far as we know, no result is available in the literature concerning Maxwell-Schrödinger systems on *non-compact Riemannian manifolds*[1]. Motivated by this fact, the purpose of the present chapter is to provide existence, uniqueness and multiplicity results in the case of the Maxwell-Schrödinger system in such a non-compact setting. Since this problem is very general, we shall restrict our study to *Cartan-Hadamard manifolds* (simply connected, complete Riemannian manifolds with non-positive sectional curvature).

To be more precise, we shall consider the Schrödinger-Maxwell system

$$\begin{cases} -\Delta_g u + u + eu\phi = \lambda\alpha(x)f(u) & \text{in} \quad M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in} \quad M, \end{cases} \tag{$\mathcal{SM}_\lambda$}$$

where (M, g) is an $n-$dimensional Cartan-Hadamard manifold ($3 \leq n \leq 5$), $e, q > 0$ are positive numbers, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha : M \to \mathbb{R}$ is a measurable function, and $\lambda > 0$ is a parameter. The solutions (u, ϕ) of $(\mathcal{SM}_\lambda)$ are sought in the Sobolev space $H^1_g(M) \times H^1_g(M)$. In order to handle the lack of compactness of (M, g), a Lions-type symmetrization argument will be used, based on the action of a suitable subgroup of the group of isometries of (M, g). More precisely, we shall adapt the main results of Skrzypczak and Tintarev [114] to our setting concerning Sobolev spaces in the presence of group-symmetries.
In the sequel, we shall formulate rigourously our main results with some comments.

Let (M, g) be an $n-$dimensional Cartan-Hadamard manifold, $3 \leq n \leq 6$. The pair $(u, \phi) \in H^1_g(M) \times H^1_g(M)$ is a *weak solution* to the system $(\mathcal{SM}_\lambda)$ if

$$\int_M (\langle \nabla_g u, \nabla_g v\rangle + uv + eu\phi v)\mathrm{d}v_g = \lambda \int_M \alpha(x) f(u) v \mathrm{d}v_g \text{ for all } v \in H^1_g(M), \tag{6.1.1}$$

$$\int_M (\langle \nabla_g \phi, \nabla_g \psi\rangle + \phi\psi)\mathrm{d}v_g = q \int_M u^2 \psi \mathrm{d}v_g \text{ for all } \psi \in H^1_g(M). \tag{6.1.2}$$

For later use, we denote by $\mathrm{Isom}_g(M)$ the group of isometries of (M, g) and let G be a subgroup of $\mathrm{Isom}_g(M)$. A function $u : M \to \mathbb{R}$ is *$G-$invariant* if $u(\sigma(x)) = u(x)$ for every $x \in M$ and $\sigma \in G$. Furthermore, $u : M \to \mathbb{R}$ is *radially symmetric w.r.t.* $x_0 \in M$ if u depends on $d_g(x_0, \cdot)$, d_g being the Riemannian distance function. The *fixed point set of* G on M is given by $\mathrm{Fix}_M(G) = \{x \in M : \sigma(x) = x \text{ for all } \sigma \in G\}$. For a given $x_0 \in M$, we introduce the following

[1] Based on the papers [52, 56]

hypothesis which will be crucial in our investigations:

$(\boldsymbol{H_G^{x_0}})$ *The group G is a compact connected subgroup of* $\mathrm{Isom}_g(M)$ *such that* $\mathrm{Fix}_M(G) = \{x_0\}$.

Remark 6.1.1. In the sequel, we provide some concrete Cartan-Hadamard manifolds and group of isometries for which hypothesis $(\boldsymbol{H_G^{x_0}})$ is satisfied:

- *Euclidean spaces.* If $(M, g) = (\mathbb{R}^n, g_{\text{euc}})$ is the usual Euclidean space, then $x_0 = 0$ and $G = SO(n_1) \times \ldots \times SO(n_l)$ with $n_j \geq 2$, $j = 1, \ldots, l$ and $n_1 + \ldots + n_l = n$, satisfy $(\boldsymbol{H_G^{x_0}})$, where $SO(k)$ is the special orthogonal group in dimension k. Indeed, we have $\mathrm{Fix}_{\mathbb{R}^n}(G) = \{0\}$.

- *Hyperbolic spaces.* Let us consider the Poincaré ball model $\mathbb{H}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the Riemannian metric $g_{\text{hyp}}(x) = (g_{ij}(x))_{i,j=1,\ldots,n} = \dfrac{4}{(1 - |x|^2)^2}\delta_{ij}$. It is well known that $(\mathbb{H}^n, g_{\text{hyp}})$ is a homogeneous Cartan-Hadamard manifold with constant sectional curvature -1. Hypothesis $(\boldsymbol{H_G^{x_0}})$ is verified with the same choices as above.

- *Symmetric positive definite matrices.* Let $\mathrm{Sym}(n, \mathbb{R})$ be the set of symmetric $n \times n$ matrices with real values, $\mathrm{P}(n, \mathbb{R}) \subset \mathrm{Sym}(n, \mathbb{R})$ be the cone of symmetric positive definite matrices, and $\mathrm{P}(n, \mathbb{R})_1$ be the subspace of matrices in $\mathrm{P}(n, \mathbb{R})$ with determinant one. The set $\mathrm{P}(n, \mathbb{R})$ is endowed with the scalar product

$$\langle U, V \rangle_X = \mathrm{Tr}(X^{-1} V X^{-1} U) \ \text{ for all } \ X \in \mathrm{P}(n, \mathbb{R}),\ U, V \in T_X(\mathrm{P}(n, \mathbb{R})) \simeq \mathrm{Sym}(n, \mathbb{R}),$$

  where $\mathrm{Tr}(Y)$ denotes the trace of $Y \in \mathrm{Sym}(n, \mathbb{R})$. One can prove that $(\mathrm{P}(n, \mathbb{R})_1, \langle \cdot, \cdot \rangle)$ is a homogeneous Cartan-Hadamard manifold (with non-constant sectional curvature) and the special linear group $SL(n)$ leaves $\mathrm{P}(n, \mathbb{R})_1$ invariant and acts transitively on it. Moreover, for every $\sigma \in SL(n)$, the map $[\sigma] : \mathrm{P}(n, \mathbb{R})_1 \to \mathrm{P}(n, \mathbb{R})_1$ defined by $[\sigma](X) = \sigma X \sigma^t$, is an isometry, where σ^t denotes the transpose of σ. If $G = SO(n)$, we can prove that $\mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G) = \{I_n\}$, where I_n is the identity matrix; for more details, see Kristály [79].

For $x_0 \in M$ fixed, we also introduce the hypothesis

$(\boldsymbol{\alpha^{x_0}})$ *The function $\alpha : M \to \mathbb{R}$ is non-zero, non-negative and radially symmetric w.r.t. x_0.*

Our results are divided into two classes:

A. Schrödinger-Maxwell systems of Poisson type. Dealing with a Poisson-type system, we set $\lambda = 1$ and $f \equiv 1$ in $(\mathcal{SM}_\lambda)$. For abbreviation, we simply denote $(\mathcal{SM}_1)$ by $(\mathcal{SM})$.

Theorem 6.1.1. *Let (M, g) be an $n-$dimensional homogeneous Cartan-Hadamard manifold $(3 \leq n \leq 6)$, and $\alpha \in L^2(M)$ be a non-negative function. Then there exists a unique, non-negative weak solution $(u_0, \phi_0) \in H_g^1(M) \times H_g^1(M)$ to problem $(\mathcal{SM})$. Moreover, if $x_0 \in M$ is fixed and α satisfies $(\boldsymbol{\alpha^{x_0}})$, then (u_0, ϕ_0) is $G-$invariant w.r.t. any group $G \subset \mathrm{Isom}_g(M)$ which satisfies $(\boldsymbol{H_G^{x_0}})$.*

Remark 6.1.2. Let (M, g) be either the $n-$dimensional Euclidean space $(\mathbb{R}^n, g_{\text{euc}})$ or hyperbolic space $(\mathbb{H}^n, g_{\text{hyp}})$, and fix $G = SO(n_1) \times \ldots \times SO(n_l)$ for a splitting of $n = n_1 + \ldots + n_l$ with $n_j \geq 2$, $j = 1, \ldots, l$. If α is radially symmetric (w.r.t. $x_0 = 0$), Theorem 6.1.1 states that the unique solution (u_0, ϕ_0) to the Poisson-type Schrödinger-Maxwell system $(\mathcal{SM})$ is not only invariant w.r.t. the group G but also with any compact connected subgroup $\tilde{G}$ of $\mathrm{Isom}_g(M)$ with the same fixed point property $\mathrm{Fix}_M(\tilde{G}) = \{0\}$; thus, in particular, (u_0, ϕ_0) is invariant w.r.t. the whole group $SO(n)$, i.e. (u_0, ϕ_0) is radially symmetric.

For $c \le 0$ and $3 \le n \le 6$ we consider the ordinary differential equations system

$$\begin{cases} -h_1''(r) - (n-1)\mathbf{ct_c}(s)h_1'(r) + h_1(r) + eh_1(r)h_2(r) - \alpha_0(r) = 0, \ r \ge 0; \\ -h_2''(r) - (n-1)\mathbf{ct_c}(r)h_2'(r) + h_2(r) - qh_1(r)^2 = 0, \ r \ge 0; \\ \displaystyle\int_0^\infty (h_1'(r)^2 + h_1^2(r))\mathbf{s}_c(r)^{n-1}\mathrm{d}r < \infty; \\ \displaystyle\int_0^\infty (h_2'(r)^2 + h_2^2(r))\mathbf{s}_c(r)^{n-1}\mathrm{d}r < \infty, \end{cases} \qquad (\mathscr{R})$$

where $\alpha_0 : [0,\infty) \to [0,\infty)$ satisfies the integrability condition $\alpha_0 \in L^2([0,\infty), \mathbf{s}_c(r)^{n-1}\mathrm{d}r)$.

We shall show (see Lemma 6.2.2) that $(\mathscr{R})$ has a unique, non-negative solution $(h_1^c, h_2^c) \in C^\infty(0,\infty) \times C^\infty(0,\infty)$. In fact, the following rigidity result can be stated:

Theorem 6.1.2. *Let (M,g) be an $n-$dimensional homogeneous Cartan-Hadamard manifold $(3 \le n \le 6)$ with sectional curvature $\mathbf{K} \le c \le 0$. Let $x_0 \in M$ be fixed, and $G \subset \mathrm{Isom}_g(M)$ and $\alpha \in L^2(M)$ be such that hypotheses $(\boldsymbol{H_G^{x_0}})$ and $(\boldsymbol{\alpha^{x_0}})$ are satisfied. If $\alpha^{-1}(t) \subset M$ has null Riemannian measure for every $t \ge 0$, then the following statements are equivalent:*

(i) $(h_1^c(d_g(x_0,\cdot)), h_2^c(d_g(x_0,\cdot)))$ *is the unique pointwise solution of* $(\mathcal{SM})$;

(ii) (M,g) *is isometric to the space form with constant sectional curvature* $\mathbf{K} = c$.

B. Schrödinger-Maxwell systems involving oscillatory terms. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function with $F(s) = \displaystyle\int_0^s f(t)\mathrm{d}t$. We assume:

(f_0^1) $-\infty < \liminf\limits_{s\to 0} \dfrac{F(s)}{s^2} \le \limsup\limits_{s\to 0} \dfrac{F(s)}{s^2} = +\infty;$

(f_0^2) there exists a sequence $\{s_j\}_j \subset (0,1)$ converging to 0 such that $f(s_j) < 0$, $j \in \mathbb{N}$.

Theorem 6.1.3. *Let (M,g) be an $n-$dimensional homogeneous Cartan-Hadamard manifold $(3 \le n \le 5)$, $x_0 \in M$ be fixed, and $G \subset \mathrm{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ be such that hypotheses $(\boldsymbol{H_G^{x_0}})$ and $(\boldsymbol{\alpha^{x_0}})$ are satisfied. If $f : [0,\infty) \to \mathbb{R}$ is a continuous function satisfying (f_0^1) and (f_0^2), then there exists a sequence $\{(u_j^0, \phi_{u_j^0})\}_j \subset H_g^1(M) \times H_g^1(M)$ of distinct, non-negative $G-$invariant weak solutions to $(\mathcal{SM})$ such that*

$$\lim_{j\to\infty} \|u_j^0\|_{H_g^1(M)} = \lim_{j\to\infty} \|\phi_{u_j^0}\|_{H_g^1(M)} = 0.$$

Remark 6.1.3. (a) Under the assumptions of Theorem 6.1.3 we consider the perturbed Schrödinger-Maxwell system

$$\begin{cases} -\Delta_g u + u + eu\phi = \lambda\alpha(x)[f(u) + \varepsilon g(u)] & \text{in } M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in } M, \end{cases} \qquad (\mathcal{SM}_\varepsilon)$$

where $\varepsilon > 0$ and $g : [0,\infty) \to \mathbb{R}$ is a continuous function with $g(0) = 0$. Arguing as in the proof of Theorem 6.1.3, a careful energy control provides the following statement: for every $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that $(\mathcal{SM}_\varepsilon)$ has at least k distinct, $G-$invariant weak solutions $(u_{j,\varepsilon}, \phi_{u_{j,\varepsilon}})$, $j \in \{1, ..., k\}$, whenever $\varepsilon \in [-\varepsilon_k, \varepsilon_k]$. Moreover, one can prove that

$$\|u_{j,\varepsilon}\|_{H_g^1(M)} < \frac{1}{j} \text{ and } \|\phi_{u_{j,\varepsilon}}\|_{H_g^1(M)} < \frac{1}{j}, \ j \in \{1, ..., k\}.$$

Note that a similar phenomenon has been described for Dirichlet problems in Kristály and Moroşanu [82].

(b) Theorem 6.1.3 complements some results from the literature where $f : \mathbb{R} \to \mathbb{R}$ has the symmetry property $f(s) = -f(-s)$ for every $s \in \mathbb{R}$ and verifies an Ambrosetti-Rabinowitz-type assumption. Indeed, in such cases, the symmetric version of the mountain pass theorem provides a sequence of weak solutions for the studied Schrödinger-Maxwell system.

(c) It is worth mentioning that the oscillation of f (condition (f_0^1)) in itself is not enough to guarantee multiple solutions: indeed in [42], de Figueiredo proves the uniqueness of positive solution of the problem $-\Delta u = \lambda \sin u$.

6.1.1. Variational framework

Let (M, g) be an $n-$dimensional Cartan-Hadamard manifold, $3 \leq n \leq 6$. We define the energy functional $\mathscr{J}_\lambda : H_g^1(M) \times H_g^1(M) \to \mathbb{R}$ associated with system $(\mathcal{SM}_\lambda)$, namely,

$$\mathscr{J}_\lambda(u, \phi) = \frac{1}{2}\|u\|^2_{H_g^1(M)} + \frac{e}{2}\int_M \phi u^2 \mathrm{d}v_g - \frac{e}{4q}\int_M |\nabla_g \phi|^2 \mathrm{d}v_g - \frac{e}{4q}\int_M \phi^2 \mathrm{d}v_g - \lambda \int_M \alpha(x) F(u) \mathrm{d}v_g.$$

In all our cases (see problems **A**, **B** and **C** above), the functional $\mathscr{J}_\lambda$ is well-defined and of class C^1 on $H_g^1(M) \times H_g^1(M)$. To see this, we have to consider the second and fifth terms from $\mathscr{J}_\lambda$; the other terms trivially verify the required properties. First, a comparison principle and suitable Sobolev embeddings give that there exists $C > 0$ such that for every $(u, \phi) \in H_g^1(M) \times H_g^1(M)$,

$$0 \leq \int_M \phi u^2 \mathrm{d}v_g \leq \left(\int_M \phi^{2^*} \mathrm{d}v_g\right)^{\frac{1}{2^*}} \left(\int_M |u|^{\frac{4n}{n+2}} \mathrm{d}v_g\right)^{1-\frac{1}{2^*}} \leq C\|\phi\|_{H_g^1(M)}\|u\|^2_{H_g^1(M)} < \infty,$$

where we used $3 \leq n \leq 6$. If $\mathcal{F} : H_g^1(M) \to \mathbb{R}$ is the functional defined by $\mathcal{F}(u) = \displaystyle\int_M \alpha(x) F(u) \mathrm{d}v_g$, we have:

- Problem **A**: $\alpha \in L^2(M)$ and $F(s) = s$, $s \in \mathbb{R}$, thus $|\mathcal{F}(u)| \leq \|\alpha\|_{L^2(M)}\|u\|_{L^2(M)} < +\infty$ for all $u \in H_g^1(M)$.
- Problem **B**: the assumptions allow to consider generically that f is subcritical, i.e., there exist $c > 0$ and $p \in [2, 2^*)$ such that

  $$|f(s)| \leq c(|s| + |s|^{p-1}) \text{forevery} s \in \mathbb{R}.$$

  Since $\alpha \in L^\infty(M)$ in every case, we have that $|\mathcal{F}(u)| < +\infty$ for every $u \in H_g^1(M)$ and $\mathcal{F}$ is of class C^1 on $H_g^1(M)$.

Step 1. *The pair $(u, \phi) \in H_g^1(M) \times H_g^1(M)$ is a weak solution of $(\mathcal{SM}_\lambda)$ if and only if (u, ϕ) is a critical point of $\mathscr{J}_\lambda$.* Indeed, due to relations (6.1.1) and (6.1.2), the claim follows.

By exploring an idea of Benci and Fortunato [20], due to the Lax-Milgram theorem (see e.g. Brezis [23, Corollary 5.8]), we introduce the map $\phi_u : H_g^1(M) \to H_g^1(M)$ by associating to every $u \in H_g^1(M)$ the unique solution $\phi = \phi_u$ of the Maxwell equation

$$-\Delta_g \phi + \phi = qu^2.$$

We recall some important properties of the function $u \mapsto \phi_u$ which are straightforward adaptations of Kristály and Repovs [85, Proposition 2.1] and Ruiz [111, Lemma 2.1] to the Riemannian setting:

$$\|\phi_u\|^2_{H_g^1(M)} = q\int_M \phi_u u^2 \mathrm{d}v_g, \quad \phi_u \geq 0; \tag{6.1.3}$$

$$u \mapsto \int_M \phi_u u^2 \mathrm{d}v_g \text{ is convex;} \tag{6.1.4}$$

$$\int_M (u\phi_u - v\phi_v)(u - v)\mathrm{d}v_g \geq 0 \text{ for all } u, v \in H_g^1(M). \tag{6.1.5}$$

The "one-variable" energy functional $\mathcal{E}_\lambda : H_g^1(M) \to \mathbb{R}$ associated with system $(\mathcal{SM}_\lambda)$ is defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|^2_{H_g^1(M)} + \frac{e}{4}\int_M \phi_u u^2 \mathrm{d}v_g - \lambda\mathcal{F}(u). \tag{6.1.6}$$

By using standard variational arguments, one has:

STEP 2. *The pair $(u,\phi)\in H_g^1(M)\times H_g^1(M)$ is a critical point of $\mathscr{J}_\lambda$ if and only if u is a critical point of $\mathcal{E}_\lambda$ and $\phi=\phi_u$.* Moreover, we have that

$$\mathcal{E}_\lambda'(u)(v)=\int_M(\langle\nabla_g u,\nabla_g v\rangle+uv+e\phi_u uv)\mathrm{d}v_g-\lambda\int_M\alpha(x)f(u)v\mathrm{d}v_g. \tag{6.1.7}$$

In the sequel, let $x_0\in M$ be fixed, and $G\subset \mathrm{Isom}_g(M)$ and $\alpha\in L^1(M)\cap L^\infty(M)$ be such that hypotheses $\boldsymbol{(H_G^{x_0})}$ and $\boldsymbol{(\alpha^{x_0})}$ are satisfied. The action of G on $H_g^1(M)$ is defined by

$$(\sigma u)(x)=u(\sigma^{-1}(x))\ \text{ for all }\sigma\in G,\ u\in H_g^1(M),\ x\in M, \tag{6.1.8}$$

where $\sigma^{-1}:M\to M$ is the inverse of the isometry σ. Let

$$H_{g,G}^1(M)=\{u\in H_g^1(M):\sigma u=u\text{ for all }\sigma\in G\}$$

be the subspace of $G-$invariant functions of $H_g^1(M)$ and $\mathcal{E}_{\lambda,G}:H_{g,G}^1(M)\to\mathbb{R}$ be the restriction of the energy functional $\mathcal{E}_\lambda$ to $H_{g,G}^1(M)$. The following statement is crucial in our investigation:

STEP 3. *If $u_G\in H_{g,G}^1(M)$ is a critical point of $\mathcal{E}_{\lambda,G}$, then it is a critical point also for $\mathcal{E}_\lambda$ and ϕ_{u_G} is $G-$invariant.*

Proof of Step 3. For the first part of the proof, we follow Kristály [79, Lemma 4.1]. Due to relation (6.1.8), the group G acts continuously on $H_g^1(M)$.

We claim that $\mathcal{E}_\lambda$ is $G-$invariant. To prove this, let $u\in H_g^1(M)$ and $\sigma\in G$ be fixed. Since $\sigma:M\to M$ is an isometry on M, we have by (6.1.8) and the chain rule that

$$\nabla_g(\sigma u)(x)=D\sigma_{\sigma^{-1}(x)}\nabla_g u(\sigma^{-1}(x))$$

for every $x\in M$, where $D\sigma_{\sigma^{-1}(x)}:T_{\sigma^{-1}(x)}M\to T_xM$ denotes the differential of σ at the point $\sigma^{-1}(x)$. The (signed) Jacobian determinant of σ is 1 and $D\sigma_{\sigma^{-1}(x)}$ preserves inner products; thus, by relation (6.1.8) and a change of variables $y=\sigma^{-1}(x)$ it turns out that

$$\begin{aligned}\|\sigma u\|^2_{H_g^1(M)} &= \int_M\left(|\nabla_g(\sigma u)(x)|_x^2+|(\sigma u)(x)|^2\right)\mathrm{d}v_g(x)\\ &= \int_M\left(|\nabla_g u(\sigma^{-1}(x))|^2_{\sigma^{-1}(x)}+|u(\sigma^{-1}(x))|^2\right)\mathrm{d}v_g(x)\\ &= \int_M\left(|\nabla_g u(y)|_y^2+|u(y)|^2\right)\mathrm{d}v_g(y)\\ &= \|u\|^2_{H_g^1(M)}.\end{aligned}$$

According to $\boldsymbol{(\alpha^{x_0})}$, one has that $\alpha(x)=\alpha_0(d_g(x_0,x))$ for some function $\alpha_0:[0,\infty)\to\mathbb{R}$. Since $\mathrm{Fix}_M(G)=\{x_0\}$, we have for every $\sigma\in G$ and $x\in M$ that

$$\alpha(\sigma(x))=\alpha_0(d_g(x_0,\sigma(x)))=\alpha_0(d_g(\sigma(x_0),\sigma(x)))=\alpha_0(d_g(x_0,x))=\alpha(x).$$

Therefore,

$$\begin{aligned}\mathcal{F}(\sigma u) &= \int_M\alpha(x)F((\sigma u)(x))\mathrm{d}v_g(x)=\int_M\alpha(x)F(u(\sigma^{-1}(x)))\mathrm{d}v_g(x)=\int_M\alpha(y)F(u(y))\mathrm{d}v_g(y)\\ &= \mathcal{F}(u).\end{aligned}$$

We now consider the Maxwell equation

$$-\Delta_g\phi_{\sigma u}+\phi_{\sigma u}=q(\sigma u)^2$$

which reads pointwisely as

$$-\Delta_g\phi_{\sigma u}(y)+\phi_{\sigma u}(y)=qu(\sigma^{-1}(y))^2,\ y\in M.$$

After a change of variables one has

$$-\Delta_g \phi_{\sigma u}(\sigma(x)) + \phi_{\sigma u}(\sigma(x)) = qu(x)^2, \ x \in M,$$

which means by the uniqueness that $\phi_{\sigma u}(\sigma(x)) = \phi_u(x)$. Therefore,

$$\int_M \phi_{\sigma u}(x)(\sigma u)^2(x)\mathrm{d}v_g(x) = \int_M \phi_u(\sigma^{-1}(x))u^2(\sigma^{-1}(x))\mathrm{d}v_g(x) \overset{x=\sigma(y)}{=} \int_M \phi_u(y)u^2(y)\mathrm{d}v_g(y),$$

which proves the $G-$invariance of $u \mapsto \int_M \phi_u u^2 \mathrm{d}v_g$, thus the claim.

Since the fixed point set of $H_g^1(M)$ for G is precisely $H_{g,G}^1(M)$, the principle of symmetric criticality of Palais [97] shows that every critical point $u_G \in H_{g,G}^1(M)$ of the functional $\mathcal{E}_{\lambda,G}$ is also a critical point of $\mathcal{E}_\lambda$. Moreover, from the above uniqueness argument, for every $\sigma \in G$ and $x \in M$ we have $\phi_{u_G}(\sigma x) = \phi_{\sigma u_G}(\sigma x) = \phi_{u_G}(x)$, i.e., ϕ_{u_G} is $G-$invariant. □

Summing up Steps 1-3, we have the following implications: for an element $u \in H_{g,G}^1(M)$,

$$\mathcal{E}_{\lambda,G}'(u) = 0 \ \Rightarrow \ \mathcal{E}_\lambda'(u) = 0 \ \Leftrightarrow \ \mathscr{J}_\lambda'(u, \phi_u) = 0 \ \Leftrightarrow (u, \phi_u) \text{ is a weak solution of } (\mathcal{SM}_\lambda). \quad (6.1.9)$$

Consequently, in order to guarantee $G-$invariant weak solutions for $(\mathcal{SM}_\lambda)$, it is enough to produce critical points for the energy functional $\mathcal{E}_{\lambda,G} : H_{g,G}^1(M) \to \mathbb{R}$. While the embedding $H_g^1(M) \hookrightarrow L^p(M)$ is only continuous for every $p \in [2, 2^*]$, we adapt the main results from Skrzypczak and Tintarev [114] in order to regain some compactness by exploring the presence of group symmetries:

Proposition 6.1.1. [114, Theorem 1.3 & Proposition 3.1] *Let (M, g) be an $n-$dimensional homogeneous Hadamard manifold and G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G)$ is a singleton. Then $H_{g,G}^1(M)$ is compactly embedded into $L^p(M)$ for every $p \in (2, 2^*)$.*

6.2. Proof of main results

6.2.1. Schrödinger-Maxwell systems of Poisson type

Consider the operator $\mathscr{L}$ on $H_g^1(M)$ given by

$$\mathscr{L}(u) = -\Delta_g u + u + e\phi_u u.$$

The following comparison principle can be stated:

Lemma 6.2.1. *Let (M, g) be an $n-$dimensional Cartan-Hadamard manifold* $(3 \leq n \leq 6)$, *$u, v \in H_g^1(M)$.*

(i) *If $\mathscr{L}(u) \leq \mathscr{L}(v)$ then $u \leq v$.*

(ii) *If $0 \leq u \leq v$ then $\phi_u \leq \phi_v$.*

Proof. (i) Assume that $A = \{x \in M : u(x) > v(x)\}$ has a positive Riemannian measure. Then multiplying $\mathscr{L}(u) \leq \mathscr{L}(v)$ by $(u - v)_+$, an integration yields that

$$\int_A |\nabla_g u - \nabla_g v|^2 \mathrm{d}v_g + \int_A (u - v)^2 \mathrm{d}v_g + e\int_A (u\phi_u - v\phi_v)(u - v)\mathrm{d}v_g \leq 0.$$

The latter inequality and relation (6.1.5) produce a contradiction.

(ii) Assume that $B = \{x \in M : \phi_u(x) > \phi_v(x)\}$ has a positive Riemannian measure. Multiplying the Maxwell-type equation $-\Delta_g(\phi_u - \phi_v) + \phi_u - \phi_v = q(u^2 - v^2)$ by $(\phi_u - \phi_v)_+$, we obtain that

$$\int_B |\nabla_g \phi_u - \nabla_g \phi_v|^2 \mathrm{d}v_g + \int_B (\phi_u - \phi_v)^2 \mathrm{d}v_g = q\int_B (u^2 - v^2)(\phi_u - \phi_v)\mathrm{d}v_g \leq 0,$$

a contradiction. $\square$

Proof of Theorem 6.1.1. Let $\lambda = 1$ and for simplicity, let $\mathcal{E} = \mathcal{E}_1$ be the energy functional from (6.1.6). First of all, the function $u \mapsto \frac{1}{2}\|u\|^2_{H^1_g(M)}$ is strictly convex on $H^1_g(M)$. Moreover, the linearity of $u \mapsto \mathcal{F}(u) = \int_M \alpha(x)u(x)\mathrm{d}v_g(x)$ and property (6.1.4) imply that the energy functional $\mathcal{E}$ is strictly convex on $H^1_g(M)$. Thus $\mathcal{E}$ is sequentially weakly lower semicontinuous on $H^1_g(M)$, it is bounded from below and coercive. Now the basic result of the calculus of variations implies that $\mathcal{E}$ has a unique (global) minimum point $u \in H^1_g(M)$, see Zeidler [128, Theorem 38.C and Proposition 38.15], which is also the unique critical point of $\mathcal{E}$, thus (u, ϕ_u) is the unique weak solution of $(\mathcal{SM})$. Since $\alpha \geq 0$, Lemma 6.2.1 (i) implies that $u \geq 0$.

Assume the function α satisfies $(\boldsymbol{\alpha^{x_0}})$ for some $x_0 \in M$ and let $G \subset \mathrm{Isom}_g(M)$ be such that $(\boldsymbol{H^{x_0}_G})$ holds. Then we can repeat the above arguments for $\mathcal{E}_{1,G} = \mathcal{E}|_{H^1_{g,G}(M)}$ and $H^1_{g,G}(M)$ instead of $\mathcal{E}$ and $H^1_g(M)$, respectively, obtaining by (6.1.9) that (u, ϕ_u) is a $G-$invariant weak solution for $(\mathcal{SM})$. $\square$

In the sequel we focus our attention to the system $(\mathscr{R})$ from §5.1; namely, we have

Lemma 6.2.2. *System $(\mathscr{R})$ has a unique, non-negative solution pair belonging to $C^\infty(0,\infty) \times C^\infty(0,\infty)$.*

Proof. Let $c \leq 0$ and $\alpha_0 \in L^2([0,\infty), \mathbf{s}_c(r)^{n-1}\mathrm{d}r)$. Let us consider the Riemannian space form (M_c, g_c) with constant sectional curvature $c \leq 0$, i.e., (M_c, g_c) is either the Euclidean space $(\mathbb{R}^n, g_{\text{euc}})$ when $c = 0$, or the hyperbolic space $(\mathbb{H}^n, g_{\text{hyp}})$ with (scaled) sectional curvature $c < 0$. Let $x_0 \in M$ be fixed and $\alpha(x) = \alpha_0(d_{g_c}(x_0, x))$, $x \in M$. Due to the integrability assumption on α_0, we have that $\alpha \in L^2(M)$. Therefore, we are in the position to apply Theorem 6.1.1 on (M_c, g_c) (see examples from Remark 6.1.1) to the problem

$$\begin{cases} -\Delta_g u + u + eu\phi = \alpha(x) & \text{in} \quad M_c, \\ -\Delta_g \phi + \phi = qu^2 & \text{in} \quad M_c, \end{cases} \tag{$\mathcal{SM}_c$}$$

concluding that it has a unique, non-negative weak solution $(u_0, \phi_{u_0}) \in H^1_{g_c}(M_c) \times H^1_{g_c}(M_c)$, where u_0 is the unique global minimum point of the "one-variable" energy functional associated with problem $(\mathcal{S}M_c)$. Since α is radially symmetric in M_c, we may consider the group $G = SO(n)$ in the second part of Theorem 6.1.1 in order to prove that (u_0, ϕ_{u_0}) is $SO(n)-$invariant, i.e., radially symmetric. In particular, we can represent these functions as $u_0(x) = h^c_1(d_{g_c}(x_0, x))$ and $\phi_0(x) = h^c_2(d_{g_c}(x_0, x))$ for some $h^c_i : [0,\infty) \to [0,\infty)$, $i = 1, 2$. By using formula (1.3.3) and the Laplace comparison theorem for $\mathbf{K} = c$ it follows that the equations from $(\mathcal{S}M_c)$ are transformed into the first two equations of $(\mathscr{R})$ while the second two relations in $(\mathscr{R})$ are nothing but the "radial" integrability conditions inherited from the fact that $(u_0, \phi_{u_0}) \in H^1_{g_c}(M_c) \times H^1_{g_c}(M_c)$. Thus, it turns out that problem $(\mathscr{R})$ has a non-negative pair of solutions (h^c_1, h^c_2). Standard regularity results show that $(h^c_1, h^c_2) \in C^\infty(0,\infty) \times C^\infty(0,\infty)$. Finally, let us assume that $(\mathscr{R})$ has another non-negative pair of solutions $(\tilde{h}^c_1, \tilde{h}^c_2)$, distinct from (h^c_1, h^c_2). Let

$$\tilde{u}_0(x) = \tilde{h}^c_1(d_{g_c}(x_0, x))$$

and

$$\tilde{\phi}_0(x) = \tilde{h}^c_2(d_{g_c}(x_0, x)).$$

There are two cases:

(a) if $h^c_1 = \tilde{h}^c_1$ then $u_0 = \tilde{u}_0$ and by the uniqueness of solution for the Maxwell equation it follows that $\phi_0 = \tilde{\phi}_0$, i.e., $h^c_2 = \tilde{h}^c_2$, a contradiction;

(b) if $h_1^c \neq \tilde{h}_1^c$ then $u_0 \neq \tilde{u}_0$. But the latter relation is absurd since both elements u_0 and $\tilde{u}_0$ appear as unique global minima of the "one-variable" energy functional associated with $(\mathcal{SM}_c)$.

□

Proof of Theorem 6.1.2. "(ii)⇒(i)": it follows directly from Lemma 6.2.2.

"(i)⇒(ii)": Let $x_0 \in M$ be fixed and assume that the pair $(h_1^c(d_g(x_0,\cdot)), h_2^c(d_g(x_0,\cdot)))$ is the unique pointwise solution to $(\mathcal{SM})$, i.e.,

$$\begin{cases} -\Delta_g h_1^c(d_g(x_0,x)) + h_1^c(d_g(x_0,x)) + eh_1^c(d_g(x_0,x))h_2^c(d_g(x_0,x)) = \alpha(d_g(x_0,x)), \ x \in M, \\ -\Delta_g h_2^c(d_g(x_0,x)) + h_2^c(d_g(x_0,x)) = qh_1^c(d_g(x_0,x))^2, \ x \in M. \end{cases}$$

By applying formula (1.3.3) to the second equation, we arrive to

$$-h_2^c(d_g(x_0,x))'' - h_2^c(d_g(x_0,x))'\Delta_g(d_g(x_0,x)) + h_2^c(d_g(x_0,x)) = qh_1^c(d_g(x_0,x))^2, \ x \in M.$$

Subtracting the second equation of the system $(\mathscr{R})$ from the above one, we have that

$$h_2^c(d_g(x_0,x))'[\Delta_g(d_g(x_0,x)) - (n-1)\mathbf{ct}_c(d_g(x_0,x))] = 0, \ x \in M. \tag{6.2.1}$$

Let us suppose that there exists a set $A \subset M$ of non-zero Riemannian measure such that $h_2^c(d_g(x_0,x))' = 0$ for every $x \in A$. By a continuity reason, there exists a non-degenerate interval $I \subset \mathbb{R}$ and a constant $c_0 \geq 0$ such that $h_2^c(t) = c_0$ for every $t \in I$. Coming back to the system $(\mathscr{R})$, we observe that

$$h_1^c(t) = \sqrt{\frac{c_0}{q}}$$

and

$$\alpha_0(t) = \sqrt{\frac{c_0}{q}}(1 + ec_0)$$

for every $t \in I$. Therefore,

$$\alpha(x) = \alpha_0(d_g(x_0,x)) = \sqrt{\frac{c_0}{q}}(1 + ec_0)$$

for every $x \in A$, which contradicts the assumption on α.

Consequently, by (6.2.1) we have $\Delta_g d_g(x_0,x) = (n-1)\mathbf{ct}_c(d_g(x_0,x))$ pointwisely on M. The latter relation can be equivalently transformed into

$$\Delta_g w_c(d_g(x_0,x)) = 1, \ x \in M,$$

where

$$w_c(r) = \int_0^r \mathbf{s}_c(s)^{-n+1} \int_0^s \mathbf{s}_c(t)^{n-1} \mathrm{d}t\mathrm{d}s. \tag{6.2.2}$$

Let $0 < \tau$ be fixed arbitrarily. The unit outward normal vector to the forward geodesic sphere $S_g(x_0,\tau) = \partial B_g(x_0,\tau) = \{x \in M : d_g(x_0,x) = \tau\}$ at $x \in S_g(x_0,\tau)$ is given by $\mathbf{n} = \nabla_g d_g(x_0,x)$. Let us denote by $\mathrm{d}\varsigma_g(x)$ the canonical volume form on $S_g(x_0,\tau)$ induced by $\mathrm{d}v_g(x)$. By Stoke's formula and $\langle \mathbf{n}, \mathbf{n} \rangle = 1$ we have that

$$\begin{aligned} \mathrm{Vol}_g(B_g(x_0,\tau)) &= \int_{B_g(x_0,\tau)} \Delta_g(w_c(d_g(x_0,x)))\mathrm{d}v_g = \int_{B_g(x_0,\tau)} \mathrm{div}(\nabla_g(w_c(d_g(x_0,x))))\mathrm{d}v_g \\ &= \int_{S_g(x_0,\tau)} \langle \mathbf{n}, w_c'(d_g(x_0,x))\nabla_g d_g(x_0,x) \rangle \mathrm{d}v_g \\ &= w_c'(\tau)\mathrm{Area}_g(S_g(x_0,\tau)). \end{aligned}$$

Therefore,

$$\frac{\mathrm{Area}_g(S_g(x_0,\tau))}{\mathrm{Vol}_g(B_g(x_0,\tau))}=\frac{1}{w_c'(\tau)}=\frac{\mathbf{s}_c(\tau)^{n-1}}{\displaystyle\int_0^\tau \mathbf{s}_c(t)^{n-1}\mathrm{d}t}.$$

Integrating the latter expression, it follows that

$$\frac{\mathrm{Vol}_g(B_g(x_0,\tau))}{V_{c,n}(\tau)}=\lim_{s\to 0^+}\frac{\mathrm{Vol}_g(B_g(x_0,s))}{V_{c,n}(s)}=1. \tag{6.2.3}$$

In fact, the Bishop-Gromov volume comparison theorem implies that

$$\frac{\mathrm{Vol}_g(B_g(x,\tau))}{V_{c,n}(\tau)}=1 \text{ for all } x\in M,\ \tau>0.$$

Now, the above equality implies that the sectional curvature is constant, $\mathbf{K}=c$, which concludes the proof. $\square$

6.2.2. Schrödinger-Maxwell systems involving oscillatory nonlinearities

Before proving Theorem 6.1.3, we need an auxiliary result. Let us consider the system

$$\begin{cases} -\Delta_g u+u+eu\phi=\alpha(x)\widetilde{f}(u) & \text{in} \quad M,\\ -\Delta_g\phi+\phi=qu^2 & \text{in} \quad M,\end{cases} \tag{$\widetilde{\mathcal{SM}}$}$$

where the following assumptions hold:

($\tilde{f}_1$) $\widetilde{f}:[0,\infty)\to\mathbb{R}$ is a bounded function such that $f(0)=0$;

($\tilde{f}_2$) there are $0<a\le b$ such that $\widetilde{f}(s)\le 0$ for all $s\in[a,b]$.

Let $x_0\in M$ be fixed, and $G\subset \mathrm{Isom}_g(M)$ and $\alpha\in L^1(M)\cap L^\infty(M)$ be such that hypotheses $(\boldsymbol{H_G^{x_0}})$ and $(\boldsymbol{\alpha^{x_0}})$ are satisfied.

Let $\widetilde{\mathcal{E}}$ be the "one-variable" energy functional associated with system $(\widetilde{\mathcal{SM}})$, and $\widetilde{\mathcal{E}_G}$ be the restriction of $\widetilde{\mathcal{E}}$ to the set $H^1_{g,G}(M)$. It is clear that $\widetilde{\mathcal{E}}$ is well defined. Consider the number $b\in\mathbb{R}$ from ($\tilde{f}_2$); for further use, we introduce the sets

$$W^b=\{u\in H^1_g(M):\|u\|_{L^\infty(M)}\le b\}\quad\text{and}\quad W^b_G=W^b\cap H^1_{g,G}(M).$$

Proposition 6.2.1. *Let (M,g) be an $n-$dimensional homogeneous Cartan-Hadamard manifold $(3\le n\le 5)$, $x_0\in M$ be fixed, and $G\subset\mathrm{Isom}_g(M)$ and $\alpha\in L^1(M)\cap L^\infty(M)$ be such that hypotheses $(\boldsymbol{H_G^{x_0}})$ and $(\boldsymbol{\alpha^{x_0}})$ are satisfied. If $\widetilde{f}:[0,\infty)\to\mathbb{R}$ is a continuous function satisfying ($\tilde{f}_1$) and ($\tilde{f}_2$) then*

(i) *the infimum of $\widetilde{\mathcal{E}_G}$ on W^b_G is attained at an element $u_G\in W^b_G$;*

(ii) *$u_G(x)\in[0,a]$ a.e. $x\in M$;*

(iii) *(u_G,ϕ_{u_G}) is a weak solution to system $(\widetilde{\mathcal{SM}})$.*

Proof. (i) One can see easily that the functional $\widetilde{\mathcal{E}_G}$ is sequentially weakly lower semicontinuous on $H^1_{g,G}(M)$. Moreover, $\widetilde{\mathcal{E}_G}$ is bounded from below. The set W^b_G is convex and closed in $H^1_{g,G}(M)$, thus weakly closed. Therefore, the claim directly follows; let $u_G\in W^b_G$ be the infimum of $\widetilde{\mathcal{E}_G}$ on W^b_G.

(ii) Let $A=\{x\in M: u_G(x)\notin[0,a]\}$ and suppose that the Riemannian measure of A is positive. We consider the function $\gamma(s)=\min(s_+,a)$ and set $w=\gamma\circ u_G$. Since γ is Lipschitz continuous, then $w \in H_g^1(M)$ (see Hebey, [66, Proposition 2.5, page 24]). We claim that $w \in H_{g,G}^1(M)$. Indeed, for every $x \in M$ and $\sigma \in G$,

$$\sigma w(x) = w(\sigma^{-1}(x)) = (\gamma \circ u_G)(\sigma^{-1}(x)) = \gamma(u_G(\sigma^{-1}(x))) = \gamma(u_G(x)) = w(x).$$

By construction, we clearly have that $w \in W_G^b$. Let

$$A_1 = \{x \in A : u_G(x) < 0\} \text{ and } A_2 = \{x \in A : u_G(x) > a\}.$$

Thus $A = A_1 \cup A_2$, and from the construction we have that $w(x) = u_G(x)$ for all $x \in M \setminus A$, $w(x) = 0$ for all $x \in A_1$, and $w(x) = a$ for all $x \in A_2$. Now we have that

$$\begin{aligned}\widetilde{\mathcal{E}_G}(w) - \widetilde{\mathcal{E}_G}(u_G) = &- \frac{1}{2}\int_A |\nabla_g u_G|^2 dv_g + \frac{1}{2}\int_A (w^2 - u_G^2)\mathrm{d}v_g + \frac{e}{4}\int_A (\phi_w w^2 - \phi_{u_G} u_G^2)\ \mathrm{d}v_g \\ &- \int_A \alpha(x)\left(\widetilde{F}(w) - \widetilde{F}(u_G)\right)\ \mathrm{d}v_g.\end{aligned}$$

Note that

$$\int_A \left(w^2 - u_G^2\right) \mathrm{d}v_g = -\int_{A_1} u_G^2 \mathrm{d}v_g + \int_{A_2} \left(a^2 - u_G^2\right) \mathrm{d}v_g \le 0.$$

It is also clear that $\int_{A_1} \alpha(x)(\widetilde{F}(w) - \widetilde{F}(u_G))\mathrm{d}v_g = 0$, and due to the mean value theorem and $(\tilde{f}_2)$ we have that $\int_{A_2} \alpha(x)(\widetilde{F}(w) - \widetilde{F}(u_G))\mathrm{d}v_g \ge 0$. Furthermore,

$$\int_A (\phi_w w^2 - \phi_{u_G} u_G^2)\mathrm{d}v_g = -\int_{A_1} \phi_{u_G} u_G^2 \mathrm{d}v_g + \int_{A_2} (\phi_w w^2 - \phi_{u_G} u_G^2)\mathrm{d}v_g,$$

thus due to Lemma 6.2.1 (ii), since $0 \le w \le u_G$, we have that $\int_{A_2} (\phi_w w^2 - \phi_{u_G} u_G^2)\mathrm{d}v_g \le 0$. Combining the above estimates, we have $\widetilde{\mathcal{E}_G}(w) - \widetilde{\mathcal{E}_G}(u_G) \le 0$.

On the other hand, since $w \in W_G^b$ then $\widetilde{\mathcal{E}_G}(w) \ge \widetilde{\mathcal{E}_G}(u_G) = \inf_{W_G^b} \widetilde{\mathcal{E}_G}$, thus we necessarily have that

$$\int_{A_1} u_G^2 \mathrm{d}v_g = \int_{A_2} (a^2 - u_G^2)\mathrm{d}v_g = 0,$$

which implies that the Riemannian measure of A should be zero, a contradiction.

(iii) The proof is divided into two steps:

Claim 1. $\widetilde{\mathcal{E}}'(u_G)(w - u_G) \ge 0$ *for all* $w \in W^b$. It is clear that the set W^b is closed and convex in $H_g^1(M)$. Let χ_{W^b} be the indicator function of the set W^b, i.e., $\chi_{W^b}(u) = 0$ if $u \in W^b$, and $\chi_{W^b}(u) = +\infty$ otherwise. Let us consider the Szulkin-type functional $\mathscr{K} : H_g^1(M) \to \mathbb{R} \cup \{+\infty\}$ given by $\mathscr{K} = \widetilde{\mathcal{E}} + \chi_{W^b}$. On account of the definition of the set W_G^b, the restriction of χ_{W^b} to $H_{g,G}^1(M)$ is precisely the indicator function $\chi_{W_G^b}$ of the set W_G^b. By (i), since u_G is a local minimum point of $\widetilde{\mathcal{E}_G}$ relative to the set W_G^b, it is also a local minimum point of the Szulkin-type functional $\mathscr{K}_G = \widetilde{\mathcal{E}_G} + \chi_{W_G^b}$ on $H_{g,G}^1(M)$. In particular, u_G is a critical point of $\mathscr{K}_G$ in the sense of Szulkin [87, 118], i.e.,

$$0 \in \widetilde{\mathcal{E}_G}'(u_G) + \partial\chi_{W_G^b}(u_G) \text{ in } \left(H_{g,G}^1(M)\right)^\star,$$

where ∂ stands for the subdifferential in the sense of convex analysis. By exploring the compactness of the group G, we may apply the principle of symmetric criticality for Szulkin-type functionals, see Kobayashi and Ôtani [70, Theorem 3.16] or Theorem (1.2.8), obtaining that

$$0 \in \widetilde{\mathcal{E}}'(u_G) + \partial\chi_{W^b}(u_G) \text{ in } \left(H_g^1(M)\right)^\star.$$

Consequently, we have for every $w \in W^b$ that

$$0 \leq \widetilde{\mathcal{E}}'(u_G)(w - u_G) + \chi_{W^b}(w) - \chi_{W^b}(u_G),$$

which proves the claim.

Claim 2. *(u_G, ϕ_{u_G}) is a weak solution to the system $(\widetilde{\mathcal{SM}})$.* By assumption $(\tilde{f}_1)$ it is clear that $C_{\mathtt{m}} = \sup_{s\in\mathbb{R}} |\widetilde{f}(s)| < \infty$. The previous step and (6.1.7) imply that for all $w \in W^b$,

$$\begin{aligned} 0 \quad \leq \quad & \int_M \langle \nabla_g u_G, \nabla_g (w - u_G)\rangle \mathrm{d}v_g + \int_M u_G(w-u_G)\mathrm{d}v_g \\ & + e\int_M u_G\phi_{u_G}(w-u_G)\mathrm{d}v_g - \int_M \alpha(x)\widetilde{f}(u_G)(w-u_G)\mathrm{d}v_g. \end{aligned}$$

Let us define the following function

$$\zeta(s) = \begin{cases} -b, & s < -b, \\ s, & -b \leq s < b, \\ b, & b \leq s. \end{cases}$$

Since ζ is Lipschitz continuous and $\zeta(0) = 0$, then for fixed $\varepsilon > 0$ and $v \in H_g^1(M)$ the function $w_\zeta = \zeta \circ (u_G + \varepsilon v)$ belongs to $H_g^1(M)$, see Hebey [66, Proposition 2.5, page 24]. By construction, $w_\zeta \in W^b$.

Let us denote by $B_1 = \{x \in M : u_G + \varepsilon v < -b\}$, $B_2 = \{x \in M : -b \leq u_G + \varepsilon v < b\}$ and $B_3 = \{x \in M : u_G + \varepsilon v \geq b\}$. Choosing $w = w_\zeta$ in the above inequality we have that

$$0 \leq I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= -\int_{B_1} |\nabla_g u_G|^2 \mathrm{d}v_g + \varepsilon\int_{B_2}\langle \nabla_g u_G, \nabla_g v\rangle \mathrm{d}v_g - \int_{B_3} |\nabla_g u_G|^2\mathrm{d}v_g, \\ I_2 &= -\int_{B_1} u_G(b+u_G)\mathrm{d}v_g + \varepsilon\int_{B_2} u_G v\mathrm{d}v_g + \int_{B_3}(b-u_G)\mathrm{d}v_g, \\ I_3 &= -e\int_{B_1} u_G\phi_{u_G}(b+u_G)\mathrm{d}v_g + \varepsilon e\int_{B_2} u_G\phi_{u_G}v\mathrm{d}v_g + e\int_{B_3} u_G\phi_{u_G}(b-u_G)\mathrm{d}v_g, \end{aligned}$$

and

$$I_4 = -\int_{B_1} \alpha(x)\widetilde{f}(u_G)(-b-u_G)\mathrm{d}v_g - \varepsilon\int_{B_2}\alpha(x)\widetilde{f}(u_G)v\mathrm{d}v_g - \int_{B_3}\alpha(x)\widetilde{f}(u_G)(b-u_G)\mathrm{d}v_g.$$

After a rearrangement we obtain that

$$\begin{aligned} I_1 + I_2 + I_3 + I_4 =& \varepsilon\int_M \langle \nabla_g u_G, \nabla_g v\rangle \mathrm{d}v_g + \varepsilon\int_M u_G v\mathrm{d}v_g + \varepsilon e\int_M u_G\phi_{u_G}v\mathrm{d}v_g - \varepsilon\int_M \alpha(x)f(u_G)v\mathrm{d}v_g \\ & - \varepsilon\int_{B_1}\langle \nabla_g u_G, \nabla_g v\rangle\mathrm{d}v_g - \varepsilon\int_{B_3}\langle \nabla_g u_G, \nabla_g v\rangle \mathrm{d}v_g - \int_{B_1}|\nabla_g u_G|^2\mathrm{d}v_g \\ & - \int_{B_3}|\nabla_g u_G|^2\mathrm{d}v_g + \int_{B_1}(b+u_G+\varepsilon v)\left(\alpha(x)\widetilde{f}(u_G) - u_G - eu_G\phi_{u_G}\right)\mathrm{d}v_g \\ & + \int_{B_3}(-b+u_G+\varepsilon v)\left(\alpha(x)\widetilde{f}(u_G) - u_G - eu_G\phi_{u_G}\right)\mathrm{d}v_g. \end{aligned}$$

Note that

$$\int_{B_1}(b+u_G+\varepsilon v)\left(\alpha(x)\widetilde{f}(u_G) - u_G - eu_G\phi_{u_G}\right)\mathrm{d}v_g \leq -\varepsilon\int_{B_1}\left(C_{\mathtt{m}}\alpha(x) + u_G + eu_G\phi_{u_G}\right)v\mathrm{d}v_g,$$

and

$$\int_{B_3}(-b+u_G+\varepsilon v)\left(\alpha(x)\widetilde{f}(u_G)-u_G-eu_G\phi_{u_G}\right)\mathrm{d}v_g\le \varepsilon C_{\mathtt{m}}\int_{B_3}\alpha(x)v\mathrm{d}v_g.$$

Now, using the above estimates and dividing by $\varepsilon>0$, we have that

$$\begin{aligned}0\le&\int_M\langle\nabla_g u_G,\nabla_g v\rangle\mathrm{d}v_g+\int_M u_G v\mathrm{d}v_g+e\int_M u_G\phi_{u_G}v\mathrm{d}v_g-\int_M\alpha(x)\widetilde{f}(u_G)v\mathrm{d}v_g\\&-\int_{B_1}\left(\langle\nabla_g u_G,\nabla_g v\rangle+C_{\mathtt{m}}\alpha(x)v+u_G v+eu_G\phi_{u_G}v\right)\mathrm{d}v_g-\int_{B_3}\left(\langle\nabla_g u_G,\nabla_g v\rangle-C_{\mathtt{m}}\alpha(x)v\right)\mathrm{d}v_g.\end{aligned}$$

Taking into account that the Riemannian measures for both sets B_1 and B_3 tend to zero as $\varepsilon\to 0$, we get that

$$0\le\int_M\langle\nabla_g u_G,\nabla_g v\rangle\mathrm{d}v_g+\int_M u_G v\mathrm{d}v_g+e\int_M u_G\phi_{u_G}v\mathrm{d}v_g-\int_M\alpha(x)\widetilde{f}(u_G)v\mathrm{d}v_g.$$

Replacing v by $(-v)$, it yields

$$0=\int_M\langle\nabla_g u_G,\nabla_g v\rangle\mathrm{d}v_g+\int_M u_G v\mathrm{d}v_g+e\int_M u_G\phi_{u_G}v\mathrm{d}v_g-\int_M\alpha(x)\widetilde{f}(u_G)v\mathrm{d}v_g,$$

i.e., $\widetilde{\mathcal{E}}'(u_G)=0$. Thus (u_G,ϕ_{u_G}) is a $G-$invariant weak solution to $(\widetilde{\mathcal{SM}})$. □

Let $s>0$, $0<r<\rho$ and $A_{x_0}[r,\rho]=B_g(x_0,\rho+r)\setminus B_g(x_0,\rho-r)$ be an annulus-type domain. For further use, we define the function $w_s:M\to\mathbb{R}$ by

$$w_s(x)=\begin{cases}0, & x\in M\setminus A_{x_0}[r,\rho],\\ s, & x\in A_{x_0}[r/2,\rho],\\ \frac{2s}{r}(r-|d_g(x_0,x)-\rho|), & x\in A_{x_0}[r,\rho]\setminus A_{x_0}[r/2,\rho].\end{cases}$$

Note that $(\boldsymbol{H_G^{x_0}})$ implies $w_s\in H^1_{g,G}(M)$.

Proof of Theorem 6.1.3. Due to (f_0^2) and the continuity of f one can fix two sequences $\{\theta_j\}_j,\{\eta_j\}_j$ such that $\lim\limits_{j\to+\infty}\theta_j=\lim\limits_{j\to+\infty}\eta_j=0$, and for every $j\in\mathbb{N}$,

$$0<\theta_{j+1}<\eta_j<s_j<\theta_j<1; \tag{6.2.4}$$

$$f(s)\le 0 \text{ for every } s\in[\eta_j,\theta_j]. \tag{6.2.5}$$

Let us introduce the auxiliary function $f_j(s)=f(\min(s,\theta_j))$. Since $f(0)=0$ (by (f_0^1) and (f_0^2)), then $f_j(0)=0$ and we may extend continuously the function f_j to the whole real line by $f_j(s)=0$ if $s\le 0$. For every $s\in\mathbb{R}$ and $j\in\mathbb{N}$, we define $F_j(s)=\displaystyle\int_0^s f_j(t)\mathrm{d}t$. It is clear that f_j satisfies the assumptions $(\tilde{f}_1)$ and $(\tilde{f}_2)$. Thus, applying Proposition 6.2.1 to the function f_j, $j\in\mathbb{N}$, the system

$$\begin{cases}-\Delta_g u+u+eu\phi=\alpha(x)f_j(u) & \text{in } M,\\ -\Delta_g\phi+\phi=qu^2 & \text{in } M,\end{cases} \tag{6.2.6}$$

has a $G-$invariant weak solution $(u_j^0,\phi_{u_j^0})\in H^1_{g,G}(M)\times H^1_{g,G}(M)$ such that

$$u_j^0\in[0,\eta_j] \text{ a.e. } x\in M; \tag{6.2.7}$$

$$u_j^0 \text{ is the infimum of the functional } \mathcal{E}_j \text{ on the set } W_G^{\theta_j}, \tag{6.2.8}$$

where

$$\mathcal{E}_j(u)=\frac{1}{2}\|u\|^2_{H_g^1(M)}+\frac{e}{4}\int_M\phi_u u^2\mathrm{d}v_g-\int_M\alpha(x)F_j(u)\mathrm{d}v_g.$$

By (6.2.7), $(u_j^0, \phi_{u_j^0}) \in H^1_{g,G}(M) \times H^1_{g,G}(M)$ is also a weak solution to the initial system $(\mathcal{SM})$.

It remains to prove the existence of infinitely many distinct elements in the sequence $\{(u_j^0, \phi_{u_j^0})\}_j$. First, due to $(\boldsymbol{\alpha^{x_0}})$, there exist $0 < r < \rho$ such that $\text{essinf}_{A_{x_0}[r,\rho]}\alpha > 0$. For simplicity, let $D = A_{x_0}[r, \rho]$ and $K = A_{x_0}[r/2, \rho]$. By (f_0^1) there exist $l_0 > 0$ and $\delta \in (0, \theta_1)$ such that

$$F(s) \geq -l_0 s^2 \text{ for every } s \in (0, \delta). \tag{6.2.9}$$

Again, (f_0^1) implies the existence of a non-increasing sequence $\{\widetilde{s}_j\}_j \subset (0, \delta)$ such that $\widetilde{s}_j \leq \eta_j$ and

$$F(\widetilde{s}_j) > L_0 \widetilde{s}_j^2 \text{ for all } j \in \mathbb{N}, \tag{6.2.10}$$

where $L_0 > 0$ is enough large, e.g.,

$$L_0 \text{essinf}_K \alpha > \frac{1}{2}\left(1 + \frac{4}{r^2}\right) \text{Vol}_g(D) + \frac{e}{4}\|\phi_\delta\|_{L^1(D)} + l_0\|\alpha\|_{L^1(M)}. \tag{6.2.11}$$

Note that

$$\mathcal{E}_j(w_{\widetilde{s}_j}) = \frac{1}{2}\|w_{\widetilde{s}_j}\|^2_{H^1_g(M)} + \frac{e}{4}I_j - J_j,$$

where

$$I_j = \int_D \phi_{w_{\widetilde{s}_j}} w^2_{\widetilde{s}_j} \text{d}v_g \text{ and } J_j = \int_D \alpha(x) F_j(w_{\widetilde{s}_j}) \text{d}v_g.$$

By Lemma 6.2.1 (ii) we have

$$I_j \leq \widetilde{s}_j^2 \|\phi_\delta\|_{L^1(D)},\ j \in \mathbb{N}.$$

Moreover, by (6.2.9) and (6.2.10) we have that

$$J_j \geq L_0 \widetilde{s}_j^2 \text{essinf}_K \alpha - l_0 \widetilde{s}_j^2 \|\alpha\|_{L^1(M)},\ j \in \mathbb{N}.$$

Therefore,

$$\mathcal{E}_j(w_{\widetilde{s}_j}) \leq \widetilde{s}_j^2 \left(\frac{1}{2}\left(1 + \frac{4}{r^2}\right) \text{Vol}_g(D) + \frac{e}{4}\|\phi_\delta\|_{L^1(D)} + l_0\|\alpha\|_{L^1(M)} - L_0 \text{essinf}_K \alpha\right).$$

Thus, in one hand, by (6.2.11) we have

$$\mathcal{E}_j(u_j^0) = \inf_{W_G^{\theta_j}} \mathcal{E}_j \leq \mathcal{E}_j(w_{\widetilde{s}_j}) < 0. \tag{6.2.12}$$

On the other hand, by (6.2.4) and (6.2.7) we clearly have

$$\mathcal{E}_j(u_j^0) \geq -\int_M \alpha(x) F_j(u_j^0) \text{d}v_g = -\int_M \alpha(x) F(u_j^0) \text{d}v_g \geq -\|\alpha\|_{L^1(M)} \max_{s \in [0,1]} |f(s)|\eta_j,\ j \in \mathbb{N}.$$

Combining the latter relations, it yields that $\lim_{j \to +\infty} \mathcal{E}_j(u_j^0) = 0$. Since $\mathcal{E}_j(u_j^0) = \mathcal{E}_1(u_j^0)$ for all $j \in \mathbb{N}$, we obtain that the sequence $\{u_j^0\}_j$ contains infinitely many distinct elements. In particular, by (6.2.12) we have that $\frac{1}{2}\|u_j^0\|^2_{H^1_g(M)} \leq \|\alpha\|_{L^1(M)} \max_{s \in [0,1]} |f(s)|\eta_j$, which implies that $\lim_{j \to \infty} \|u_j^0\|_{H^1_g(M)} = 0$. Recalling (6.1.3), we also have $\lim_{j \to \infty} \|\phi_{u_j^0}\|_{H^1_g(M)} = 0$, which concludes the proof. $\square$

Remark 6.2.1. Using Proposition 6.2.1 (i) and $\lim_{j \to \infty} \eta_j = 0$, it follows that $\lim_{j \to \infty} \|u_j^0\|_{L^\infty(M)} = 0$.

6.3. Remarks

We point out that, there are other conditions (on the nonlinearity) which ensure infinitely many solutions for a quasi-linear problem, see [52]. Indeed, let us consider the following Dirichlet problem

$$\begin{cases} -\Delta_p u = h(x)f(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{$\mathscr{P}$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p > 1$, Δ_p is the p-Laplacian operator, i.e, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $h : \Omega \to \mathbb{R}$ is a bounded, non negative function. If $N \geq p$, $\mathcal{A}$ denotes the class of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{1+|t|^\gamma} < +\infty,$$

where $0 < \gamma < p^* - 1$ if $p < N$ (being $p^* = \frac{pN}{N-p}$) and $0 < \gamma < +\infty$ if $p = N$, while if $N < p$, $\mathcal{A}$ is the class of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Denote by F the primitive of f, i.e.

$$F(t) = \int_0^t f(s)ds.$$

Let also $0 \leq a < b \leq +\infty$. For a pair of functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$, if $\lambda \in [a, b]$, we denote by $M(\varphi, \psi, \lambda)$ the set of all global minima of the function $\lambda\psi - \varphi$ or the empty set according to whether $\lambda < +\infty$ or $\lambda = +\infty$. We adopt the conventions $\sup\emptyset = -\infty$, $\inf\emptyset = +\infty$. We also put

$$\alpha(\varphi, \psi, b) = \max\left\{\inf_{\mathbb{R}} \psi, \sup_{M(\varphi,\psi,b)} \psi\right\}$$

and

$$\beta(\varphi, \psi, a) = \min\left\{\sup_{\mathbb{R}} \psi, \inf_{M(\varphi,\psi,a)} \psi\right\}.$$

Furthermore, let $q \in]0, p^*]$ if $N > p$ or $q \in]0, +\infty[$ if $N \leq p$ and

$$c_q = \sup_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |u(x)|^q dx}{\left(\displaystyle\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{q}{p}}}.$$

Denote by $\mathcal{F}_q$ the family of all lower semicontinuous functions $\psi : \mathbb{R} \to \mathbb{R}$, with $\sup_{\mathbb{R}} \psi > 0$, such that

$$\inf_{t\in\mathbb{R}} \frac{\psi(t)}{1+|t|^q} > -\infty$$

and

$$\gamma_\psi := \sup_{t\in\mathbb{R}\setminus\{0\}} \frac{\psi(t)}{|t|^q} < +\infty.$$

Theorem 6.3.1. *Let $f \in \mathcal{A}$ and $h \in L^\infty(\Omega) \setminus \{0\}$, with $h \geq 0$. Assume that there exists $\psi \in \mathcal{F}_q$ such that, for each $\lambda \in]a, b[$, the function $\lambda\psi - F$ is coercive and has a unique global minimum in $\mathbb{R}$. Finally, suppose that*

$$\alpha(F, \psi, b) \leq 0 < \beta(F, \psi, a),$$

$$\liminf_{r\to 0^+} \frac{\sup_{\psi^{-1}(r)} F}{r^{\frac{p}{q}}} < \frac{1}{p(\gamma_\psi \operatorname{ess\,sup}_\Omega h c_q)^{\frac{p}{q}} \left(\displaystyle\int_\Omega h(x)dx\right)^{\frac{q-p}{q}}}, \tag{6.3.1}$$

and 0 *is not a local minimum of* $\mathscr{E}$.

Under such hypotheses, problem $(\mathscr{P})$ *has a sequence of non-zero weak solutions* $(u_n)_n$ *with*

$$\lim_{n\to\infty} \|u_n\|_{W_0^{1,p}(\Omega)} = 0.$$

Also, $\mathscr{E}(u_n) < 0$ *for any* $n \in \mathbb{N}$ *and* $\{\mathscr{E}(u_n)\}$ *is increasing.*

To ensure that 0 is not a local minimum of the energy functional, we propose the following lemma:

Lemma 6.3.1. *Assume one of the following conditions:*

(i_{0^+}) $-\infty < \liminf_{t\to 0^+} \dfrac{F(t)}{t^p} \le \limsup_{t\to 0^+} \dfrac{F(t)}{t^p} = +\infty$;

(i_{0^-}) $-\infty < \liminf_{t\to 0^-} \dfrac{F(t)}{|t|^p} \le \limsup_{t\to 0^-} \dfrac{F(t)}{|t|^p} = +\infty$.

Then, 0 *is not a local minimum of* $\mathscr{E}$.

From the proof of the Lemma 6.3.1, we can weaken condition (i_{0^+}) assuming that:

(j_{0^+}) $\liminf_{t\to 0^+} \dfrac{F(t)}{t^p} > -\infty$, and $\limsup_{t\to 0^+} \dfrac{F(t)}{t^p} > (2^N-1)\left[p \liminf_{t\to 0^+} \dfrac{F(t)}{t^p} + \dfrac{2^p}{\|h\|_\infty \theta^p}\right]$.

Analogously, we can replace (i_{0^-}) with:

(j_{0^-}) $\liminf_{t\to 0^-} \dfrac{F(t)}{t^p} > -\infty$, and $\limsup_{t\to 0^-} \dfrac{F(t)}{t^p} > (2^N-1)\left[p \liminf_{t\to 0^-} \dfrac{F(t)}{t^p} + \dfrac{2^p}{\|h\|_\infty \theta^p}\right]$.

From Theorem 6.3.1 easily follows:

Corollary 6.3.1. *Let* $f \in \mathcal{A}$ *and* $\psi \in \mathcal{F}_q$ *such that, for each* $\lambda \in]a,b[$, *the function* $\lambda\psi - F$ *is coercive and has a unique global minimum in* $\mathbb{R}$ *and one of the conditions* $(i_{0^+}), (i_{0^-})$ *hold. Finally, suppose that*

$$\alpha(F,\psi,b) \le 0 < \beta(F,\psi,a)$$

and

$$\liminf_{r\to 0^+} \frac{\sup_{\psi^{-1}(r)} F}{r^{\frac{p}{q}}} < +\infty. \tag{6.3.2}$$

Under such hypotheses, there exists $\mu^\star > 0$ *such that for every* $\mu \in]0,\mu^\star]$, *the problem*

$$\begin{cases} -\Delta_p u = \mu f(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

has a sequence of non-zero weak solutions, $(u_n)_n$ *with*

$$\lim_{n\to\infty} \|u_n\|_{W_0^{1,p}(\Omega)} = 0.$$

Theorem 6.3.2. *Let* $f \in \mathcal{A}$ *and* $h \in L^\infty(\Omega)\setminus\{0\}$, *with* $h \ge 0$. *Assume that there exists* $\psi \in \mathcal{F}_q$ *such that, for each* $\lambda \in]a,b[$, *the function* $\lambda\psi - F$ *is coercive and has a unique global minimum in* $\mathbb{R}$. *Finally, suppose that*

$$\alpha(F,\psi,b) < +\infty \qquad \text{and} \qquad \beta(F,\psi,a) = +\infty,$$

$$\liminf_{r\to+\infty} \frac{\sup_{\psi^{-1}(r)} F}{r^{\frac{p}{q}}} < \frac{1}{p(\gamma_\psi \operatorname{ess\,sup}_\Omega h c_q)^{\frac{p}{q}} \left(\int_\Omega h(x)dx\right)^{\frac{q-p}{q}}}, \tag{6.3.3}$$

and $\mathscr{E}$ *is unbounded from below.*

Under such hypotheses, problem $(\mathcal{SM}_\lambda)$ *has a sequence of weak solutions* $(u_n)_n$ *with*

$$\lim_{n\to\infty} \|u_n\|_{W_0^{1,p}(\Omega)} = +\infty.$$

Also, $\mathscr{E}(u_n) < 0$ *for any* $n \in \mathbb{N}$ *and* $\{\mathscr{E}(u_n)\}$ *is decreasing.*

Notice that it is crucial to require that $\mathscr{E}$ has no global minima.

Lemma 6.3.2. *Assume one of the following conditions:*

$(i_{+\infty})$ $-\infty < \liminf_{t\to+\infty} \frac{F(t)}{t^p} < \limsup_{t\to+\infty} \frac{F(t)}{t^p} = +\infty$;

$(i_{-\infty})$ $-\infty < \liminf_{t\to-\infty} \frac{F(t)}{|t|^p} < \limsup_{t\to-\infty} \frac{F(t)}{|t|^p} = +\infty$;

$(k_{+\infty})$ $\mathrm{essinf}_\Omega h > 0$, *and* $\liminf_{t\to+\infty} \frac{F(t)}{t^p} > \frac{1}{pc_p \mathrm{essinf}_\Omega h}$;

$(k_{-\infty})$ $\mathrm{essinf}_\Omega h > 0$, *and* $\liminf_{t\to-\infty} \frac{F(t)}{|t|^p} > \frac{1}{pc_p \mathrm{essinf}_\Omega h}$.

Then, $\mathscr{E}$ is unbounded from below.

We also point out that, in the paper [52] we developed a variant of a recent existence and localization theorem by Ricceri [107] in order to prove the existence of infinitely many solutions for $(\mathscr{P})$ under new conditions on the nonlinearity. First of all, our result can be applied when

$$\lim_{t\to\ell} \frac{F(t)}{|t|^p} \in \mathbb{R}.$$

This is not the unique novelty. Notice that the result of Ricceri [107] is a consequence of the variational methods contained in Ricceri [104]. The applicability of Ricceri's variational principle (see Ricceri [104]) in the framework of infinitely many weak solutions for quasilinear problems is only known in low dimension, i.e. for $p > N$. We gave a positive contribution also when $p \leq N$, which seems to provide the very first example in this direction. In conclusion, our result represents a step forward in the research of new conditions for finding infinitely many weak solutions for $(\mathscr{P})$. The previous discussion can be adapted also for the Schrödinger-Maxwell systems.

7.

Singular Schrödinger type equations on Cartan-Hadamard manifolds

Simplicity is the ultimate sophistication.

(Leonardo da Vinci)

7.1. Statement of main results

In this chapter we present some application of inequalities presented in Chapter 4[1].

In the sequel, let (M, g) be an n-dimensional Cartan-Hadamard manifold ($n \geq 3$) with $\mathbf{K} \geq k_0$ for some $k_0 \leq 0$, and $S = \{x_1, x_2\} \subset M$ be the set of poles. In this section we deal with the Schrödinger-type equation

$$-\Delta_g u + V(x)u = \lambda \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1)\mathbf{s}_{k_0}(d_2)} u + \mu W(x) f(u) \quad \text{in } M, \tag{$\mathscr{P}_M^\mu$}$$

where $\lambda \in \left[0, (n-2)^2\right)$ is fixed, $\mu \geq 0$ is a parameter, and the continuous function $f : [0, \infty) \to \mathbb{R}$ verifies

(f_1) $f(s) = o(s)$ as $s \to 0^+$ and $s \to \infty$;

(f_2) $F(s_0) > 0$ for some $s_0 > 0$, where $F(s) = \displaystyle\int_0^s f(t)\mathrm{d}t$.

According to (f_1) and (f_2), the number $c_f = \max_{s>0} \frac{f(s)}{s}$ is well defined and positive.

On the potential $V : M \to \mathbb{R}$ we require that

(V_1) $V_0 = \inf\limits_{x \in M} V(x) > 0$;

(V_2) $\lim\limits_{d_g(x_0,x)\to\infty} V(x) = +\infty$ for some $x_0 \in M$,

and $W : M \to \mathbb{R}$ is assumed to be positive. Elliptic problems with similar assumptions on V have been studied on Euclidean spaces, see e.g. Bartsch, Pankov and Wang [15], Bartsch and Wang [14], Rabinowitz [102] and Willem [124].

Before to state our result, let us consider the functional space

$$H_V^1(M) = \left\{ u \in H_g^1(M) : \int_M \left(|\nabla_g u|^2 + V(x)u^2\right) \mathrm{d}v_g < +\infty \right\}$$

endowed with the norm

$$\|u\|_V = \left(\int_M |\nabla_g u|^2 \,\mathrm{d}v_g + \int_M V(x)u^2 \,\mathrm{d}v_g \right)^{1/2}.$$

The main result of this subsection is as follows.

[1] Based on the paper [54, 57]

Theorem 7.1.1. *Let (M, g) be an n-dimensional Cartan-Hadamard manifold ($n \geq 3$) with $\mathbf{K} \geq k_0$ for some $k_0 \leq 0$ and let $S = \{x_1, x_2\} \subset M$ be the set of distinct poles. Let $V, W : M \to \mathbb{R}$ be positive potentials verifying (V_1), (V_2) and $W \in L^1(M) \cap L^\infty(M) \setminus \{0\}$, respectively. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function verifying (f_1) and (f_2), and $\lambda \in \big[0, (n-2)^2\big)$ be fixed. Then the following statements hold:*

(i) *Problem $(\mathscr{P}_M^\mu)$ has only the zero solution whenever $0 \leq \mu < V_0 \|W\|^{-1}_{L^\infty(M)} c_f^{-1}$;*

(ii) *There exists $\mu_0 > 0$ such that problem $(\mathscr{P}_M^\mu)$ has at least two distinct non-zero, non-negative weak solutions in $H^1_V(M)$ whenever $\mu > \mu_0$.*

7.2. Proof of main results

Proof of the Theorem 7.1.1. According to (f_1), one has $f(0) = 0$. Thus, we may extend the function f to the whole $\mathbb{R}$ by $f(s) = 0$ for $s \leq 0$, which will be considered throughout the proof. Fix $\lambda \in \big[0, (n-2)^2\big)$.

(i) Assume that $u \in H^1_V(M)$ is a non-zero weak solution of problem $(\mathscr{P}_M^\mu)$. Multiplying $(\mathscr{P}_M^\mu)$ by u, an integration on M gives that

$$\int_M |\nabla_g u|^2 \, \mathrm{d}v_g + \int_M V(x) u^2 \, \mathrm{d}v_g = \lambda \int_M \frac{\mathbf{s}^2_{k_0}\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u^2 \mathrm{d}v_g + \mu \int_M W(x) f(u) u \mathrm{d}v_g.$$

By the latter relation, Corollary 4.3.1 (see relation (4.3.2)) and the definition of c_f, it yields that

$$\begin{aligned} \int_M |\nabla_g u|^2 \, \mathrm{d}v_g + V_0 \int_M u^2 \, \mathrm{d}v_g &\leq \int_M |\nabla_g u|^2 \, \mathrm{d}v_g + \int_M V(x) u^2 \, \mathrm{d}v_g \\ &= \lambda \int_M \frac{\mathbf{s}^2_{k_0}\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u^2 \mathrm{d}v_g + \mu \int_M W(x) f(u) u \mathrm{d}v_g \\ &\leq \int_M |\nabla_g u|^2 \, \mathrm{d}v_g + \mu \|W\|_{L^\infty(M)} c_f \int_M u^2 \, \mathrm{d}v_g. \end{aligned}$$

Consequently, if $0 \leq \mu < V_0 \|W\|^{-1}_{L^\infty(M)} c_f^{-1}$, then u is necessarily 0, a contradiction.

(ii) Let us consider the energy functional associated with problem $(\mathscr{P}_M^\mu)$, i.e., $\mathcal{E}_\mu : H^1_V(M) \to \mathbb{R}$ defined by

$$\mathcal{E}_\mu(u) = \frac{1}{2} \int_M \left(|\nabla_g u|^2 + V(x) u^2\right) \mathrm{d}v_g - \frac{\lambda}{2} \int_M \frac{\mathbf{s}^2_{k_0}\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u^2 \, \mathrm{d}v_g - \mu \int_M W(x) F(u) \mathrm{d}v_g.$$

One can show that $\mathcal{E}_\mu \in C^1(H^1_V(M), \mathbb{R})$ and for all $u, w \in H^1_V(M)$ we have

$$\mathcal{E}'_\mu(u)(w) = \int_M \left(\langle \nabla_g u, \nabla_g w \rangle + V(x) u w\right) \mathrm{d}v_g - \lambda \int_M \frac{\mathbf{s}^2_{k_0}\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1) \mathbf{s}_{k_0}(d_2)} u w \, \mathrm{d}v_g - \mu \int_M W(x) f(u) w \mathrm{d}v_g.$$

Therefore, the critical points of $\mathcal{E}_\mu$ are precisely the weak solutions of problem $(\mathscr{P}_M^\mu)$ in $H^1_V(M)$. By exploring the sublinear character of f at infinity, Corollary 4.3.1 and Lemma 2.2.1, one can see that $\mathcal{E}_\mu$ is bounded from below, coercive and satisfies the usual Palais-Smale condition for every $\mu \geq 0$. Moreover, by an elementary computation one can see that assumption (f_1) is inherited as a sub-quadratic property in the sense that

$$\lim_{\|u\|_V \to 0} \frac{\int_M W(x) F(u) \mathrm{d}v_g}{\|u\|_V^2} = \lim_{\|u\|_V \to \infty} \frac{\int_M W(x) F(u) \mathrm{d}v_g}{\|u\|_V^2} = 0. \tag{7.2.1}$$

Due to (f_2) and $W \neq 0$, we can construct a non-zero truncation function $u_0 \in H^1_V(M)$ such that $\displaystyle\int_M W(x)F(u_0)\mathrm{d}v_g > 0$. Thus, we may define

$$\mu_0 = \frac{1}{2}\inf\left\{ \frac{\|u\|_V^2}{\displaystyle\int_M W(x)F(u)\mathrm{d}v_g} : u \in H^1_V(M), \int_M W(x)F(u)\mathrm{d}v_g > 0 \right\}.$$

By the relations in (7.2.1), we clearly have that $0 < \mu_0 < \infty$.

Let us fix $\mu > \mu_0$. Then there exists $\tilde{u}_\mu \in H^1_V(M)$ with $\displaystyle\int_M W(x)F(\tilde{u}_\mu)\mathrm{d}v_g > 0$ such that $\mu > \dfrac{\|\tilde{u}_\mu\|_V^2}{2\displaystyle\int_M W(x)F(\tilde{u}_\mu)\mathrm{d}v_g} \geq \mu_0$. Consequently,

$$c^1_\mu := \inf_{H^1_V(M)} \mathcal{E}_\mu \leq \mathcal{E}_\mu(\tilde{u}_\mu) \leq \frac{1}{2}\|\tilde{u}_\mu\|_V^2 - \mu\int_M W(x)F(\tilde{u}_\mu) < 0.$$

Since $\mathcal{E}_\mu$ is bounded from below and satisfies the Palais-Smale condition, the number c^1_μ is a critical value of $\mathcal{E}_\mu$, i.e., there exists $u^1_\mu \in H^1_V(M)$ such that $\mathcal{E}_\mu(u^1_\mu) = c^1_\mu < 0$ and $\mathcal{E}'_\mu(u^1_\mu) = 0$. In particular, $u^1_\mu \neq 0$ is a weak solution of problem $(\mathscr{P}^\mu_M)$.

Standard computations based on Corollary 4.3.1 and the embedding $H^1_V(M) \hookrightarrow L^p(M)$ for $p \in (2, 2^*)$ show that there exists a sufficiently small $\rho_\mu \in (0, \|\tilde{u}_\mu\|_V)$ such that

$$\inf_{\|u\|_V = \rho_\mu} \mathcal{E}_\mu(u) = \eta_\mu > 0 = \mathcal{E}_\mu(0) > \mathcal{E}_\mu(\tilde{u}_\mu),$$

which means that the functional $\mathcal{E}_\mu$ has the mountain pass geometry. Therefore, we may apply the mountain pass theorem, see Rabinowitz [102], showing that there exists $u^2_\mu \in H^1_V(M)$ such that $\mathcal{E}'_\mu(u^2_\mu) = 0$ and $\mathcal{E}_\mu(u^2_\mu) = c^2_\mu$, where $c^2_\mu = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{E}_\mu(\gamma(t))$, and $\Gamma = \{\gamma \in C([0,1]; H^1_V(M)) : \gamma(0) = 0,\ \gamma(1) = \tilde{u}_\mu\}$. Due to the fact that $c^2_\mu \geq \inf_{\|u\|_V=\rho_\mu}\mathcal{E}_\mu(u) > 0$, it is clear that $0 \neq u^2_\mu \neq u^1_\mu$. Moreover, since $f(s) = 0$ for every $s \leq 0$, the solutions u^1_μ and u^2_μ are non-negative. □

Remark 7.2.1. Theorem 7.1.1 can be applied on the hyperbolic space $\mathbb{H}^n = \{y = (y_1, ..., y_n) : y_n > 0\}$ endowed with the metric $g_{ij}(y_1, ..., y_n) = \frac{\delta_{ij}}{y_n^2}$; it is new even on the Euclidean space $\mathbb{R}^n$, $n \geq 3$.

Remark 7.2.2. Let us assume that (M, g) is a *Hadamard manifold* in Theorem 4.1.2. In particular, a Laplace comparison principle yields that

(b) Limiting cases:

- If $k_0 \to 0$, then
$$\frac{\mathbf{s}^2_{k_0}\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i)\mathbf{s}_{k_0}(d_j)} \to \frac{d^2_{ij}}{4d_i^2 d_j^2} \text{ and } R_{ij}(k_0) \to 0,$$
thus (4.1.5) reduces to (4.1.2).

- If $k_0 \to -\infty$, then basic properties of the sinh function shows that for a.e. on M we have
$$\frac{\mathbf{s}^2_{k_0}\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i)\mathbf{s}_{k_0}(d_j)} \to 0 \text{ and } R_{ij}(k_0) \to \left(\frac{1}{d_i} - \frac{1}{d_j}\right)^2;$$
therefore, (4.1.5) reduces to
$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \sum_{1\leq i<j\leq m}\int_M u^2\left(\frac{1}{d_i} - \frac{1}{d_j}\right)^2 \mathrm{d}v_g, \quad \forall u \in C_0^\infty(M).$$

Remark 7.2.3. *Based on the previous chapters, the result of the Theorem 7.1.1 can be extended to the Schrödinger-Maxwell systems, taking into account the Proposition 6.1.1.*

7.3. Remarks

In [57], we investigated and elliptic PDE which involve the so called Finsler-Laplace operator associated with asymmetric Minkowski norms modeling for instance the Matsumoto mountain slope metric or various Randers-type norms coming from mathematical physics (see Bao, Chern and Shen [12], Belloni, Ferone and Kawohl [18], Matsumoto [93], and Randers [103]). More precisely, we proved a multiplicity result for an anisotropic sub-linear elliptic problem with Dirichlet boundary condition, depending on a positive parameter λ, see Theorem 7.3.1. In what follows, we give some details about this result:

Many anisotropic problems are studied via variational arguments, by considering the functional

$$\mathscr{E}_H(u) = \int_\Omega H(\nabla u)^2, \quad u \in W^{1,2}(\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is a regular open domain, and $H : \mathbb{R}^n \to [0, \infty)$ is a convex function of class $C^2(\mathbb{R}^n \setminus \{0\})$ which is absolutely homogeneous of degree one, i.e.,

$$H(tx) = |t|H(x) \quad \text{for all} \quad t \in \mathbb{R}, \; x \in \mathbb{R}^n. \tag{7.3.1}$$

It is clear that there exists $c_1, c_2 > 0$ such that for every $x \in \mathbb{R}^n$,

$$c_1|x| \leq H(x) \leq c_2|x|,$$

where $|x|$ denotes the Euclidean norm.

In fact, the energy functional $\mathscr{E}_H$ is associated with highly nonlinear equations which involve the so-called *Finsler-Laplace operator*

$$\Delta_H u = \operatorname{div}(H(\nabla u)\nabla H(\nabla u)).$$

In the sequel $H : \mathbb{R}^n \to [0, \infty)$ will be called a *positively homogeneous Minkowski norm* if H is a positive homogeneous function and verifies the properties:

- $H \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
- The Hessian matrix $\nabla^2(H^2/2)(x)$ is positive definite for all $x \neq 0$.

Note that, in this case the pair $(\mathbb{R}^n, H)$ is a *Minkowski space*, see Bao, Chern and Shen [12]. In the paper [57], we considered the following nonlinear equation coupled with the Dirichlet boundary condition:

$$\begin{cases} -\Delta_H u = \lambda\kappa(x)f(u) \quad \text{in} \quad \Omega; \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{$\mathscr{P}_\lambda$}$$

Here λ is a positive parameter, $\Omega \subset \mathbb{R}^n$ is an open bounded domain, $\kappa \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

(f_1) $f(s) = o(s)$ as $s \to 0^+$ and $s \to \infty$;

(f_2) $F(s_0) > 0$ for some $s_0 > 0$, where $F(s) = \int\limits_0^s f(t)dt$.

Due to the assumptions, the number $c_f = \max\limits_{s>0} \dfrac{f(s)}{s}$ is well-defined and positive.

Let us define the following Rayleigh-type quotient:

$$\lambda_1 = \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega H^2(\nabla u(x))dx}{\displaystyle\int_\Omega u^2(x)dx}.$$

It is clear that $0 < \lambda_1 < \infty$ (see for example Belloni, Ferone and Kawohl [18]). Let Ω^* be the anisotropic symmetrization of Ω. Our results read as follows:

Theorem 7.3.1. *Let $H : \mathbb{R}^n \to [0, \infty)$ be a positively homogeneous Minkowski norm, $\Omega \subset \mathbb{R}^n$ be a bounded open domain and $\kappa \in L^\infty(\Omega)_+ \setminus \{0\}$. Then*

(a) if $0 \leq \lambda < c_f^{-1} \|\kappa\|_{L^\infty(\Omega)}^{-1} \lambda_1$, problem $(\mathscr{P}_\lambda)$ has only the zero solution;

(b) there exists $\tilde{\lambda} > 0$ such that for every $\lambda > \tilde{\lambda}$, problem $(\mathscr{P}_\lambda)$ has at least two distinct non-zero, non-negative solutions;

(c) if $\Omega = \Omega^\star$ and $\kappa \equiv 1$, at least one of the solutions in (b) has level sets homothetic to the Wulff set

$$B_{H_0}^{-}(1) = \{x \in \mathbb{R}^n : H_0(-x) < r\}.$$

Remark 7.3.1. *We emphasize that our result is still valid for positively homogeneous convex functions $H : \mathbb{R}^n \to [0, \infty)$ of class $C^2(\mathbb{R}^n \setminus \{0\})$.*

8. A characterization related to Schrödinger equations on Riemannian manifolds

> I hear and I forget. I see and I remember. I do and I understand.
>
> *(Confucius)*

8.1. Introduction and statement of main results

The existence of standing waves solutions for the nonlinear Schrödinger equation[1]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - f(x,|\psi|), \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \setminus \{0\},$$

has been intensively studied in the last decades. The Schrödinger equation plays a central role in quantum mechanic as it predicts the future behavior of a dynamic system. Indeed, the wave function $\psi(x,t)$ represents the quantum mechanical probability amplitude for a given unit-mass particle to have position x at time t. Such equation appears in several fields of physics, from Bose–Einstein condensates and nonlinear optics, to plasma physics (see for instance Byeon and Wang [25] and Cao, Noussair and Yan [28] and reference therein).

A Lyapunov-Schmidt type reduction, i.e. a separation of variables of the type $\psi(x,t) = u(x)e^{-i\frac{E}{\hbar}t}$, leads to the following semilinear elliptic equation

$$-\Delta u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^n.$$

With the aid of variational methods, the existence and multiplicity of nontrivial solutions for such problems have been extensively studied in the literature over the last decades. For instance, the existence of positive solutions when the potential V is coercive and f satisfies standard mountain pass assumptions, are well known after the seminal paper of Rabinowitz [102]. Moreover, in the class of bounded from below potentials, several attempts have been made to find general assumptions on V in order to obtain existence and multiplicity results (see for instance Bartsch, Pankov and Wang [16], Bartsch and Wang [14], Benci and Fortunato [19] Willem [124] and Strauss [116]). In such papers the nonlinearity f is required to satisfy the well-know Ambrosetti-Rabinowitz condition, thus it is superlinear at infinity. For a sublinear growth of f see also Kristály [74].

Most of the aforementioned papers provide *sufficient* conditions on the nonlinear term f in order to prove existence/multiplicity type results. The novelty of the present chapter is to establish a *characterization* result for stationary Schrödinger equations on unbounded domains; even more, our arguments work on not necessarily linear structures. Indeed, our results fit the research direction where the solutions of certain PDEs are influenced by the geometry of the ambient structure (see for instance Farkas, Kristály and Varga [58], Farkas and Kristály [56], Kristály [75], Li and Yau [89], Ma [92] and reference therein). Accordingly, we deal with a Riemannian setting, the results on $\mathbb{R}^n$ being a particular consequence of our general achievements.

[1] Based on the paper [53]

Let $x_0 \in M$ be a fixed point, $\alpha : M \to \mathbb{R}_+ \setminus \{0\}$ a bounded function and $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$ such that there exist two constants $C > 0$ and $q \in (1, 2^\star)$ (being $2^\star$ the Sobolev critical exponent) such that

$$f(\xi) \leq k\left(1 + \xi^{q-1}\right) \text{ for all } \xi \geq 0. \tag{8.1.1}$$

Denote by $F : \mathbb{R}_+ \to \mathbb{R}_+$ the function $F(\xi) = \displaystyle\int_0^\xi f(t)dt$.

We assume that $V : M \to \mathbb{R}$ is a measurable function satisfying the following conditions:

(V_1) $V_0 = \operatorname{essinf}_{x \in M} V(x) > 0$;

(V_2) $\displaystyle\lim_{d_g(x_0,x)\to\infty} V(x) = +\infty$, for some $x_0 \in M$.

The problem we deal with is written as:

$$\begin{cases} -\Delta_g u + V(x)u = \lambda\alpha(x)f(u), & \text{in } M \\ u \geq 0, & \text{in } M \\ u \to 0, & \text{as } d_g(x_0, x) \to \infty. \end{cases} \tag{$\mathscr{P}_\lambda$}$$

Our result reads as follows:

Theorem 8.1.1. *Let $n \geq 3$ and (M, g) be a complete, non-compact $n-$dimensional Riemannian manifold satisfying the curvature condition* $(\mathbf{C})$, *and* $\inf_{x\in M} \mathrm{Vol}_g(B_x(1)) > 0$. *Let also $\alpha : M \to \mathbb{R}_+ \setminus \{0\}$ be in $L^\infty(M) \cap L^1(M)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$ verifying (8.1.1) and $V : M \to \mathbb{R}$ be a potential verifying (V_1), (V_2). Assume that for some $a > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is non-increasing in $(0, a]$. Then, the following conditions are equivalent:*

(i) *for each $b > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is not constant in $(0, b]$;*

(ii) *for each $r > 0$, there exists an open interval $I_r \subseteq (0, +\infty)$ such that for every $\lambda \in I_r$, problem $(\mathscr{P}_\lambda)$ has a nontrivial solution $u_\lambda \in H^1_g(M)$ satisfying*

$$\int_M \left(|\nabla_g u_\lambda(x)|^2 + V(x)u_\lambda^2\right) \mathrm{d}v_g < r.$$

Remark 8.1.2. (a) One can replace the assumption $\inf_{x\in M} \mathrm{Vol}_g(B_x(1)) > 0$ with a curvature restriction, requiring that the sectional curvature is bounded from above. Indeed, using the Bishop-Gromov theorem one can easily get that $\inf_{x\in M} \mathrm{Vol}_g(B_x(1)) > 0$.

(b) A more familiar form of Theorem 8.1.1 can be obtained when $Ric_{(M,g)} \geq 0$; it suffices to put $H \equiv 0$ in $(\mathbf{C})$.

The following potentials V fulfills assumptions (V_1) and (V_2):

(i) Let $V(x) = d_g^\theta(x, x_0) + 1$, where $x_0 \in M$ and $\theta > 0$.

(ii) More generally, if $z : [0, +\infty) \to [0, +\infty)$ is a bijective function, with $z(0) = 0$, let $V(x) = z(d_g(x, x_0)) + c$, where $x_0 \in M$ and $c > 0$.

The work is motivated by a result of Ricceri [109], where a similar theorem is stated for one-dimensional Dirichlet problem; more precisely, (i) from Theorem 8.1.1 characterizes the existence of the solutions for the following problem

$$\begin{cases} -u'' = \lambda\alpha(x)f(u), & \text{in } (0,1) \\ u > 0, & \text{in } (0,1) \\ u(1) = u(0) = 0. \end{cases}$$

In the above theorem it is crucial the embedding of the Sobolev space $H_0^1((0,1))$ into $C^0([0,1])$.

Recently, this result has been extended by Anello to higher dimension, i.e. when the interval $(0,1)$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ with smooth boundary (see Anello [7]). The generalization follows by direct minimization procedures and contains a more precise information on the interval of parameters I. See also Bisci and Rădulescu [94], for a similar characterization in the framework of fractal sets.

Let us note that in our setting the situation is much more delicate with respect to those treated in the papers Anello [7], Ricceri [109]. Indeed, the Riemannian framework produces several technical difficulties that we overcome by using an appropriate variational formulation.

One of the main tools in our investigation is a recent result by Ricceri [108], see Theorem 1.2.11. The main difficulty in the implication $(i) \Rightarrow (ii)$ in Theorem 8.1.1, consists in proving the boundedness of the solutions. To overcome this difficulty we use the Nash-Moser iteration method adapted to the Riemannian setting.

In proving $(ii) \Rightarrow (i)$, we make use of a recent result by Poupaud [101], see Theorem 1.3.4 concerning the discreteness of the spectrum of the operator $u \mapsto -\Delta_g u + V(x)u$.

8.2. Proof of main results

Let us consider the functional space

$$H_V^1(M) = \left\{ u \in H_g^1(M) : \int_M \left(|\nabla_g u|^2 + V(x)u^2\right) \mathrm{d}v_g < +\infty \right\}$$

endowed with the norm

$$\|u\|_V = \left(\int_M |\nabla_g u|^2 \,\mathrm{d}v_g + \int_M V(x)u^2 \,\mathrm{d}v_g \right)^{1/2}.$$

If V is bounded from below by a positive constant, it is clear that the embedding $H_V^1(M) \hookrightarrow H_g^1(M)$ is continuous

The energy functional associated to problem $(\mathscr{P}_\lambda)$ is the functional $\mathcal{E} : H_V^1 \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_V^2 - \lambda \int_M \alpha(x)F(u)\mathrm{d}v_g,$$

which is of class C^1 in H_V^1 with derivative, at any $u \in H_V^1$, given by

$$\mathcal{E}'(u)(v) = \int_M (\langle \nabla_g u, \nabla_g v\rangle + V(x)uv)\mathrm{d}v_g - \lambda \int_M \alpha(x)f(u)v\mathrm{d}v_g, \quad \text{for all } v \in H_V^1.$$

Weak solutions of problem $(\mathcal{SM}_\lambda)$ are precisely critical points of $\mathcal{E}$.

8.2.1. Regularity of weak solutions via Nash-Moser iteration

Because of the sign of f, it is clear that critical points of $\mathscr{E}$ are non negative functions. More properties of critical points of $\mathscr{E}$ can be deduced by the following regularity theorem which is crucial in the proof of the Theorem 8.1.1. We adapt to our setting the classical Nash Moser iteration techniques.

Theorem 8.2.1. *Let $n \geq 3$ and (M,g) be a complete, non-compact $n-$dimensional Riemannian manifold satisfying the curvature condition* $(\mathbf{C})$, *and* $\inf_{x\in M} \mathrm{Vol}_g(B_x(1)) > 0$. *Let also $\varphi : M \times \mathbb{R}_+ \to \mathbb{R}$ be a continuous function with primitive $\Phi(x,t) = \displaystyle\int_0^t \varphi(x,\xi)d\xi$ such that, for some constants $k > 0$ and $q \in (2, 2^*)$ one has*

$$|\varphi(x,\xi)| \leq k(\xi + \xi^{q-1}), \quad \textit{for all } \xi \geq 0, \quad \textit{uniformly in } x \in M.$$

Let $u \in H^1_V(M)$ be a non negative critical point of the functional $\mathcal{G} : H_V \to \mathbb{R}$

$$\mathcal{G}(u) = \frac{1}{2}\|u\|_V^2 - \int_M \Phi(x,u)\mathrm{d}v_g.$$

and $x_0 \in M$. Then,

(i) for every $\rho > 0$, $u \in L^\infty(B_{x_0}(\rho))$;

(ii) $u \in L^\infty(M)$ and $\lim_{d_g(x_0,x)\to\infty} u(x) = 0$.

Proof. Let u be a critical point of $\mathcal{G}$. Then,

$$\int_M (\langle \nabla_g u, \nabla_g v\rangle + V(x)uv)\,\mathrm{d}v_g = \int_M \varphi(x,u)v\mathrm{d}v_g \qquad \text{for all } v \in H_V. \tag{8.2.1}$$

For each $L > 0$, define

$$u_L(x) = \begin{cases} u(x) & \text{if } u(x) \le L, \\ L & \text{if } u(x) > L. \end{cases}$$

Let also $\tau \in C^\infty(M)$ with $0 \le \tau \le 1$.

For $\beta > 1$, set $v_L = \tau^2 u u_L^{2(\beta-1)}$ and $w_L = \tau u u_L^{\beta-1}$ which are in $H^1_V(M)$. Thus, plugging v_L into (8.2.1), we get

$$\int_M (\langle \nabla_g u, \nabla_g v_L\rangle + V(x)uv_L)\,\mathrm{d}v_g = \int_M \varphi(x,u)v_L\mathrm{d}v_g. \tag{8.2.2}$$

A direct calculation yields that

$$\nabla_g v_L = 2\tau\ u\ u_L^{2(\beta-1)}\ \nabla_g\tau + \tau^2\ u_L^{2(\beta-1)}\ \nabla_g u\ + 2(\beta-1)\tau^2\ u\ u_L^{2\beta-3}\ \nabla_g u_L,$$

and

$$\begin{aligned}\int_M \langle \nabla_g u, \nabla_g v_L\rangle \mathrm{d}v_g &= \int_M \left[2\tau\ u\ u_L^{2(\beta-1)}\ \langle\nabla_g u, \nabla_g \tau\rangle + \tau^2\ u_L^{2(\beta-1)}\ |\nabla_g u|^2\right]\mathrm{d}v_g \\ &\quad + \int_M 2(\beta-1)\tau^2\ u\ u_L^{2\beta-3}\ \langle \nabla_g u, \nabla_g u_L\rangle \mathrm{d}v_g \\ &\ge \int_M \left[2\tau\ u\ u_L^{2(\beta-1)}\langle\nabla_g u, \nabla_g\tau\rangle\ + \tau^2 u_L^{2(\beta-1)}|\nabla_g u|^2\right]\mathrm{d}v_g,\end{aligned} \tag{8.2.3}$$

since

$$2(\beta-1)\int_M \tau^2\ u\ u_L^{2\beta-3}\langle \nabla_g u_L, \nabla_g u\rangle \mathrm{d}v_g = \int_{\{u\le L\}} \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g \ge 0.$$

Notice that

$$\begin{aligned}|\nabla_g w_L|^2 &= u^2\ u_L^{2(\beta-1)}\ |\nabla_g\tau|^2 + \tau^2\ u_L^{2(\beta-1)}\ |\nabla_g u|^2 + (\beta-1)^2\ \tau^2 u^2\ u_L^{2(\beta-2)}|\nabla_g u_L|^2 \\ &\quad + 2\tau\ u\ u_L^{2(\beta-1)}\ \langle \nabla_g\tau, \nabla_g u\rangle + 2(\beta-1)\tau\ u^2\ u_L^{2\beta-3}\langle\nabla_g\tau, \nabla_g u_L\rangle + 2(\beta-1)\ \tau^2\ u\ u_L^{2\beta-3}\langle\nabla_g u, \nabla_g u_L\rangle.\end{aligned}$$

Then, one can observe that

$$\int_M \tau^2\ u^2\ u_L^{2(\beta-2)}|\nabla_g u_L|^2\mathrm{d}v_g = \int_{\{u\le L\}} \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g \le \int_M \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g,$$

and

$$\int_M \tau^2 u\ u_L^{2\beta-3}\langle\nabla_g u, \nabla_g u_L\rangle\mathrm{d}v_g = \int_{\{u\le L\}} \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g \le \int_M \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g,$$

and also that

$$
\begin{aligned}
2\int_M \tau\ u^2\ u_L^{2\beta-3}\ \langle\nabla_g\tau,\nabla_g u_L\rangle \mathrm{d}v_g &\leq 2\int_M \tau\ u^2\ u_L^{2\beta-3}\ |\nabla_g\tau|\cdot|\nabla_g u_L|\mathrm{d}v_g\\
&= 2\int_M (\tau\ u\ u_L^{\beta-2}\ |\nabla_g u_L|)\cdot(u\ u_L^{\beta-1}\ |\nabla_g\tau|)\mathrm{d}v_g\\
&\leq \int_M \tau^2\ u^2\ u_L^{2(\beta-2)}\ |\nabla_g u_L|^2\mathrm{d}v_g + \int_M u^2\ u_L^{2(\beta-1)}\ |\nabla_g\tau|^2\mathrm{d}v_g\\
&\leq \int_M \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g + \int_M u^2\ u_L^{2(\beta-1)}\ |\nabla_g\tau|^2\mathrm{d}v_g.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_M |\nabla_g w_L|^2\mathrm{d}v_g \leq &\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + \beta^2\int_M \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g+\\
&+2\int_M \tau\ u\ u_L^{2(\beta-1)}\ \langle\nabla_g\tau,\nabla_g u\rangle\mathrm{d}v_g + 2(\beta-1)\int_M \tau\ u^2\ u_L^{2\beta-3}\ \langle\nabla_g\tau,\nabla u_L\rangle\mathrm{d}v_g\\
&\leq \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + (\beta^2+\beta-1)\int_M \tau^2\ u_L^{2(\beta-1)}|\nabla_g u|^2\mathrm{d}v_g+\\
&+2\int_M \tau\ u\ u_L^{2(\beta-1)}\ \langle\nabla_g\tau,\nabla_g u\rangle\mathrm{d}v_g.
\end{aligned}
\tag{8.2.4}
$$

In the sequel we will need the constant $\gamma = \frac{2\cdot 2^\star}{2^\star-q+2}$. It is clear that $2<\gamma<2^\star$.
Proof of i). Putting together (8.2.3), (8.2.4), with (8.2.2), recalling that $\beta>1$, and bearing in mind the growth of the function φ, we obtain that

$$
\begin{aligned}
\|w_L\|_V^2 &= \int_M \left(|\nabla_g w_L|^2 + V(x)w_L^2\right)\mathrm{d}v_g\\
&\leq \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + 2\beta^2\int_M \left(\langle\nabla_g u,\nabla_g v_L\rangle + V(x)\tau^2u^2\ u_L^{2(\beta-1)}\right)\mathrm{d}v_g\\
&= \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + 2\beta^2\int_M \left(\langle\nabla_g u,\nabla_g v_L\rangle + V(x)uv_L\right)\mathrm{d}v_g\\
&= \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + 2\beta^2\int_M \varphi(x,u)v_L\mathrm{d}v_g\\
&\leq \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + 2\beta^2k\int_M \left(\tau^2u^2\ u_L^{2(\beta-1)} + \tau^2u^qu_L^{2(\beta-1)}\right)\mathrm{d}v_g\\
&= \underbrace{\beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g}_{I_1} + 2\beta^2k\underbrace{\int_M w_L^2\mathrm{d}v_g}_{I_2} + 2\beta^2k\underbrace{\int_M u^{q-2}w_L^2\mathrm{d}v_g}_{I_3}.
\end{aligned}
$$

Let $R,r>0$. In the proof of case i), τ verifies the further following properties: $|\nabla\tau|\leq\frac{2}{r}$ and

$$
\tau(x) = \begin{cases} 1 & \text{if } d_g(x_0,x)\leq R,\\ 0 & \text{if } d_g(x_0,x)>R+r.\end{cases}
$$

Then, applying Hölder inequality yields that

$$
\begin{aligned}
I_1 &\leq \frac{4}{r^2}\int_{R\leq d_g(x_0,x)\leq R+r} u^2\ u_L^{2(\beta-1)}\mathrm{d}v_g\\
&\leq \frac{4}{r^2}\left(\mathrm{Vol}_g\left(A[R,R+r]\right)\right)^{1-\frac{2}{\gamma}}\left(\int_{A[R,R+r]} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}},
\end{aligned}
$$

where $A[R,R+r] = \{x\in M:\ R\leq d_g(x_0,x)\leq R+r\}$. Then, from Theorem 1.3.3, we have that

$$
I_1 \leq 4\omega_n^{1-\frac{2}{\gamma}}e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\frac{(R+r)^{n\left(1-\frac{2}{\gamma}\right)}}{r^2}\left(\int_{d_g(x_0,x)\leq R+r} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}.
$$

In a similar way, we obtain that

$$\begin{aligned} I_2 &\leq \int_{d_g(x_0,x)\leq R+r} u^2\ u_L^{2(\beta-1)}\mathrm{d}v_g \\ &\leq \omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\ (R+r)^{n(1-\frac{2}{\gamma})}\left(\int_{d_g(x,x_0)\leq R+r} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}, \end{aligned}$$

and also that

$$\begin{aligned} I_3 &= \int_M u^{q-2}w_L^2\mathrm{d}v_g \leq \left(\int_M u^{2^\star}\mathrm{d}v_g\right)^{\frac{q-2}{2^\star}}\left(\int_M w_L^\gamma\mathrm{d}v_g\right)^{\frac{2}{\gamma}} \\ &= \|u\|_{L^{2^\star}(M)}^{q-2}\left(\int_{d_g(x_0,x)\leq R+r} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}. \end{aligned}$$

In the sequel we will use the notation $\mathscr{J} = \left(\int_{d_g(x_0,x)\leq R+r} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}$. Therefore, summing up the above computations, we obtain that

$$\begin{aligned} \|w_L\|_V^2 &\leq 4\beta\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2}\mathscr{J} + 2\beta^2 k\|u\|_{L^{2^\star}(M)}^{q-2}\mathscr{J} \\ &+ 2\beta^2 k\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\ (R+r)^{n(1-\frac{2}{\gamma})}\mathscr{J}. \end{aligned} \tag{8.2.5}$$

Moreover, if $C_\star$ denotes the embedding constant of $H_V^1(M)$, one has into $L^{2^\star}(M)$,

$$\|w_L\|_V^2 \geq C_\star\|w_L\|_{L^{2^\star}(M)}^2 = C_\star\left(\int_M(\tau\ u\ u_L^{\beta-1})^{2^\star}\mathrm{d}v_g\right)^{\frac{2}{2^\star}} \geq C_\star\left(\int_{d_g(x_0,x)\leq R}(u\ u_L^{\beta-1})^{2^\star}\mathrm{d}v_g\right)^{\frac{2}{2^\star}}.$$

Combining the above computations with (8.2.5), and bearing in mind that $\beta>1$, we get

$$\begin{aligned} \left(\int_{d_g(x_0,x)\leq R}(u\ u_L^{\beta-1})^{2^\star}\mathrm{d}v_g\right)^{\frac{2}{2^\star}} &\leq 4C_\star^{-1}\beta^2\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\ \frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2}\mathscr{J} + 2kC_\star^{-1}\beta^2\|u\|_{2^\star}^{q-2}\mathscr{J} \\ &+ 2kC_\star^{-1}\beta^2\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\ (R+r)^{n(1-\frac{2}{\gamma})}\mathscr{J}. \end{aligned} \tag{8.2.6}$$

Taking the limit as $L\to+\infty$ in (8.2.6), we obtain

$$\begin{aligned} \left(\int_{d_g(x_0,x)\leq R} u^{2^\star\beta}\mathrm{d}v_g\right)^{\frac{2}{2^\star}} \leq & 4C_\star^{-1}\beta^2\omega_n^{1-\frac{2}{\gamma}}\ \frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2}\left(\int_{d_g(x_0,x)\leq R+r} u^{\gamma\beta}\mathrm{d}v_g\right)^{\frac{2}{\gamma}} + \\ & + 2kC_\star^{-1}\beta^2\omega_n^{1-\frac{2}{\gamma}}(R+r)^{n(1-\frac{2}{\gamma})}\left(\int_{d_g(x_0,x)\leq R+r} u^{\gamma\beta}\mathrm{d}v_g\right)^{\frac{2}{\gamma}} + \\ & + 2C_\star^{-1}\beta^2 k\|u\|_{2^\star}^{q-2}\left(\int_{d_g(x_0,x)\leq R+r} u^{\gamma\beta}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}. \end{aligned}$$

Thus, for every $R>0$, $r>0$, $\beta>1$ one has

$$\|u\|_{L^{2^\star\beta}(d_g(x_0,x)\leq R)} \leq (C_\star^{-1})^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}\left(C_1\ \frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2} + C_2\ (R+r)^{n(1-\frac{2}{\gamma})} + C_3\right)^{\frac{1}{2\beta}}\|u\|_{L^{\gamma\beta}(d_g(x_0,x)\leq R+r)}, \tag{8.2.7}$$

where $C_1 = 4\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}$, $C_2 = 2k\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}$, $C_3 = 2k\|u\|^{q-2}_{L^{2^\star}(M)}$.

Fix $\rho > 0$. We are going to apply (8.2.7) choosing first $\beta = \dfrac{2^\star}{\gamma}$, $R = \rho + \dfrac{\rho}{2}$, $r = \dfrac{\rho}{2}$, to get

$$\|u\|_{L^{2^\star\beta}(d_g(x_0,x)\le\rho+\frac{\rho}{2})} \le (C_\star^{-1})^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}\left(C_1\, 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^2 + C_2\,(2\rho)^{n(1-\frac{2}{\gamma})} + C_3\right)^{\frac{1}{2\beta}}\|u\|_{L^{2^\star}(d_g(x_0,x)\le 2\rho)}$$

Noticing that $\gamma\beta^2 = 2^\star\beta$, we can apply (8.2.7) with β^2 in place of β and $R = \rho + \frac{\rho}{2^2}$, $r = \frac{\rho}{2^2}$. We obtain

$$\begin{aligned}\|u\|_{L^{2^\star\beta^2}(d_g(x_0,x)\le\rho+\frac{\rho}{2^2})} &\le (C_\star^{-1})^{\frac{1}{2\beta}+\frac{1}{2\beta^2}}\beta^{\frac{1}{\beta}+\frac{2}{\beta^2}}e^{\frac{1}{2\beta}\log\left(C_1\, 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^2+C_2\,(2\rho)^{n(1-\frac{2}{\gamma})}+C_3\right)}\,.\\ &\quad\cdot e^{\frac{1}{2\beta^2}\log\left(C_1\, 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}(2^2)^2+C_2\,(2\rho)^{n(1-\frac{2}{\gamma})}+C_3\right)}\|u\|_{L^{2^\star}(d_g(x_0,x)\le 2\rho)}\end{aligned}$$

Iterating this procedure, for every integer k we obtain

$$\begin{aligned}&\|u\|_{L^{2^\star\beta^k}(d_g(x_0,x)\le\rho)} \le \|u\|_{L^{2^\star\beta^k}(d_g(x_0,x)\le\rho+\frac{\rho}{2^k})}\\ &\le (C_\star^{-1})^{\sum_{i=1}^k\frac{1}{2\beta^i}}\beta^{\sum_{i=1}^k\frac{i}{\beta^i}}e^{\sum_{i=1}^k\frac{\log\left(C_1\, 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^{2i}+C_2(2\rho)^{n(1-\frac{2}{\gamma})}+C_3\right)}{2\beta^i}}\|u\|_{L^{2^\star}(d_g(x_0,x)\le 2\rho)}.\end{aligned}$$

If

$$\sigma = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\beta^k} = \frac{1}{2(\beta-1)},\ \vartheta = \sum_{k=1}^{\infty}\frac{k}{\beta^k},\ \eta = \sum_{k=1}^{\infty}\frac{\log\left(C_1\, 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^{2k}+C_2\,(2\rho)^{n(1-\frac{2}{\gamma})}+C_3\right)}{2\beta^k}.$$

Passing to the limit as $n\to\infty$, we obtain

$$\|u\|_{L^\infty(d_g(x_0,x)\le\rho)} \le (C_\star^{-1})^\sigma\beta^\vartheta e^\eta\|u\|_{L^{2^\star}(d_g(x_0,x)\le 2\rho)}.$$

Since $u\in L^{2^\star}(M)$, claim (i) follows at once. Notice that η depends on ρ.
Proof of (ii). Since V is coercive, we can find $\bar{R} > 0$ such that

$$V(x) \ge 2k \qquad \text{for} \qquad d_g(x_0,x)\ge\bar{R}$$

(where k is from the growth of φ. Without loss of generality we can assume that $k\ge 1$.)

Let $R > \max\{\bar{R},1\}$, $0 < r \le \frac{R}{2}$. In the proof of case ii), τ verifies the further following properties: $|\nabla\tau| \le \frac{2}{r}$ and τ is such that

$$\tau(x) = \begin{cases} 0 & \text{if } d_g(x_0,x)\le R,\\ 1 & \text{if } d_g(x_0,x) > R+r.\end{cases}$$

From (8.2.2), we get

$$\begin{aligned}\int_M(\langle\nabla_g u,\nabla_g v_L\rangle + 2kuv_L)\mathrm{d}v_g &= \int_{d_g(x_0,x)\ge R}(\langle\nabla_g u,\nabla_g v_L\rangle + 2kuv_L)\mathrm{d}v_g\\ &\le \int_{d_g(x_0,x)\ge R}(\langle\nabla_g u,\nabla_g v_L\rangle + V(x)uv_L)\mathrm{d}v_g\\ &= \int_M(\langle\nabla_g u,\nabla_g v_L\rangle + V(x)uv_L)\mathrm{d}v_g = \int_M\varphi(x,u)v_L\mathrm{d}v_g\\ &\le k\int_M(uv_L + u^{q-1}v_L)\mathrm{d}v_g,\end{aligned}$$

thus,

$$\int_M (\langle \nabla_g u, \nabla_g v_L\rangle + uv_L)\mathrm{d}v_g \le \int_M (\langle \nabla_g u, \nabla_g v_L\rangle + kuv_L)\mathrm{d}v_g \le k\int_M u^{q-1}v_L \mathrm{d}v_g.$$

From (8.2.3) and (8.2.4), and since $w_L^2 = u\cdot v_L$,

$$\begin{aligned}
\int_M (|\nabla_g w_L|^2 + w_L^2)\mathrm{d}v_g &\le \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + 2\beta^2\int_M \langle\nabla_g u, \nabla_g v_L\rangle \mathrm{d}v_g + \int_M uv_L\mathrm{d}v_g\\
&\le \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + 2\beta^2\int_M (\langle\nabla_g u, \nabla_g v_L\rangle + uv_L)\mathrm{d}v_g\\
&\le \beta\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g + 2\beta^2 k\int_M u^{q-1}v_L\mathrm{d}v_g.
\end{aligned}$$

Thus,

$$\|w_L\|^2_{H^1_g(M)} \le \beta\underbrace{\int_M u^2\ u_L^{2(\beta-1)}|\nabla_g\tau|^2\mathrm{d}v_g}_{I_1} + 2\beta^2 k\underbrace{\int_M u^{q-2}w_L^2\mathrm{d}v_g}_{I_2}.$$

As in the proof of $i)$ one has

$$I_1 \le 4\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\ \frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2}\left(\int_{d_g(x_0,x)\ge R} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}},$$

and

$$I_2 \le \|u\|^{q-2}_{L^{2^\star}(M)}\left(\int_{d_g(x_0,x)\ge R} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}.$$

Since,

$$\|w_L\|^2_{H^1_g(M)} \ge C^\star\|w_L\|^2_{L^{2^\star}(M)} = C^\star\left(\int_M (\tau\ u\ u_L^{\beta-1})^{2^\star}\mathrm{d}v_g\right)^{\frac{2}{2^\star}} \ge C^\star\left(\int_{d_g(x_0,x)\ge R+r} (u\ u_L^{\beta-1})^{2^\star}\mathrm{d}v_g\right)^{\frac{2}{2^\star}},$$

where $C^\star$ denotes the embedding constant of $H^1_g(M)$ into $L^{2^\star}(M)$, we obtain

$$\begin{aligned}
\left(\int_{d_g(x_0,x)\ge R+r} (u\ u_L^{\beta-1})^{2^\star}\mathrm{d}v_g\right)^{\frac{2}{2^\star}} \le & 4(C^\star)^{-1}\beta^2\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\ \frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2}\cdot\\
&\cdot\left(\int_{d_g(x_0,x)\ge R} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}} +\\
&+ 2(C^\star)^{-1}\beta^2 k\|u\|^{q-2}_{L^{2^\star}(M)}\cdot\left(\int_{d_g(x_0,x)\ge R} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}.
\end{aligned}$$

Taking the limit as $L\to+\infty$ in the above inequality, we obtain

$$\begin{aligned}
\left(\int_{d_g(x_0,x)\ge R+r} u^{2^\star\beta}\mathrm{d}v_g\right)^{\frac{2}{2^\star}} \le & 4(C^\star)^{-1}\beta^2\omega_n^{1-\frac{2}{\gamma}} e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}\frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2}\left(\int_{d_g(x_0,x)\ge R} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}} +\\
&+ 2(C^\star)^{-1}\beta^2 k\|u\|^{q-2}_{2^\star}\left(\int_{d_g(x_0,x)\ge R} u^\gamma\ u_L^{\gamma(\beta-1)}\mathrm{d}v_g\right)^{\frac{2}{\gamma}}.
\end{aligned}$$

Thus, for every $R > \max\{\bar{R}, 1\}$, $0 < r \le \frac{R}{2}$, $\beta > 1$ one has

$$\|u\|_{L^{2^\star\beta}(d_g(x_0,x)\ge R+r)} \le ((C^\star)^{-1})^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}\left(C_1\ \frac{(R+r)^{n(1-\frac{2}{\gamma})}}{r^2}+C_2\right)^{\frac{1}{2\beta}}\|u\|_{L^{\gamma\beta}(d_g(x_0,x)\ge R)}, \quad (8.2.8)$$

where $C_1 = 4\omega_n^{1-\frac{2}{\gamma}}e^{(n-1)b_0\left(1-\frac{2}{\gamma}\right)}$, $C_2 = 2k\|u\|^{q-2}_{L^{2^\star}(M)}$. Fix $\rho > \max\{\bar{R}, 1\}$. We are going to apply (8.2.8) choosing first $\beta = \dfrac{2^\star}{\gamma}$, $R = \rho + \dfrac{\rho}{2}$, $r = \dfrac{\rho}{2}$, to get

$$\|u\|_{L^{2^\star\beta}(d_g(x_0,x)\ge 2\rho)} \le ((C^\star)^{-1})^{\frac{1}{2\beta}}\beta^{\frac{1}{\beta}}\left(C_1\ 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^2+C_2\right)^{\frac{1}{2\beta}}\|u\|_{L^{2^\star}(d_g(x_0,x)\ge\rho+\frac{\rho}{2})}.$$

Noticing that $\gamma\beta^2 = 2^\star\beta$, let us apply (8.2.8) with β^2 in place of β and $R = \rho + \frac{\rho}{2^2}$, $r = \frac{\rho}{2^2}$, to obtain

$$\begin{aligned}
\|u\|_{L^{2^\star\beta^2}(d_g(x_0,x)\ge\rho+\frac{\rho}{2})} &\le ((C^\star)^{-1})^{\frac{1}{2\beta^2}}\beta^{\frac{2}{\beta^2}}\left(C_1\ \frac{(\rho+\frac{\rho}{2})^{n(1-\frac{2}{\gamma})}}{\rho^2}(2^2)^2+C_2\right)^{\frac{1}{2\beta^2}}\|u\|_{L^{2^\star\beta}(d_g(x_0,x)\ge\rho+\frac{\rho}{2^2})}\\
&\le ((C^\star)^{-1})^{\frac{1}{2\beta^2}}\beta^{\frac{2}{\beta^2}}\left(C_1\ 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}(2^2)^2+C_2\right)^{\frac{1}{2\beta^2}}\|u\|_{L^{2^\star\beta}(d_g(x_0,x)\ge\rho+\frac{\rho}{2^2})}
\end{aligned}$$

Thus, combining the previous two inequalities we get

$$\begin{aligned}
\|u\|_{L^{2^\star\beta}(d_g(x_0,x)\ge 2\rho)} &\le ((C^\star)^{-1})^{\frac{1}{2\beta}+\frac{1}{2\beta^2}}\beta^{\frac{1}{\beta}+\frac{2}{\beta^2}}e^{\frac{1}{2\beta}\log\left(C_1\ 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^2+C_2\right)}\cdot\\
&\quad\cdot e^{\frac{1}{2\beta^2}\log\left(C_1\ 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}(2^2)^2+C_2\right)}\|u\|_{L^{2^\star}(d_g(x_0,x)\ge\rho+\frac{\rho}{2^2})}
\end{aligned}$$

Iterating this procedure, for every integer m we obtain

$$\begin{aligned}
&\|u\|_{L^{2^\star\beta^m}(d_g(x_0,x)\ge 2\rho)}\\
\le\ &((C^\star)^{-1})^{\sum_{i=1}^{m}\frac{1}{2\beta^i}}\cdot\beta^{\sum_{i=1}^{m}\frac{i}{\beta^i}}\cdot e^{\sum_{i=1}^{m}\frac{\log\left(C_1\ 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^{2i}+C_2\right)}{2\beta^i}}\|u\|_{L^{2^\star}(d_g(x_0,x)\ge\rho+\frac{\rho}{2^m})}\\
\le\ &((C^\star)^{-1})^{\sum_{i=1}^{m}\frac{1}{2\beta^i}}\cdot\beta^{\sum_{i=1}^{m}\frac{i}{\beta^i}}\cdot e^{\sum_{i=1}^{m}\frac{\log\left(C_1\ 2^{n(1-\frac{2}{\gamma})}\rho^{n(1-\frac{2}{\gamma})-2}2^{2i}+C_2\right)}{2\beta^i}}\|u\|_{L^{2^\star}(d_g(x_0,x)\ge\rho)}.
\end{aligned}$$

Since $n(1-\frac{2}{\gamma}) < 2$ and $\rho > 1$, one has $\rho^{n(1-\frac{2}{\gamma})-2} < 1$, and the previous estimate implies

$$\|u\|_{L^{2^\star\beta^m}(d_g(x_0,x)\ge 2\rho)} \le ((C^\star)^{-1})^{\sum_{i=1}^{m}\frac{1}{2\beta^i}}\cdot\beta^{\sum_{i=1}^{m}\frac{i}{\beta^i}}\cdot e^{\sum_{i=1}^{m}\frac{\log\left(C_1\ 2^{n(1-\frac{2}{\gamma})}2^{2i}+C_2\right)}{2\beta^i}}\|u\|_{L^{2^\star}(d_g(x_0,x)\ge\rho)}.$$

If

$$\sigma = \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{\beta^m} = \frac{1}{2(\beta-1)},\ \vartheta = \sum_{m=1}^{\infty}\frac{m}{\beta^m},\ \zeta = \sum_{m=1}^{\infty}\frac{\log\left(C_1\ 2^{n(1-\frac{2}{\gamma})}2^{2m}+C_2\right)}{2\beta^m},$$

passing to the limit as $m \to \infty$, we obtain

$$\|u\|_{L^\infty(d_g(x_0,x)\geq 2\rho)} \leq C_0\|u\|_{L^{2^\star}(d_g(x_0,x)\geq\rho)}$$

where $C_0 = (C^\star)^{-\sigma}\beta^\vartheta e^\zeta$ does not depend on ρ. Taking into account that $u \in L^{2^\star}(M)$, and combining the above inequality with claim $i)$, we obtain that $u \in L^\infty(M)$. Moreover, as $\lim\limits_{\rho\to\infty} \|u\|_{L^{2^\star}(d_g(x_0,x)\geq\rho)} = 0$, we deduce also that $\lim\limits_{d_g(x_0,x)\to\infty} u(x) = 0$. □

8.2.2. A minimization problem

Now, we consider the following minimization problem:

(M) $\min\left\{\|u\|_V^2 :\ u \in H_V^1(M),\ \|\alpha^{\frac{1}{2}}u\|_{L^2(M)} = 1\right\}.$

Lemma 8.2.1. *Problem* **(M)** *has a non negative solution* $\varphi_\alpha \in L^\infty(M)$ *such that for every* $x_0 \in M$, $\lim\limits_{d_g(x_0,x)\to\infty} \varphi_\alpha(x) = 0$. *Moreover,* φ_α *is an eigenfunction of the equation*

$$-\Delta_g u + V(x)u = \lambda\alpha(x)u, \qquad u \in H_V^1(M)$$

corresponding to the eigenvalue $\|\varphi_\alpha\|_V^2$.

Proof. Notice first that $\alpha^{\frac{1}{2}}u \in L^2(M)$ for any $u \in H_V^1(M)$. Fix a minimizing sequence $\{u_n\}$ for problem **(M)**, that is $\|u_n\|_V^2 \to \lambda_\alpha$, being

$$\lambda_\alpha = \inf\left\{\|u\|_V^2 :\ u \in H_V^1(M), \|\alpha^{\frac{1}{2}}u\|_{L^2(M)} = 1\right\}.$$

Then, there exists a subsequence (still denoted by $(u_j)_j$) weakly converging in $H_V^1(M)$ to some $\varphi_\alpha \in H_V^1(M)$. By the weak lower semicontinuity of the norm, we obtain that

$$\|\varphi_\alpha\|_V^2 \leq \liminf_n \|u_j\|_V^2 = \lambda_\alpha.$$

In order to conclude, it is enough to prove that $\|\alpha^{\frac{1}{2}}\varphi_\alpha\|_{L^2(M)} = 1$. Since $(u_j)_j$ converges strongly to φ_α in $L^2(M)$ and $\alpha \in L^\infty(M)$,

$$\alpha^{\frac{1}{2}}u_n \to \alpha^{\frac{1}{2}}\varphi_\alpha \qquad \text{in } L^2(M),$$

thus, by the continuity of the norm, $\|\alpha^{\frac{1}{2}}\varphi_\alpha\|_{L^2(M)} = 1$ and the claim is proved. Clearly, $\varphi_\alpha \neq 0$. Replacing eventually φ_α with $|\varphi_\alpha|$ we can assume that φ_α is non negative. Equivalently, we can write

$$\lambda_\alpha = \inf_{u\in H_V^1(M)\setminus\{0\}} \frac{\|u\|_V^2}{\|\alpha^{\frac{1}{2}}u\|_{L^2(M)}^2}.$$

This means that φ_α is a global minimum of the function $u \to \dfrac{\|u\|_V^2}{\|\alpha^{\frac{1}{2}}u\|_{L^2(M)}^2}$, hence its derivative at φ_α is zero, i.e.

$$\int_M (\langle\nabla_g\varphi_\alpha, \nabla_g v\rangle + V(x)\varphi_\alpha v)\mathrm{d}v_g - \|\varphi_\alpha\|_V^2 \int_M \alpha(x)\varphi_\alpha v\mathrm{d}v_g = 0 \text{ for any } v \in H_V$$

(recall that $\|\alpha^{\frac{1}{2}}\varphi_\alpha\|_{L^2(M)} = 1$). The above equality implies that φ_α is an eigenfunction of the problem

$$-\Delta_g u + V(x)u = \lambda\alpha(x)u, \qquad u \in H_V^1(M)$$

corresponding to the eigenvalue $\|\varphi_\alpha\|_V^2$. From Theorem 8.2.1 we also have that φ_α is a bounded function and $\lim\limits_{d_g(x,x_0)\to\infty} \varphi_\alpha(x) = 0$. □

8.2.3. Characterization of weak solutions

Now we are in the position to prove our main theorem.

Proof of Theorem 8.1.1. $(i) \Rightarrow (ii)$.

From the assumption, we deduce the existence of $\sigma_1 \in (0, +\infty]$ defined as

$$\sigma_1 \equiv \lim_{\xi \to 0} \frac{F(\xi)}{\xi^2}.$$

Assume first that $\sigma_1 < \infty$.

Define the following continuous truncation of f,

$$\tilde{f}(\xi) = \begin{cases} 0, & \text{if } \xi \in (-\infty, 0] \\ f(\xi), & \text{if } \xi \in (0, a] \\ f(a), & \text{if } \xi \in (a, +\infty) \end{cases}$$

and let $\tilde{F}$ its primitive, that is $\tilde{F}(\xi) = \int_0^\xi \tilde{f}(t)dt$, i.e.

$$\tilde{F}(\xi) = \begin{cases} F(\xi), & \text{if } \xi \in (-\infty, a] \\ F(a) + f(a)(\xi - a), & \text{if } \xi \in (a, +\infty). \end{cases}$$

Observe that, from the monotonicity assumption on the function $\xi \to \frac{F(\xi)}{\xi^2}$, the derivative of the latter is non-positive, that is

$$f(\xi)\xi \leq 2F(\xi) \qquad \text{for all } \xi \in [0, a].$$

This implies

$$\tilde{f}(\xi)\xi \leq 2\tilde{F}(\xi) \qquad \text{for all } \xi \in \mathbb{R}, \tag{8.2.9}$$

or that the function $\xi \to \frac{\tilde{F}(\xi)}{\xi^2}$ is not increasing in $(0, +\infty)$. Then,

$$\sigma_1 \equiv \lim_{\xi \to 0} \frac{F(\xi)}{\xi^2} = \lim_{\xi \to 0} \frac{\tilde{F}(\xi)}{\xi^2} = \sup_{\xi > 0} \frac{\tilde{F}(\xi)}{\xi^2}. \tag{8.2.10}$$

Moreover,

$$\tilde{F}(\xi) \leq \sigma_1 \xi^2 \ \text{ and } \ \tilde{f}(\xi) \leq 2\sigma_1 \xi, \qquad \text{for all } \xi \in \mathbb{R} \tag{8.2.11}$$

Define now the functional

$$J : H^1_V(M) \to \mathbb{R}, \qquad J(u) = \int_M \alpha(x)\tilde{F}(u)\mathrm{d}v_g,$$

which is well defined, sequentially weakly continuous, Gâteaux differentiable with derivative given by

$$J'(u)(v) = \int_M \alpha(x)\tilde{f}(u)v\mathrm{d}v_g \qquad \text{for all } v \in H^1_V(M).$$

Moreover, $J(0) = 0$ and

$$\sup_{u \in H^1_V(M)\setminus\{0\}} \frac{J(u)}{\|u\|^2_V} = \frac{\sigma_1}{\lambda_\alpha}. \tag{8.2.12}$$

Indeed, from (8.2.11) immediately follows that

$$\frac{J(u)}{\|u\|_V^2} \leq \frac{\sigma_1}{\lambda_\alpha} \text{ for every } \ u \in H^1_V(M) \setminus \{0\}.$$

Also, using the monotonicity assumption, for every $t > 0$, and for every $x \in M$, such that $\varphi_\alpha(x) > 0$

$$\frac{\tilde{F}(t\varphi_\alpha(x))}{(t\varphi_\alpha(x))^2} \geq \frac{\tilde{F}\left(t\|\varphi_\alpha\|_{L^\infty(M)}\right)}{t^2\|\varphi_\alpha\|^2_{L^\infty(M)}},$$

thus

$$\begin{aligned} J(t\varphi_\alpha) &= \int_{\{\varphi_\alpha>0\}} \alpha(x)\frac{\tilde{F}(t\varphi_\alpha)}{(t\varphi_\alpha)^2}(t\varphi_\alpha)^2 \mathrm{d}v_g \geq \frac{\tilde{F}(t\|\varphi_\alpha\|_{L^\infty(M)})}{\|\varphi_\alpha\|^2_{L^\infty(M)}}\int_M \alpha(x)\varphi_\alpha^2 \mathrm{d}v_g \\ &= \frac{\tilde{F}(t\|\varphi_\alpha\|_{L^\infty(M)})}{\|\varphi_\alpha\|^2_{L^\infty(M)}} > 0. \end{aligned}$$

Thus,

$$\frac{J(t\varphi_\alpha)}{\|t\varphi_\alpha\|_V^2} = \frac{J(t\varphi_\alpha)}{t^2\lambda_\alpha} \geq \frac{\tilde{F}(t\|\varphi_\alpha\|_{L^\infty(M)})}{\|t\varphi_\alpha\|^2_{L^\infty(M)}}\frac{1}{\lambda_\alpha}.$$

Passing to the limit as $t \to 0^+$, from (8.2.10), condition (8.2.12) follows at once. Let us now apply Theorem 1.2.11 with $X = H^1_V(M)$ and J as above. Let $r > 0$ and denote by $\hat{u}$ the global maximum of $J_{|B_{x_0}(r)}$. We observe that $\hat{u} \neq 0$ as $J(t\varphi_\alpha) > 0$ for every t small enough, thus $J(\hat{u}) > 0$. If $\hat{u} \in \operatorname{int}(B_{x_0}(r))$, then, it turns out to be a critical point of J, that is $J'(\hat{u}) = 0$ and (1.2.1) is satisfied. If $\|\hat{u}\|_V^2 = r$, then, from the Lagrange multiplier rule, there exists $\mu > 0$ such that $J'(\hat{u}) = \mu\hat{u}$, that is, $\hat{u}$ is a solution of the equation

$$-\Delta_g u + V(x)u = \frac{1}{\mu}\alpha(x)\tilde{f}(u), \text{ in } M.$$

Also, by Theorem 8.2.1, $\hat{u} \in L^\infty(M)$ and $\lim\limits_{d_g(x_0,x)\to\infty} \hat{u}(x) = 0$. Condition (8.2.9) implies in addition that

$$J'(\hat{u})(\hat{u}) - 2J(\hat{u}) = \int_M \alpha(x)[\tilde{f}(\hat{u})\hat{u} - 2\tilde{F}(\hat{u})]\mathrm{d}v_g \leq 0.$$

If the latter integral is zero, then, being $\alpha > 0$, $\tilde{f}(\hat{u}(x))\hat{u}(x) - 2\tilde{F}(\hat{u}(x)) = 0$ for all $x \in M$, which in turn implies that $\tilde{f}(\xi)\xi - 2\tilde{F}(\xi) = 0$ for all $\xi \in [0, \|\hat{u}\|_{L^\infty(M)}]$, that is, the function $\xi \to \frac{\tilde{F}(\xi)}{\xi^2}$ is constant in the interval $]0, \|\hat{u}\|_{L^\infty(M)}]$. In particular it would be constant in a small neighborhood of zero which is in contradiction with the assumption (i). This means that (1.2.1) is fulfilled and the thesis applies: there exists an interval $I \subseteq (0, +\infty)$ such that for every $\lambda \in I$ the functional

$$u \to \frac{\|u\|_V^2}{2} - \lambda J(u)$$

has a non-zero critical point u_λ with $\displaystyle\int_M (|\nabla u_\lambda|^2 + V(x)u_\lambda^2)\mathrm{d}v_g < r$. In particular, u_λ turns out to be a nontrivial solution of the problem

$$\begin{cases} -\Delta_g u + V(x)u = \lambda\alpha(x)\tilde{f}(u), & \text{in } M \\ u \geq 0, & \text{in } M \\ u \to 0, & \text{as } d_g(x, x_0) \to \infty. \end{cases} \tag{$\tilde{\mathcal{P}}_\lambda$}$$

From Remark 1.2.1, we know that $I = \dfrac{1}{2}\left(\eta(r\delta_r), \lim\limits_{s\to\beta_r}\eta(s)\right)$. It is clear that

$$\eta(r\delta_r) = \sup_{y\in B_r}\frac{r - \|y\|_V^2}{r\delta_r - J(y)} \geq \frac{1}{\delta_r}$$

and by the definition of δ_r,

$$\frac{r-\|y\|^2}{r\delta_r - J(y)} \leq \frac{r-\|y\|_V^2}{r\delta_r - \delta_r\|y\|_V^2} = \frac{1}{\delta_r}$$

for every $y \in B_r$. Thus, recalling (8.2.12),

$$\eta(r\delta_r) = \frac{1}{\delta_r} = \frac{\lambda_\alpha}{\sigma_1}.$$

Notice also that from Theorem 8.2.1, $u_\lambda \in L^\infty(M)$. Let us prove that

$$\lim_{\lambda \to \frac{\lambda_\alpha}{2\sigma_1}} \|u_\lambda\|_{L^\infty(M)} = 0.$$

Fix a sequence $\lambda_j \to \left(\frac{\lambda_\alpha}{2\sigma_1}\right)^+$. Since $\|u_{\lambda_j}\|_V^2 \leq r$, $(u_{\lambda_j})_j$ admits a subsequence still denoted by $(u_{\lambda_j})_j$ which is weakly convergent to some $u_0 \in B_{x_0}(r)$. Moreover, from the compact embedding of $H_V^1(M)$ in $L^2(M)$, $(u_{\lambda_j})_j$ converges (up to a subsequence) strongly to u_0 in $L^2(M)$. Thus, being u_{λ_j} a solution of $(\mathcal{P}_{\lambda_n})$,

$$\int_M (\langle \nabla_g u_{\lambda_j}, \nabla_g v\rangle + V(x)u_{\lambda_j}v)\mathrm{d}v_g = \lambda_j \int_M \alpha(x)\tilde{f}(u_{\lambda_j})v\mathrm{d}v_g \qquad \text{for all } \; v \in H_V^1(M), \tag{8.2.13}$$

passing to the limit we obtain that u_0 is a solution of the equation

$$-\Delta_g u + V(x)u = \frac{\lambda_\alpha}{2\sigma_1}\alpha(x)\tilde{f}(u) \text{ in } M.$$

Assume $u_0 \neq 0$. Thus, testing (8.2.13) with $v = u_{\lambda_j}$,

$$\|u_{\lambda_j}\|_V^2 = \lambda_j \int_M \alpha(x)\tilde{f}(u_{\lambda_j})u_{\lambda_j}\mathrm{d}v_g,$$

and passing to the limit,

$$\begin{aligned}
\|u_0\|_V^2 &\leq \liminf_{n\to\infty} \|u_{\lambda_j}\|_V^2 = \frac{\lambda_\alpha}{2\sigma_1}\int_M \alpha(x)\tilde{f}(u_0)u_0\mathrm{d}v_g \\
&< \frac{\lambda_\alpha}{\sigma_1}\int_M \alpha(x)\tilde{F}(u_0)\mathrm{d}v_g \leq \lambda_\alpha \int_M \alpha(x)u_0^2\mathrm{d}v_g \\
&\leq \|u_0\|_V^2.
\end{aligned}$$

The above contradiction implies that $u_0 = 0$, and that $\lim_{j\to\infty} \|u_{\lambda_j}\|_V = 0$. Thus, in particular, because of the embedding into $L^{2^\star}(M)$, we deduce that $\lim_{j\to\infty} \|u_{\lambda_j}\|_{L^{2^\star}(M)} = 0$ and from Theorem 8.2.1, $\lim_{j\to\infty} \|u_{\lambda_j}\|_{L^\infty(M)} = 0$. Therefore,

$$\lim_{\lambda \to \frac{\lambda_\alpha}{2\sigma_1}^+} \|u_\lambda\|_{L^\infty(M)} = 0.$$

This implies that there exists a number $\varepsilon_r > 0$ such that for every $\lambda \in \left(\frac{\lambda_\alpha}{2\sigma_1}, \frac{\lambda_\alpha}{2\sigma_1} + \varepsilon_r\right)$, $\|u_\lambda\|_{L^\infty(M)} \leq a$. Hence, u_λ turns out to be a solution of the original problem $(\mathscr{P}_\lambda)$ and the proof of this first case is concluded.

Assume now $\sigma_1 = +\infty$. The functional

$$K : H_V^1(M) \to \mathbb{R}, \;\; K(u) = \int_M \alpha(x)F(u)dv_g.$$

is well defined and sequentially weakly continuous. Let $r > 0$ and fix $\lambda \in I = \frac{1}{2}\left(0, \frac{1}{\lambda^*}\right)$ where

$$\lambda^* = \inf_{\|y\|_V^2 < r} \frac{\sup_{\|u\|_V^2 \leq r} K(u) - K(y)}{r - \|y\|_V^2}$$

(with the convention $\frac{1}{\lambda^*} = +\infty$ if $\lambda^* = 0$). Denote by u_λ the global minimum of the restriction of the functional $\mathcal{E}$ to B_r. Then, since

$$\lim_{t \to 0} \frac{K(t\varphi_\alpha)}{\|t\varphi_\alpha\|_V^2} = +\infty,$$

it is easily seen that $\mathcal{E}(u_\lambda) < 0$, therefore, $u_\lambda \neq 0$. The choice of λ implies, via easy computations, that $\|u_\lambda\|_V^2 < r$. So, u_λ is a critical point of $\mathcal{E}$, thus a weak solution of $(\mathcal{SM}_\lambda)$.

$(ii) \Rightarrow (i)$. Assume by contradiction that there exist two positive constants b, c such that

$$\frac{F(\xi)}{\xi^2} = c \qquad \text{for all } \ \xi \in (0, b].$$

Thus,

$$f(\xi) = 2c\xi \qquad \text{for all } \ \xi \in [0, b]. \tag{8.2.14}$$

Let $(r_m)_m$ be a sequence of positive numbers such that $r_m \to 0^+$. Then, for every $m \in \mathbb{N}$ there exists an interval I_m such that for every $\lambda \in I_m$, $(\mathscr{P}_\lambda)$ has a solution $u_{\lambda,m}$ with $\|u_{\lambda,m}\|_V^2 < r_m$. Thus,

$$\lim_m \sup_{\lambda \in I_m} \|u_{\lambda,m}\|_V = 0.$$

Since $f(\xi) \leq k(\xi + \xi^{q-1})$ for all $\xi \geq 0$ (this follows from the growth assumption (8.1.1) and equality (8.2.14)), and being $u_{\lambda,m}$ a critical point of $\mathscr{E}$, from the continuous embedding of $H_V^1(M)$ into $L^{2^\star}(M)$ and by Theorem 8.2.1 we obtain that

$$\lim_m \sup_{\lambda \in I_m} \|u_{\lambda,m}\|_{L^\infty(M)} = 0.$$

Let us fix m_0 big enough, such that $\sup_{\lambda \in I_m} \|u_{\lambda,m}\|_{L^\infty(M)} < b$. We deduce that for every $\lambda \in I_{m_0}$, u_{λ,m_0} is a solution of the equation

$$-\Delta_g u + V(x)u = 2\lambda c\alpha(x)u, \ \text{in } M,$$

against the discreteness of the spectrum of the Schrödinger operator $-\Delta_g + V(x)$ established in Theorem 1.3.4. $\square$

Remark 8.2.1. *Notice that without the growth assumption (8.1.1) the result holds true replacing the norm of the solutions u_λ in the Sobolev space with the norm in $L^\infty(M)$.*

We conclude the section with a corollary of the main result in the euclidean setting. We propose a more general set of assumption on V which implies both the compactness of the embedding of $H_V^1(\mathbb{R}^n)$ into and the discreteness of the spectrum of the Schrödinger operator, see Benci and Fortunato [19]. Namely, let $n \geq 3$, $\alpha : \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ be in $L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $f(0) = 0$ such that there exist two constants $k > 0$ and $q \in (1, 2^\star)$ such that

$$f(\xi) \leq k(1 + \xi^{q-1}) \text{ for all } \xi \geq 0.$$

Let also $V : \mathbb{R}^n \to \mathbb{R}$ be in $L^\infty_{\text{loc}}(\mathbb{R}^n)$, such that $\operatorname{essinf}_{\mathbb{R}^n} \mathrm{V} \equiv \mathrm{V}_0 > 0$ and

$$\int_{B(x)} \frac{1}{V(y)} dy \to 0 \qquad \text{as } |x| \to \infty,$$

where $B(x)$ denotes the unit ball in $\mathbb{R}^n$ centered at x. In particular, if V is a strictly positive ($\inf_{\mathbb{R}^n} V > 0$), continuous and coercive function, the above conditions hold true.

Corollary 8.2.1. *Assume that for some $a > 0$ the function $\xi \to \frac{F(\xi)}{\xi^2}$ is non-increasing in $(0, a]$. Then, the following conditions are equivalent:*

(i) for each $b > 0$, the function $\xi \to \frac{F(\xi)}{\xi^2}$ is not constant in $(0, b]$;

(ii) for each $r > 0$, there exists an open interval $I \subseteq (0, +\infty)$ such that for every $\lambda \in I$, problem

$$\begin{cases} -\Delta u + V(x)u = \lambda\alpha(x)f(u), & \text{in } \mathbb{R}^n \\ u \geq 0, & \text{in } \mathbb{R}^n \\ u \to 0, & \text{as } |x| \to \infty \end{cases}$$

has a nontrivial solution $u_\lambda \in H^1(\mathbb{R}^n)$ satisfying $\displaystyle\int_{\mathbb{R}^n} \left(|\nabla u_\lambda|^2 + V(x)u_\lambda^2\right) dx < r$.

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