On Estimating the Risk of an Investment Portfolio

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***Abstract* — According to Markowitz's classic approach, the risk of an investment portfolio is determined by the standard deviation of its return. Two other risk assessment methods are considered in this paper. One calculates the so-called *VaR* (Value at Risk), while the other calculates the standard deviation for a portfolio return not exceeding the mean return. A computational formula is derived for the latter. In the typical case of continuous return distribution, some simpler and more convenient definition for *VaR* is given. These three risk assessment methods are compared between each other.**

Keywords* — *Markowitz approach, investment portfolio, return, risk, standard deviation, semi-deviation, VaR.

An investment portfolio is a set of securities acquired by an investor in the present to get income in the future. Such a portfolio is defined by a set of numbers $(p_1; \dots; p_i; \dots; p_N)$ representing the shares of capital invested in each of securities, where N is the number of securities in the investment portfolio. Herewith

$$p_1 + \dots + p_i + \dots + p_N = 1 \qquad p_i \geq 0$$

The return of the i^{th} security is expressed by the following formula:

$$r_{it} = \frac{q_{it} - q_{i(t-1)}}{q_{i(t-1)}}$$

where q_{it} and $q_{i(t-1)}$ are the security quotes at times t and $t-1$.

The return r_{pt} of the investment portfolio at time t is expressed as follows:

$$r_{pt} = \sum_{i=1}^{N} p_i r_{it}$$

Assuming that T is the current moment in time, it is necessary to predict the portfolio return at the next time $T+1$. Denote this random variable as $R_{p(T+1)}$. The predicted value of $R_{p(T+1)}$ is the mathematical expectation:

$$E[R_{p(T+1)}] = E\left[\sum_{i=1}^{N} p_i R_{i(T+1)}\right] = \sum_{i=1}^{N} p_i \cdot E[R_{i(T+1)}]$$

where $R_{i(T+1)}$ is a random variable representing the i^{th} security return at time $T+1$.

Using historical data, it is possible to calculate a statistical estimate for the value $E[R_{i(T+1)}]$ as the mean $\bar{r}_i$ of the i^{th} security returns over the entire previous period:

$$E[R_{i(T+1)}] \approx \bar{r}_i = \frac{1}{T-1}\sum_{t=2}^{T} r_{it}$$

H. Markowitz proposed using the standard deviation σ of the investment portfolio return as a measure of the risk [1]. Herewith the variance σ_i^2 of the i^{th} security return at time $T+1$ is given by the formula:

$$\sigma_i^2 = E\left[\left(R_{i(T+1)} - E[R_{i(T+1)}]\right)^2\right]$$

Sample estimate of the variance of the i^{th} security looks like:

$$\sigma_i^2 \approx s_i^2 = \frac{1}{T-2}\sum_{t=2}^{T}(r_{it} - \bar{r}_i)^2$$

When using all the above formulas, the time series r_{it} of each security returns was implicitly assumed to be ergodic and, accordingly, stationary [2]. It is difficult to verify the ergodicity in practice, so they usually limit themselves to checking only stationarity. A new method for checking the stationarity of time series is presented in [3].

It is important for an investor to have a forecast of a possible loss. If the standard deviation σ is used as a measure of the risk, and the distribution of the investment portfolio returns is asymmetric, not quite correct estimate for the return decrease (or the loss) may be obtained.

In the case of asymmetry in the return distribution, a more adequate measure of the risk is the standard deviation σ of such the portfolio returns that do not exceed their mean return [4]. Let's call this indicator the Standard Down Deviation (SDD). If $\sigma_- > \sigma$, then using σ as a risk measure would result in the underestimation of return decrease from its expected (i.e. mean) value. If $\sigma_- < \sigma$, then using σ as a risk measure would result in the unjustified overestimation of return decrease.

Let's calculate the SDD of the i^{th} security returns. Suppose that among the returns r_{it} at $t > 1$ there are $\tilde{T}$ values $r_{it_1}, r_{it_2}, \dots, r_{it_{\tilde{T}}}$ satisfying the condition $r_{it} < \bar{r}_i$. The following statistical estimate is used for calculating SDD σ_-:

$$\sigma_- \approx \tilde{s}_i, \qquad \tilde{s}_i^2 = \frac{1}{\tilde{T}-1}\sum_{j=1}^{\tilde{T}}(r_{it_j} - \bar{r}_i)^2$$

The variance of portfolio returns at the future time $T+1$ is given by the formula:

$$\sigma^2[R_{p(T+1)}] = \sum_{i=1}^{N}\sum_{j=1}^{N} p_i p_j cov_{ij}$$

where cov_{ij} is the covariance of the i^{th} and j^{th} security returns (remark $cov_{ii} = \sigma_i^2$).

Using historical data on securities returns, the covariance can be estimated by means of the following approximate formula:

$$cov_{ij} \approx \frac{1}{T-2}\sum_{t=2}^{T}(r_{it} - \bar{r}_i)\cdot(r_{jt} - \bar{r}_j)$$

In this paper one proposes a method for calculating SDD σ_- for a securities portfolio. It consists of the following.

Let's denote by R_{pt} the investment portfolio return and by R_{it} the i^{th} security return at time $t = 1, 2, \dots, T, \dots$ for each $i = 1, 2, \dots, N$. These random functions (random processes) are assumed to be ergodic. It was shown in [3] that the ergodicity condition is necessary for application of the effective portfolio theory [1] to be mathematically correct [5]. We have the following equations:

$$R_{pt} = \sum_{i=1}^{N} p_i R_{it} \qquad E[R_{pt}] = \sum_{i=1}^{N} p_i E[R_{it}]$$

Due to the supposed ergodicity of the time series of securities returns (i.e., the ergodicity of random functions R_{it} [2]), the mean returns $E[R_{it}]$ and $E[R_{pt}]$ as well as the standard deviations $\sigma_i = \sigma[R_{it}]$ and $\sigma = \sigma[R_{pt}]$ do not depend on t.

Now we are ready to describe a statistical estimate s_- for the SDD σ_- of portfolio returns. If among the actual returns r_{pt} at $t > 1$ there are $\tilde{T}$ values $r_{pt_1}, r_{pt_2}, \dots, r_{pt_{\tilde{T}}}$ satisfying the condition $r_{pt} < r_{\bar{p}}$, the following statistical estimate is used for calculating σ_-:

$$\sigma_- \approx s_-, \quad s_-^2 = \frac{1}{\tilde{T}-1}\sum_{j=1}^{\tilde{T}}(r_{pt_j} - r_{\bar{p}})^2, \quad \bar{r}_p = \sum_{i=1}^{N} p_i \bar{r}_i$$

The above considered risk measures for an investment portfolio allow us to estimate the deviation of its return from the mean. Another measure of risk that directly indicates the loss is *VaR* (Value at Risk). It is interpreted as the maximum possible loss value for a given confidence probability. In other words, for a given significance level α the probable loss does not exceed *VaR* with the probability $1 - \alpha$. However, in general case this definition is mathematically ambiguous. Consider examples 12.3, 12.4 on page 258 [6]. “A one-year project has a 98% chance of leading to a gain of $2 million, a 1.5% chance of leading to a loss of $4 million and a 0.5% chance of leading to a loss of $10 million. There is a probability of 0.5% of any specified loss level between $4 and $10 million being exceeded. *VaR* is therefore not uniquely defined. One reasonable convention in this type of situation is to set *VaR* equal to the midpoint of the range of possible *VaR* values. This means that, in this case, *VaR* would equal $7 million”.

However, it is difficult agree this opinion of the author [6], since from the practical point of view it would be much more informative to state that *VaR* is equal to 4 million dollars. Accordingly, well-known formula (1) should be used:

$$VaR_{1-\mathrm{d}} = \inf\{x: P(-r_p > x) \le \alpha\} \qquad (1)$$

where r_p is the portfolio return ($-r_p$ is correspondingly the loss), and α is the significance level at which $VaR_{1-\mathrm{d}}$ is calculated.

However, when analyzing formula (1) a hypothesis has arisen that the risk measure $VaR_{1-\mathrm{d}}$ is nothing else than the quantile of the loss distribution for a given probability $1 - \alpha$. As it turned out, this hypothesis is true under some restrictions on the loss distribution function. This result is presented in the following statement

Proposition 1: *Suppose the losses of an investment portfolio have a continuous distribution, and different values of the loss (which could occur in reality) are corresponded by different values of the losses distribution function. Then, for any given significance level* α *the risk estimate* $VaR_{1-\mathrm{d}}$ *is the* $(1-\alpha)$*-quantile of the losses distribution.*

Proof:

It follows from the condition of Proposition 1 that all the possible loss values fill an interval (half-interval, segment). In all further considerations the variables $x, x', y, \pm r_p$ are assumed to take their values in this interval (half-interval, segment).

Let the inequality $P(-r_p > x) \le \alpha$ holds for some x. We can rewrite this as $P(r_p < -x) \le \alpha$. If $P(r_p < -x) = \alpha$, then $-x$ is the α-quantile of the portfolio returns distribution while x is the $(1-\alpha)$-quantile of the losses distribution.

Let's verify the following equality:

$$x = \inf\{x': P(-r_p > x') \le \alpha\} = VaR_{1-\mathrm{d}} \qquad (2)$$

Since $P(-r_p > x) = \alpha$, then when $P(-r_p > y) \le \alpha$ there takes place $x \le y$ (if $x > y$, then $P(-r_p > y) > P(-r_p > x) = \alpha$ which contradicts to $P(-r_p > y) \le \alpha$.

The case $P(-r_p > y) = P(-r_p > x)$ is impossible here, otherwise the values of the distribution function at points x and y would be equal which contradicts to Proposition 1).

Thus, it has been proved that x is a lower bound for the set $\{x': P(-r_p > x') \le \alpha\}$. Since $x \in \{x': P(-r_p > x') \le \alpha\}$, then $x = \inf\{x': P(-r_p > x') \le \alpha\}$.

Assume that $P(r_p < -x) < \alpha$, i.e., $P(-r_p > x) < \alpha$.

It is clear, the smaller x the higher the probability of the event $-r_p > x$. With a continuous decrease in x the value of $P(-r_p > x)$ increases continuously and, it would seem, the value should become equal to α at a certain value of x. However, the situation is theoretically possible when, for a certain number $0 < \beta < \alpha$, for all values of x obtained during the decreasing process we have $P(-r_p > x) \le \beta$. That is, $P(r_p < -x) \le \beta$ and the value of $P(r_p < -x)$ indefinitely approaches β as x decreases. In such a situation, for any sufficiently small x we have $P(r_p < -x) \leq \beta$. This contradicts to the fact that

$$\lim_{-x \to +\infty} F_{r_p}(-x) = 1$$

where F_{r_p} is the portfolio returns distribution function.

Thus, for some x satisfying $P(-r_p > x) \leq \alpha$ there takes place $P(-r_p > x) = \alpha$. Since this x is the $(1-\alpha)$-quantile of the losses distribution, the Proposition 1 is proved.

The condition imposed on continuously distributed portfolio losses in Proposition 1 is not restrictive and, obviously, fulfilled in all economically realistic situations. Therefore, for the practical application of Proposition 1 it is sufficient the portfolio losses could be modelled by a random variable with continuous distribution. Proposition 1 is not applicable in the case of discrete losses distribution, formula (1) should be used instead.

Among the considered measures for the risk of loss (or decrease of return) resulted from investing in a securities portfolio, SDD and *VaR* should be highlighted. If SDD allows the investor to estimate the possible decrease of expected return, then *VaR* gives a representation of how significant the loss may be. The combination of these two indicators will help in making more informed decision.

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